算法设计与分析(2024年春季学期)

Greedy Algorithms

授课老师: 栗师

南京大学计算机科学与技术系

Def. In an optimization problem, our goal of is to find a valid solution with the minimum cost (or maximum value).

Trivial Algorithm for an Optimization Problem

Enumerate all valid solutions, compare them and output the best one.

- However, trivial algorithm often runs in exponential time, as the number of potential solutions is often exponentially large.
- $f(n)$ is a polynomial if $f(n) = O(n^k)$ for some constant $k > 0$.
- convention: polynomial time $=$ efficient

Goals of algorithm design

1. Design efficient algorithms to solve problems
2. Design more efficient algorithms to solve problems

Common Paradigms for Algorithm Design

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming

- Greedy algorithms are often for optimization problems.
- They often run in polynomial time due to their simplicity.

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

Def. A strategy is safe: there is always an optimum solution that agrees with the decision made according to the strategy.

Outline

Box Packing

Input: n boxes of capacities $c_1, c_2, \cdots, c_n$

m items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item j can be put into box i if $s_j \leq c_i$

Output: A way to put as many items as possible in the boxes.

Example:

- Box capacities: 60, 40, 25, 15, 12
- Item sizes: 45, 42, 20, 19, 16
- Can put 3 items in boxes: $45 \rightarrow 60, 20 \rightarrow 40, 19 \rightarrow 25$

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?
- A: The item of the largest size that can be put into the box.

A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

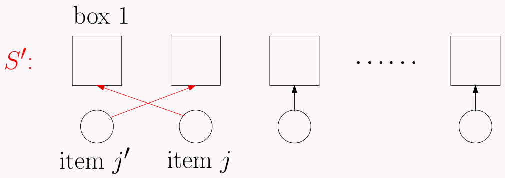
Lemma The strategy that put into box 1 the largest item it can hold is "safe": There is an optimum solution in which box 1 contains the largest item it can hold.

- Intuition: putting the item gives us the easiest residual problem.
- formal proof via exchanging argument:

Lemma There is an optimum solution in which box 1 contains the largest item it can hold.

Proof.

- Let $j =$ largest item that box 1 can hold.
- Take any optimum solution S. If j is put into Box 1 in S, done.
- Otherwise, assume this is what happens in S:



- $s_{j'} \le s_j$, and swapping gives another solution S'
- S' is also an optimum solution. In S', j is put into Box 1. $\square$

- Notice that the exchanging operation is only for the sake of analysis; it is not a part of the algorithm.

A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

- Trivial: we decided to put Item j into Box 1, and the remaining instance is obtained by removing Item j and Box 1.

Generic Greedy Algorithm

```
while the instance is non-trivial do
    make the choice using the greedy strategy
    reduce the instance
```

Greedy Algorithm for Box Packing

$T \leftarrow \{1, 2, 3, \cdots, m\}$

for $i \leftarrow 1$ to n **do**

  if some item in T can be put into box i **then**

    $j \leftarrow$ the largest item in T that can be put into box i

    print("put item j in box i")

    $T \leftarrow T \setminus \{j\}$

Exchange argument: Proof of Safety of a Strategy

- let S be an arbitrary optimum solution.
- if S is consistent with the greedy choice, done.
- otherwise, show that it can be modified to another optimum solution S' that is consistent with the choice.

- The procedure is not a part of the algorithm.

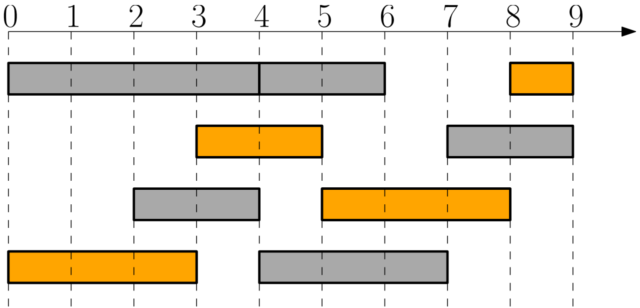
Outline

Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

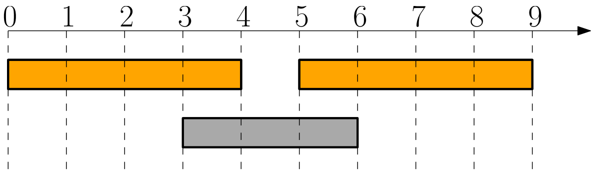
i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: A maximum-size subset of mutually compatible jobs



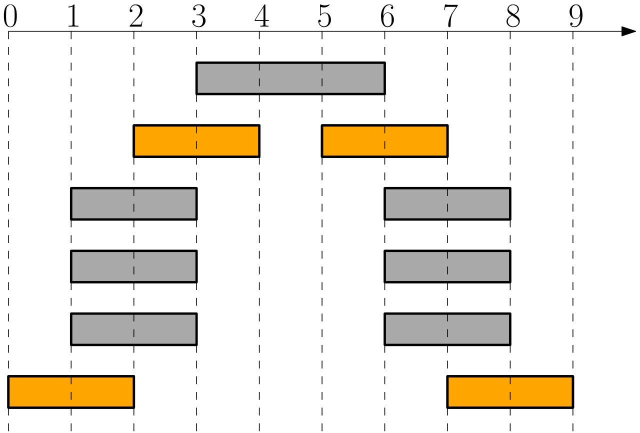
Greedy Algorithm for Interval Scheduling

- Which of the following strategies are safe?
- Schedule the job with the smallest size? No!



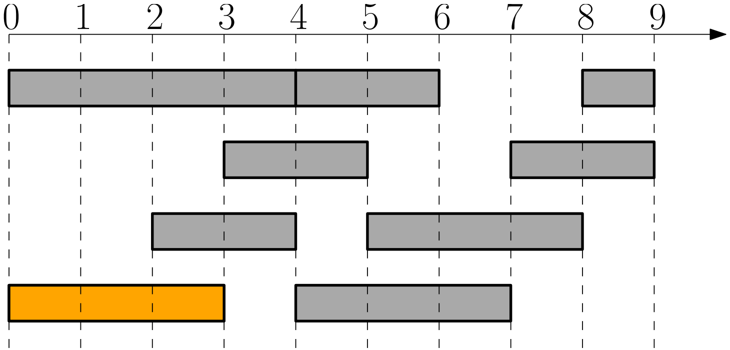
Greedy Algorithm for Interval Scheduling

- Which of the following strategies are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!



Greedy Algorithm for Interval Scheduling

- Which of the following strategies are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!
- Schedule the job with the earliest finish time? Yes!

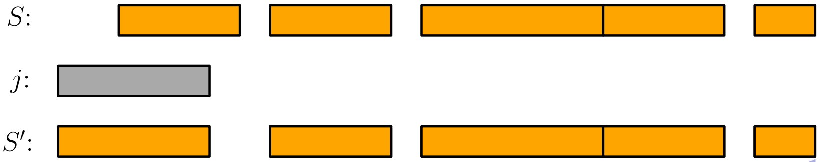


Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: There is an optimum solution where the job j with the earliest finish time is scheduled.

Proof.

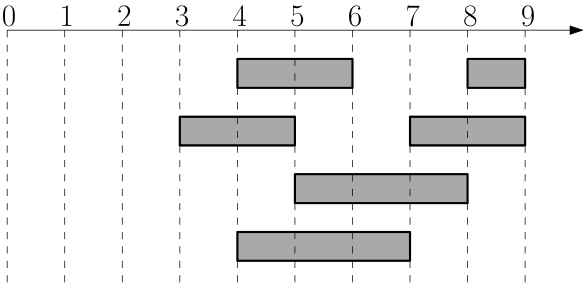
- Take an arbitrary optimum solution S
- If it contains j, done
- Otherwise, replace the first job in S with j to obtain another optimum schedule S'. $\square$



Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: There is an optimum solution where the job j with the earliest finish time is scheduled.

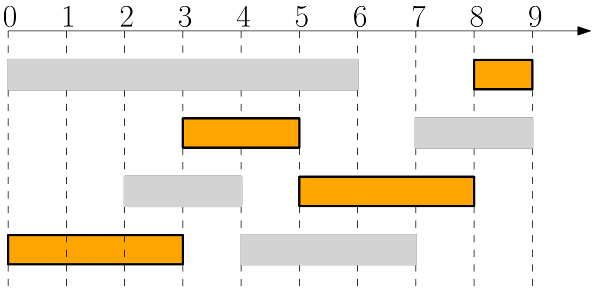
- What is the remaining task after we decided to schedule j?
- Is it another instance of interval scheduling problem? Yes!



Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

1: $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
2: **while** $A \neq \emptyset$ **do**
3: $\quad j \leftarrow \arg\min_{j' \in A} f_{j'}$
4: $\quad S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
5: **return** S



Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
while $A \neq \emptyset$ **do**
    $j \leftarrow \arg\min_{j' \in A} f_{j'}$
    $S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
return S

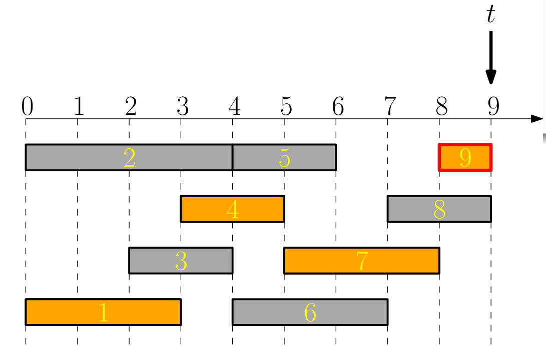
Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time

Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

sort jobs according to f values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of f_j **do**
    if $s_j \geq t$ **then**
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return S



Outline

Scheduling to minimize lateness

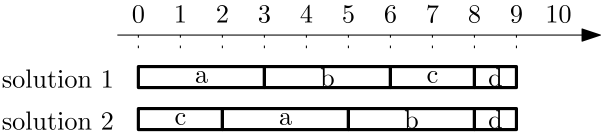
Input: n jobs, each job $j \in [n]$ with a processing time p_j and deadline d_j

Output: schedule jobs on 1 machine, to minimize the max. lateness

C_j: completion time of j lateness $l_j := \max\{C_j - d_j, 0\}$

- Example input:

j	a	b	c	d
p_j	3	3	2	1
d_j	5	7	4	8



- solution 1: max lateness $= \max\{0, 3-5, 6-7, 8-4, 9-8\} = 4$
- solution 2: max lateness $= \max\{0, 2-4, 5-5, 8-7, 9-8\} = 1$
- solution 2 is better

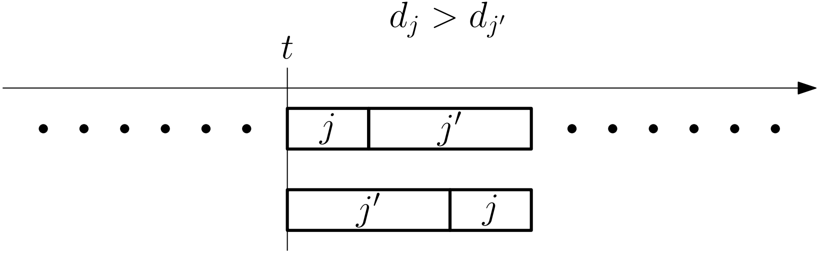
Candidate algorithms

Schedule the jobs in some natural order. Which order should we choose?

A. Ascending order of processing times p_j
B. Ascending order of slackness $d_j - p_j$
C. Ascending order of deadline d_j.

Lemma The ascending order of deadlines d_j (the Earliest Deadline First order or the EDF order) is the optimum schedule.

- maximum lateness $= \max\left\{0, \max_{j\in[n]}\{C_j - d_j\}\right\}$.



- before: $\max\{t + p_j - d_j, t + p_j + p_{j'} - d_{j'}\} = t + p_j + p_{j'} - d_{j'}$
- after: $\max\{t + p_{j'} - d_{j'}, t + p_j + p_{j'} - d_j\}$
- $p_{j'} - d_{j'} < p_j + p_{j'} - d_{j'}$ and $p_j + p_{j'} - d_j < p_j + p_{j'} - d_{j'}$
- $\max\{t + p_{j'} - d_{j'}, t + p_j + p_{j'} - d_j\} < t + p_j + p_{j'} - d_{j'}$
- after swapping, the maximum of the two terms strictly decreases

Repeated Swapping (for Analysis Only)

let S be any schedule (i.e, a permutation of $[n]$)
while there are two adjacent jobs j and j' in S, with j before j' and $d_j > d_{j'}$ **do**
    swap j and j' in S

Q: Does the algorithm terminate?

A: Yes. Number of inversions go down!

- (j, j') is an inversion in S if j appears before j' and $d_j > d_{j'}$.
- So the algorithm converges to an EDF order.

Q: What if there are multiple EDF orders, i.e., some jobs have the same deadline?

A: All EDF orders have the same maximum lateness.

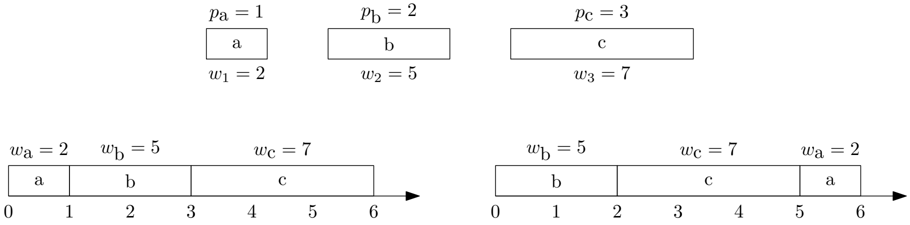
Outline

Scheduling to Minimize Weighted Completion Time

Input: A set of n jobs $[n] := \{1, 2, 3, \cdots, n\}$

each job j has a weight w_j and processing time p_j

Output: an ordering of jobs so as to minimize the total weighted completion time of jobs



$\text{cost} = 2 \times 1 + 5 \times 3 + 7 \times 6 = 59$

$\text{cost} = 5 \times 2 + 7 \times 5 + 2 \times 6 = 57$

Candidate algorithms

Schedule the jobs in some natural order. Which order should we choose?

- A. Ascending order of processing times p_j
- B. Descending order of slackness w_j
- C. Ascending order of $p_j - w_j$
- D. Ascending order of p_j/w_j

Def. The Smith ratio of a job is w_j/p_j.

Lemma The descending order of Smith ratios (the Smith rule) is optimum.

- A schedule S, j is right before j'.

Q: How does the total weighted completion time change if we swap j and j'? $(\cdots, j, j', \cdots) \Longrightarrow (\cdots, j', j, \cdots)$

A: $w_{j'}p_j \quad \Longrightarrow \quad w_j p_{j'}$

- Therefore, swapping decrease the weighted completion time if $\frac{p_{j'}}{w_{j'}} < \frac{p_j}{w_j}$.
- Using the same argument as for the maximum lateness problem: ascending order of p_j/w_j is optimum.
- Indeed, optimum weighted completion time is

$$\sum_{j \in [n]} w_j p_j + \sum_{1 \le j < j' \le n} \min\{w_j p_{j'}, w_{j'} p_j\}.$$

Outline

Offline Caching

- Cache that can store k pages
- Sequence of page requests
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
- Cache hit happens if requested page already in cache.
- Goal: minimize the number of cache misses.

page sequence		cache		
1	✗	1		
5	✗	1	5	
4	✗	1	5	4
2	✗	1	2	4
5	✗	1	2	5
3	✗	1	2	3
2	✓	1	2	3
1	✓	1	2	3

misses $= 6$

A Better Solution for Example

page sequence		cache				cache		
1	✗	1			✗	1		
5	✗	1	5		✗	1	5	
4	✗	1	5	4	✗	1	5	4
2	✗	1	2	4	✗	1	5	2
5	✗	1	2	5	✓	1	5	2
3	✗	1	2	3	✗	1	3	2
2	✓	1	2	3	✓	1	3	2
1	✓	1	2	3	✓	1	3	2
		misses $= 6$				misses $= 5$		

Offline Caching Problem

Input: k : the size of cache

n : number of pages

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

We use $[n]$ for $\{1, 2, 3, \cdots, n\}$.

Output: $i_1, i_2, i_3, \cdots, i_T \in \{\mathrm{hit}, \mathrm{empty}\} \cup [n]$: indices of pages to evict ("hit" means evicting no page, "empty" means evicting empty page)

- Offline Caching: we know the whole sequence ahead of time.
- Online Caching: we have to make decisions on the fly, before seeing future requests.

Q: Which one is more realistic?

A: Online caching

- Offline Caching: we know the whole sequence ahead of time.
- Online Caching: we have to make decisions on the fly, before seeing future requests.

Q: Which one is more realistic?

A: Online caching

Q: Why do we study the offline caching problem?

A: Use the offline solution as a benchmark to measure the "competitive ratio" of online algorithms

Offline Caching: Potential Greedy Algorithms

- FIFO(First-In-First-Out): always evict the first page in cache
- LRU(Least-Recently-Used): Evict page whose most recent access was earliest
- LFU(Least-Frequently-Used): Evict page that was least frequently requested

- All the above algorithms are not optimum!
- Indeed all the algorithms are "online", i.e, the decisions can be made without knowing future requests. Online algorithms can not be optimum.

FIFO is not optimum

requests		FIFO				Furthest-in-Future		
1	✗	1			✗	1		
2	✗	1	2		✗	1	2	
3	✗	1	2	3	✗	1	2	3
4	✗	4	2	3	✗	1	4	3
1	✗	4	1	3	✓	1	4	3
		misses $= 5$				misses $= 4$		

Optimum Offline Caching

Furthest-in-Future (FF)

- Algorithm: every time, evict the page that is not requested until furthest in the future, if we need to evict one.
- The algorithm is not an online algorithm, since the decision at a step depends on the request sequence in the future.

Furthest-in-Future (FF)

requests	FIFO				Furthest-in-Future			
1	✗	1			✗	1		
2	✗	1	2		✗	1	2	
3	✗	1	2	3	✗	1	2	3
4	✗	4	2	3	✗	1	4	3
1	✗	4	1	3	✓	1	4	3
	misses = 5				misses = 4			

Example

requests	1	5	4	2	5	3	2	4	3	1	5	3
	✗	✗	✗	✗	✓	✗	✓	✓	✓	✗	✗	✓
	1	1	1	2	2	2	2	2	2	1	5	5
		5	5	5	5	3	3	3	3	3	3	3
			4	4	4	4	4	4	4	4	4	4

Recall: Designing and Analyzing Greedy Algorithms

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

Offline Caching Problem

Input: k : the size of cache

n : number of pages

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

$p_1, p_2, \cdots, p_k \in \{\text{empty}\} \cup [n]$: initial set of pages in cache

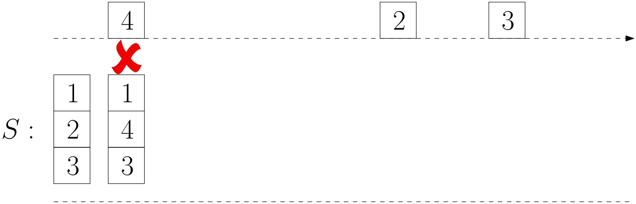
Output: $i_1, i_2, i_3, \cdots, i_t \in \{\mathrm{hit}, \text{empty}\} \cup [n]$

- empty stands for an empty page
- "hit" means evicting no pages

A Common Way to Analyze Greedy Algorithms

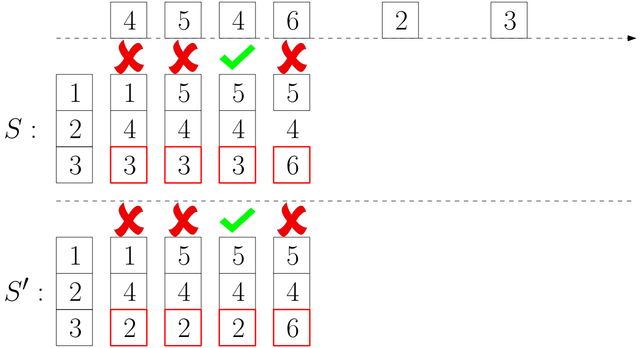
- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. There is an optimum solution in which p^* is evicted at time 1.



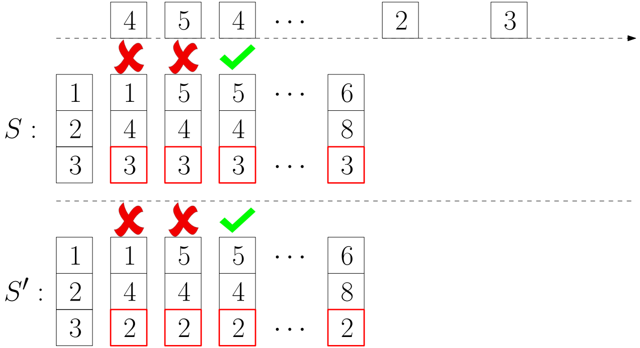
Proof.

1. S: any optimum solution
2. p^*: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$.
3. Assume S evicts some $p' \neq p^*$ at time 1; otherwise done.
   - In the example, $p' = 2$.



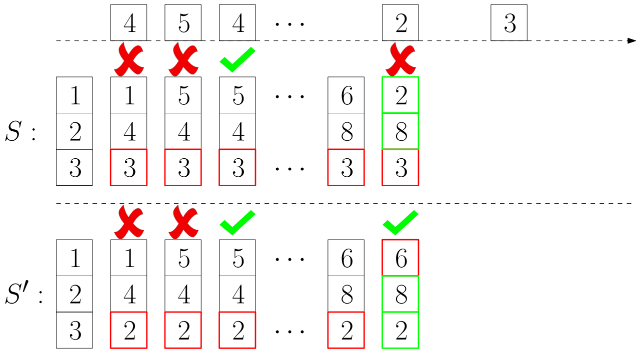
Proof.

4. Create S'. S' evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.
5. After time 1, cache status of S and that of S' differ by only 1 page. S' contains $p'(=2)$ and S contains $p^*(=3)$.
6. From now on, S' will "copy" S.



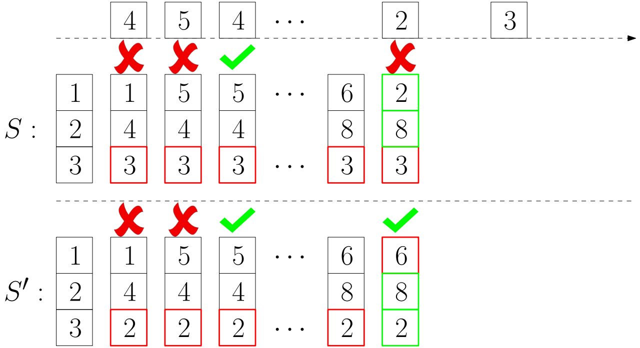
Proof.

7. If S evicted the page p^*, S' will evict the page p'. Then, the cache status of S and that of S' will be the same. S and S' will be exactly the same from now on.
8. Assume S did not evict $p^*(=3)$ before we see $p'(=2)$.



Proof.

9. If S evicts $p^*(=3)$ for $p'(=2)$, then S won't be optimum. Assume otherwise.
10. So far, S' has 1 less page-miss than S does.
11. The status of S' and that of S only differ by 1 page.



Proof.

12 We can then guarantee that S' make at most the same number of page-misses as S does.

- Idea: if S has a page-hit and S' has a page-miss, we use the opportunity to make the status of S' the same as that of S. □

- Thus, we have shown how to create another solution S' with the same number of page-misses as that of the optimum solution S. Thus, we proved

Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. It is safe to evict p^* at time 1.

Theorem The furthest-in-future strategy is optimum.

for $t \leftarrow 1$ to T **do**
    if ρ_t is in cache **then** do nothing
    else if there is an empty page in cache **then**
        evict the empty page and load ρ_t in cache
    else
        $p^* \leftarrow$ page in cache that is not used furthest in the future
        evict p^* and load ρ_t in cache

Q: How can we make the algorithm as fast as possible?

A:

- The running time can be made to be $O(n + T \log k)$.
- For each page p, use a linked list (or an array with dynamic size) to store the time steps in which p is requested.
  - We can find the next time a page is requested easily.
- Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓	✗	✗	✓

P1:	1	10		
P2:	4	7		
P3:	6	9	12	
P4:	3	8		
P5:	2	5	11	

priority queue

pages	priority values
P5	∞
P3	∞
P4	∞

for every $p \leftarrow 1$ to n **do**
    $times[p] \leftarrow$ array of times in which p is requested, in increasing order ▷ put ∞ at the end of array
    $pointer[p] \leftarrow 1$
$Q \leftarrow$ empty priority queue
for every $t \leftarrow 1$ to T **do**
    $pointer[\rho_t] \leftarrow pointer[\rho_t] + 1$
    if $\rho_t \in Q$ **then**
        Q.increase-key$(\rho_t, times[\rho_t, pointer[\rho_t]])$, **print** "hit", **continue**
    if $Q.size() < k$ **then**
        print "load ρ_t to an empty page"
    else
        $p \leftarrow Q$.extract-max(), **print** "evict p and load ρ_t"
    Q.insert$(\rho_t, times[\rho_t, pointer[\rho_t]])$ ▷ add ρ_t to Q with key value $times[\rho_t, pointer[\rho_t]]$

Outline

- Let V be a ground set of size n.

Def. A priority queue is an abstract data structure that maintains a set $U \subseteq V$ of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element $v \in V \setminus U$, with associated key value key_value.
- decrease_key(v, new_key_value): decrease the key value of an element $v \in U$ to new_key_value
- extract_min$()$: return and remove the element in U with the smallest key value
- $\cdots$

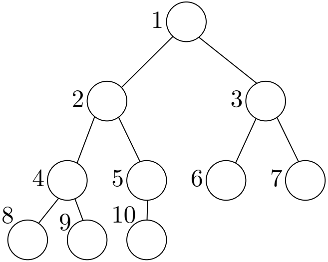
Simple Implementations for Priority Queue

- $n =$ size of ground set V

data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Heap

The elements in a heap is organized using a complete binary tree:

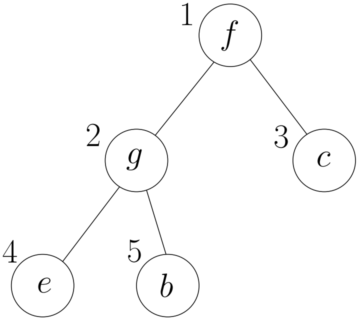


- Nodes are indexed as $\{1, 2, 3, \cdots, s\}$
- Parent of node i: $\lfloor i/2 \rfloor$
- Left child of node i: $2i$
- Right child of node i: $2i + 1$

Heap

A heap H contains the following fields

- s: size of U (number of elements in the heap)
- $A[i], 1 \leq i \leq s$: the element at node i of the tree
- $p[v], v \in U$: the index of node containing v
- $key[v], v \in U$: the key value of element v

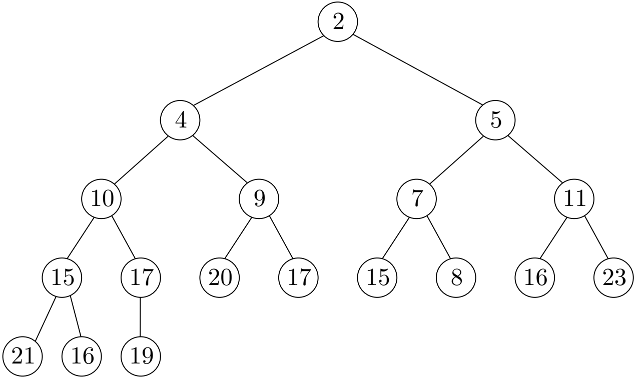


- $s = 5$
- $A = (\text{‘}f\text{’}, \text{‘}g\text{’}, \text{‘}c\text{’}, \text{‘}e\text{’}, \text{‘}b\text{’})$
- $p[\text{‘}f\text{’}] = 1, p[\text{‘}g\text{’}] = 2, p[\text{‘}c\text{’}] = 3,$
  $p[\text{‘}e\text{’}] = 4, p[\text{‘}b\text{’}] = 5$

Heap

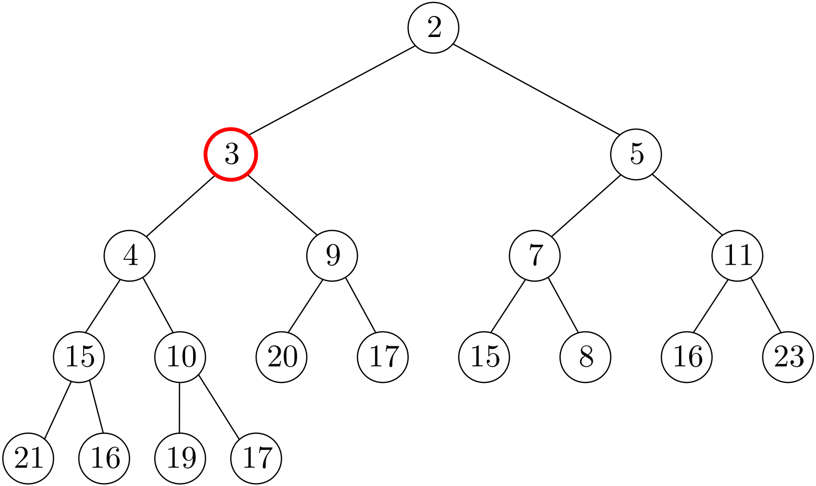
The following heap property is satisfied:

- for any two nodes i, j such that i is the parent of j, we have $key[A[i]] \leq key[A[j]]$.



A heap. Numbers in the circles denote key values of elements.

insert(v, key_value)



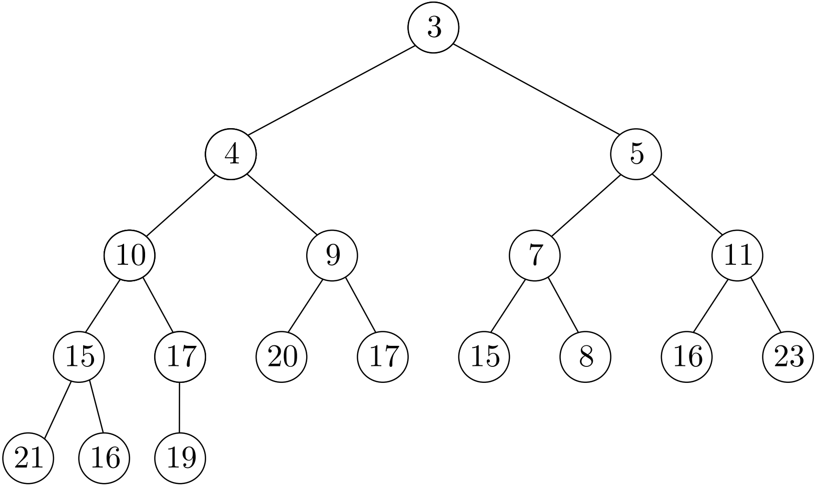
insert(v, key_value)

$s \leftarrow s + 1$
$A[s] \leftarrow v$
$p[v] \leftarrow s$
$key[v] \leftarrow key\_value$
heapify_up(s)

heapify-up(i)

while $i > 1$ **do**
  $j \leftarrow \lfloor i/2 \rfloor$
  if $key[A[i]] < key[A[j]]$ **then**
    swap $A[i]$ and $A[j]$
    $p[A[i]] \leftarrow i$, $p[A[j]] \leftarrow j$
    $i \leftarrow j$
  else break

extract_min()



extract_min()

$ret \leftarrow A[1]$
$A[1] \leftarrow A[s]$
$p[A[1]] \leftarrow 1$
$s \leftarrow s - 1$
if $s \geq 1$ **then**
    heapify_down(1)
return ret

decrease_key$(v, key_val$

$key[v] \leftarrow key\_value$
heapify-up$(p[v])$

heapify-down(i)

while $2i \leq s$ **do**
    if $2i = s$ or $key[A[2i]] \leq key[A[2i+1]]$ **then**
        $j \leftarrow 2i$
    else
        $j \leftarrow 2i + 1$
    if $key[A[j]] < key[A[i]]$ **then**
        swap $A[i]$ and $A[j]$
        $p[A[i]] \leftarrow i$, $p[A[j]] \leftarrow j$
        $i \leftarrow j$
    else break

- Running time of heapify_up and heapify_down: $O(\lg n)$
- Running time of insert, exact_min and decrease_key: $O(\lg n)$

data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Two Definitions Needed to Prove that the Procedures Maintain Heap Property

Def. We say that H is almost a heap except that $key[A[i]]$ is too small if we can increase $key[A[i]]$ to make H a heap.

Def. We say that H is almost a heap except that $key[A[i]]$ is too big if we can decrease $key[A[i]]$ to make H a heap.

Outline

Encoding Letters Using Bits

- 8 letters a, b, c, d, e, f, g, h in a language
- need to encode a message using bits
- idea: use 3 bits per letter

a	b	c	d	e	f	g	h
000	001	010	011	100	101	110	111

$$deacfg \to 011100000010101110$$

Q: Can we have a better encoding scheme?

- Seems unlikely: must use 3 bits per letter

Q: What if some letters appear more frequently than the others?

Q: If some letters appear more frequently than the others, can we have a better encoding scheme?

A: Using variable-length encoding scheme might be more efficient.

Idea

- using fewer bits for letters that are more frequently used, and more bits for letters that are less frequently used.

Q: What is the issue with the following encoding scheme?

- a: 0 b: 1 c: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to aa or c.

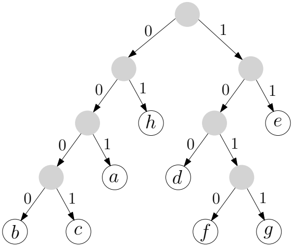
Solution

Use prefix codes to guarantee a unique decoding.

Prefix Codes

Def. A prefix code for a set S of letters is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

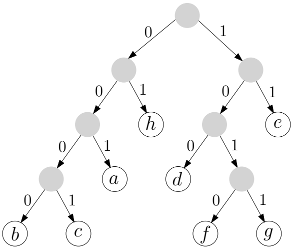
a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



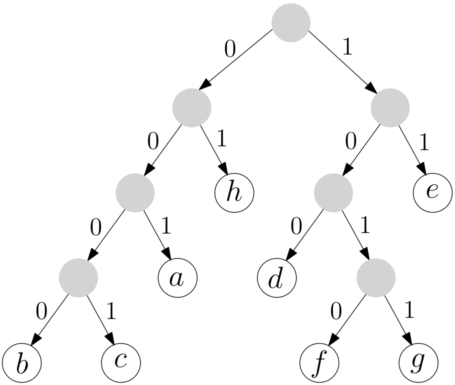
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



- 0001/001/100/0000/01/01/11/1010/0001/001/
- cadbhhefca



Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some letter
- If coding scheme is not wasteful: a non-leaf has exactly two children

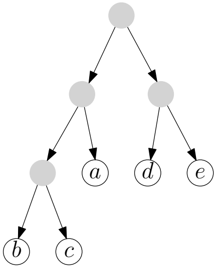
Best Prefix Codes

Input: frequencies of letters in a message

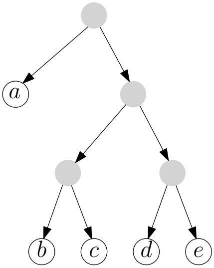
Output: prefix coding scheme with the shortest encoding for the message

example

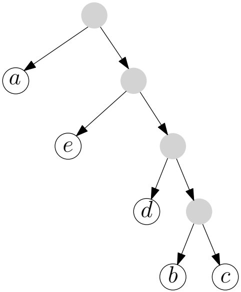
letters	a	b	c	d	e	
frequencies	18	3	4	6	10	
scheme 1 length	2	3	3	2	2	total $=$ 89
scheme 2 length	1	3	3	3	3	total $=$ 87
scheme 3 length	1	4	4	3	2	total $=$ 84



scheme 1



scheme 2



scheme 3

- Example Input: (a: 18, b: 3, c: 4, d: 6, e: 10)

Q: What types of decisions should we make?

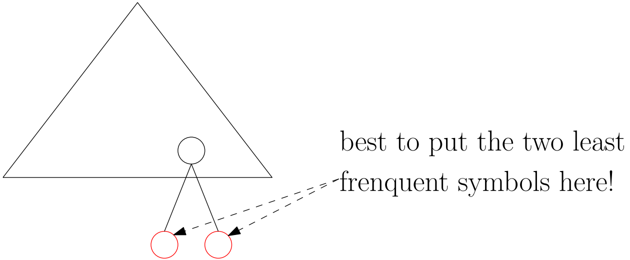
- Can we directly give a code for some letter?
- Hard to design a strategy; residual problem is complicated.

- Can we partition the letters into left and right sub-trees?
- Not clear how to design the greedy algorithm

A: We can choose two letters and make them brothers in the tree.

Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the "structure" of the optimum encoding tree
- There are two deepest leaves that are brothers



Lemma It is safe to make the two least frequent letters brothers.

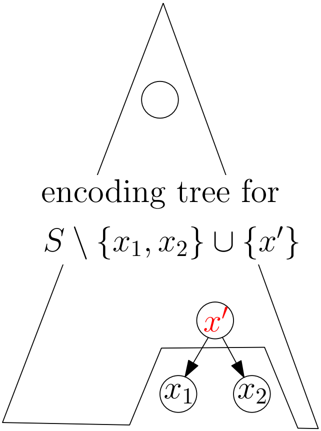
Lemma There is an optimum encoding tree, where the two least frequent letters are brothers.

- So we can irrevocably decide to make the two least frequent letters brothers.

Q: Is the residual problem another instance of the best prefix codes problem?

A: Yes, though it is not immediate to see why.

- f_x: the frequency of the letter x in the support.
- x_1 and x_2: the two letters we decided to put together.
- d_x the depth of letter x in our output encoding tree.



$$
\begin{aligned}
&\sum_{x \in S} f_x d_x \\
&= \sum_{x \in S \setminus \{x_1,x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
&= \sum_{x \in S \setminus \{x_1,x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1} \\
&= \sum_{x \in S \setminus \{x_1,x_2\}} f_x d_x + f_{x'} (d_{x'} + 1) \\
&= \sum_{x \in S \setminus \{x_1,x_2\} \cup \{x'\}} f_x d_x + f_{x'}
\end{aligned}
$$

Def: $f_{x'} = f_{x_1} + f_{x_2}$

In order to minimize

$$\sum_{x \in S} f_x d_x,$$

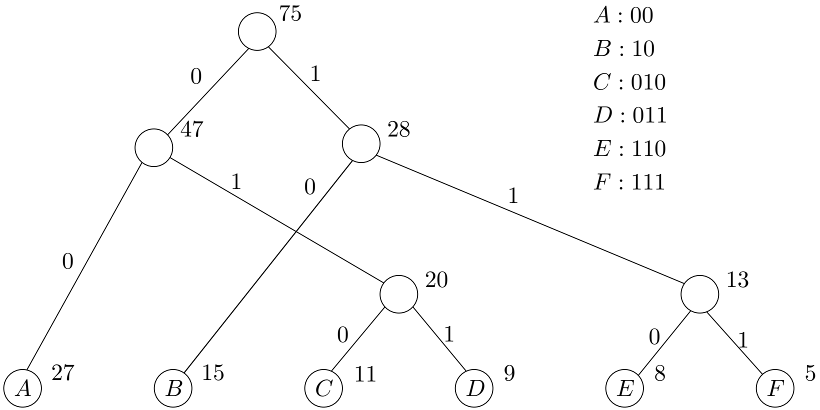
we need to minimize

$$\sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x,$$

subject to that d is the depth function for an encoding tree of $S \setminus \{x_1, x_2\}$.

- This is exactly the best prefix codes problem, with letters $S \setminus \{x_1, x_2\} \cup \{x'\}$ and frequency vector f!

Example



$A : 00$

$B : 10$

$C : 010$

$D : 011$

$E : 110$

$F : 111$

Def. The codes given the greedy algorithm is called the Huffman codes.

Huffman(S, f)

while $|S| > 1$ **do**
- let x_1, x_2 be the two letters with the smallest f values
- introduce a new letter x' and let $f_{x'} = f_{x_1} + f_{x_2}$
- let x_1 and x_2 be the two children of x'
- $S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$

return the tree constructed

Algorithm using Priority Queue

Huffman(S, f)

```
Q ← build-priority-queue(S)
while Q.size > 1 do
    x_1 ← Q.extract-min()
    x_2 ← Q.extract-min()
    introduce a new letter x' and let f_{x'} = f_{x_1} + f_{x_2}
    let x_1 and x_2 be the two children of x'
    Q.insert(x', f_{x'})
return the tree constructed
```

Outline

Summary for Greedy Algorithms

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

- Interval scheduling problem: schedule the job j^* with the earliest deadline
- Offline Caching: evict the page that is used furthest in the future
- Huffman codes: make the two least frequent letters brothers

Summary for Greedy Algorithms

A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

Def. A strategy is "safe" if there is always an optimum solution that "agrees with" the decision made according to the strategy.

Proving a Strategy is Safe

- Take an arbitrary optimum solution S
- If S agrees with the decision made according to the strategy, done
- So assume S does not agree with decision
- Change S slightly to another optimum solution S' that agrees with the decision
  - Interval scheduling problem: exchange j^* with the first job in an optimal solution
  - Offline caching: a complicated "copying" algorithm
  - Huffman codes: move the two least frequent letters to the deepest leaves.

Summary for Greedy Algorithms

A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

- Interval scheduling problem: remove j^* and the jobs it conflicts with
- Offline caching: trivial
- Huffman codes: merge two letters into one

- Two problems that do not fall into the category: lateness, weighted completion time