算法设计与分析(2024年春季学期)

Greedy Algorithms

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Goals of algorithm design

1. Design efficient algorithms to solve problems
2. Design more efficient algorithms to solve problems

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- Greedy algorithms are often for optimization problems.
- They often run in polynomial time due to their simplicity.

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- At each step, make an irrevocable decision using a "reasonable" strategy

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Def. A strategy is safe: there is always an optimum solution that agrees with the decision made according to the strategy.

Outline

Box Packing

Input: n boxes of capacities $c_1, c_2, \cdots, c_n$

m items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item j can be put into box i if $s_j \leq c_i$

Output: A way to put as many items as possible in the boxes.

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Example:

- Box capacities: $60,\ 40,\ 25,\ 15,\ 12$
- Item sizes: $45,\ 42,\ 20,\ 19,\ 16$
- Can put 3 items in boxes: $45 \to 60, 20 \to 40, 19 \to 25$

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- A: The item of the largest size that can be put into the box.

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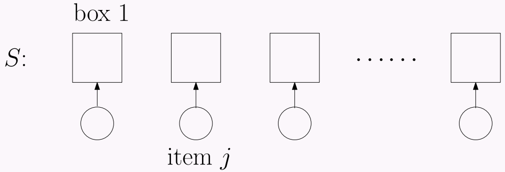

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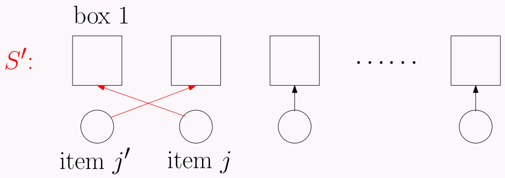

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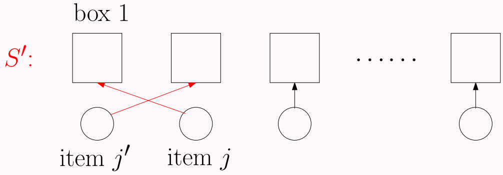

- $s_{j'} \leq s_j$, and swapping gives another solution S'
- S' is also an optimum solution. In S', j is put into Box 1. □

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- Trivial: we decided to put Item j into Box 1, and the remaining instance is obtained by removing Item j and Box 1.

Generic Greedy Algorithm

while the instance is non-trivial **do**

  make the choice using the greedy strategy

  reduce the instance

Greedy Algorithm for Box Packing

$T \leftarrow \{1, 2, 3, \cdots, m\}$

for $i \leftarrow 1$ to n **do**

  if some item in T can be put into box i **then**

    $j \leftarrow$ the largest item in T that can be put into box i

    print("put item j in box i")

    $T \leftarrow T \setminus \{j\}$

Exchange argument: Proof of Safety of a Strategy

- let S be an arbitrary optimum solution.
- if S is consistent with the greedy choice, done.
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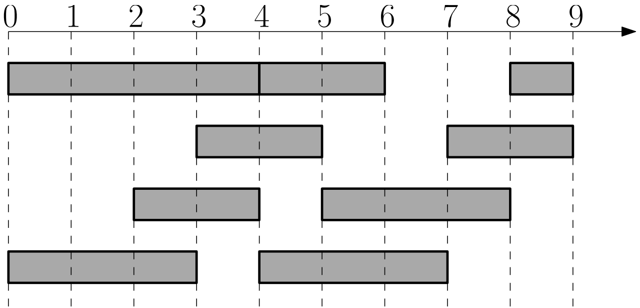
Outline

Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: A maximum-size subset of mutually compatible jobs



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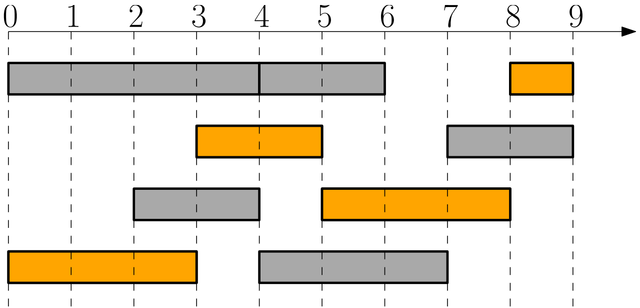

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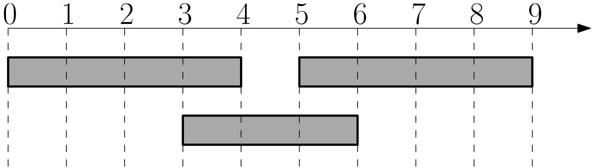

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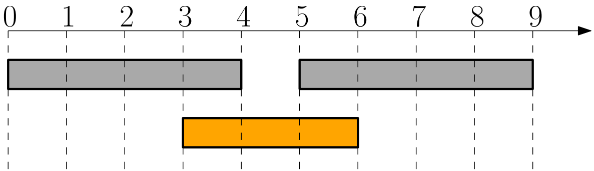

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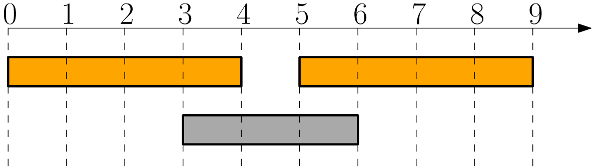

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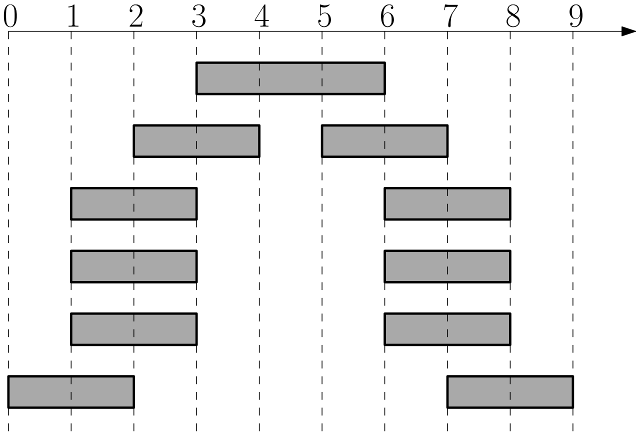

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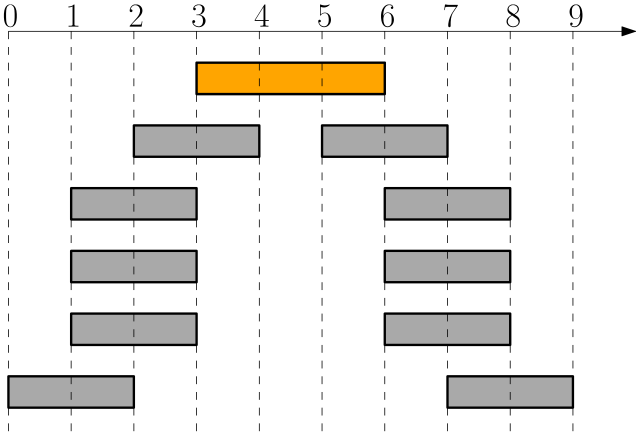

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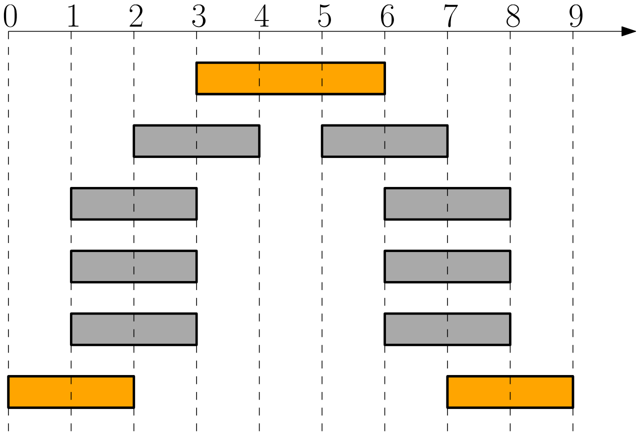

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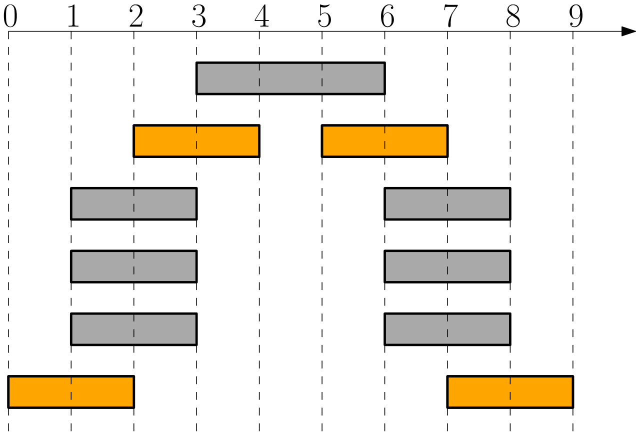

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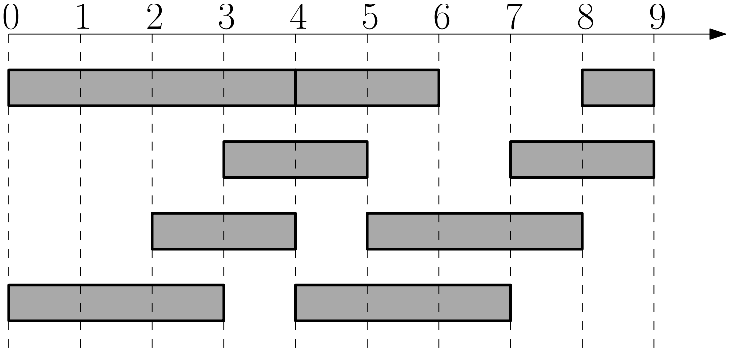

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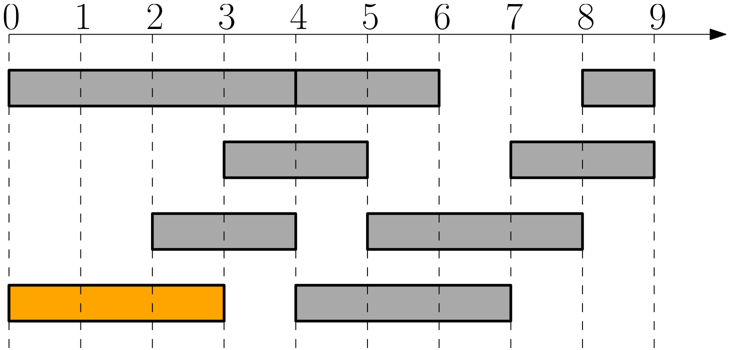

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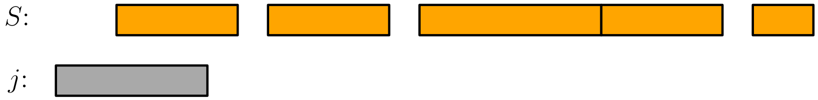

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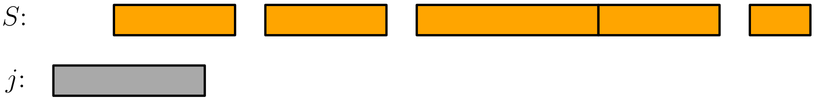

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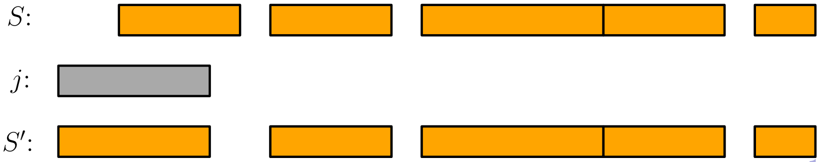

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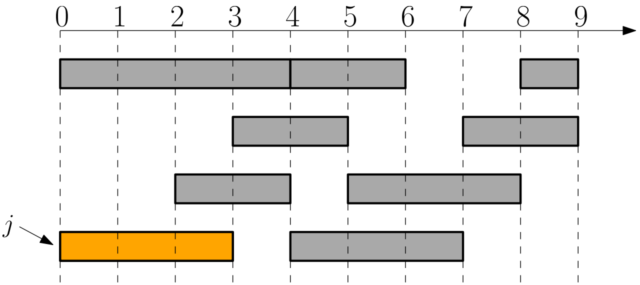
- What is the remaining task after we decided to schedule j?
- Is it another instance of interval scheduling problem?



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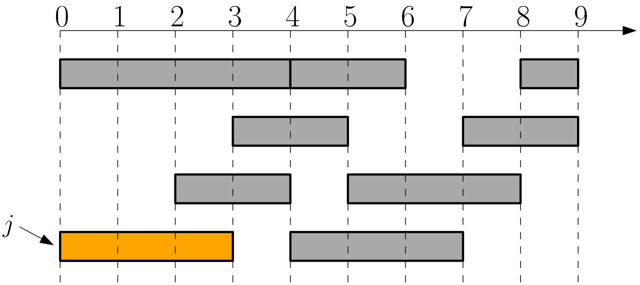

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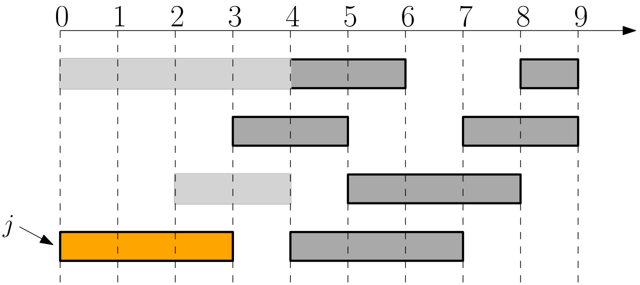

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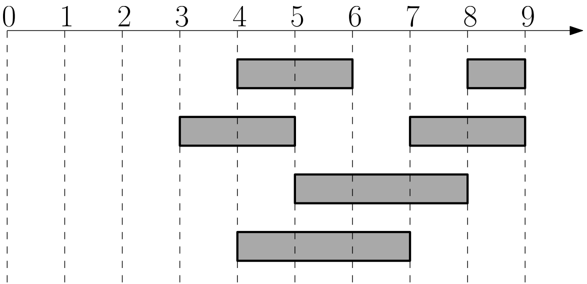

Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

```
A ← {1, 2, ⋯, n}, S ← ∅
while A ≠ ∅ do
    j ← argmin_{j'∈A} f_{j'}
    S ← S ∪ {j}; A ← {j' ∈ A : s_{j'} ≥ f_j}
return S
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2: **while** $A \neq \emptyset$ **do**
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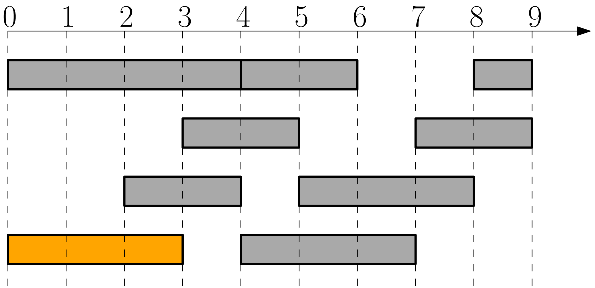

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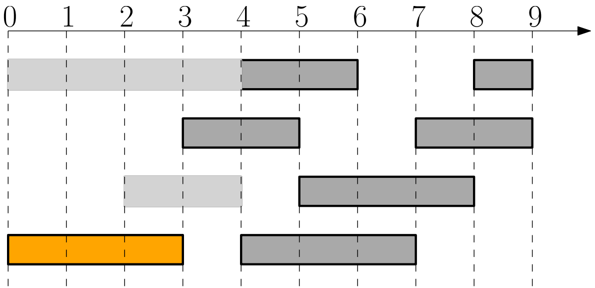

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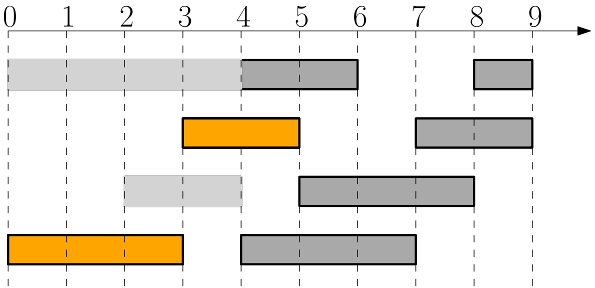

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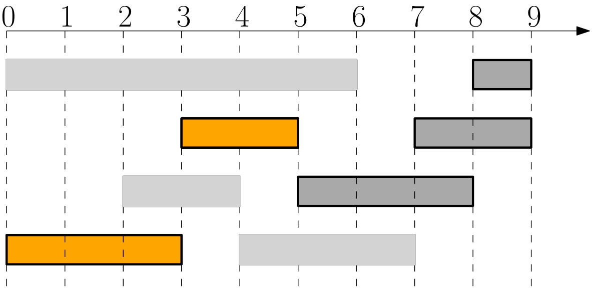

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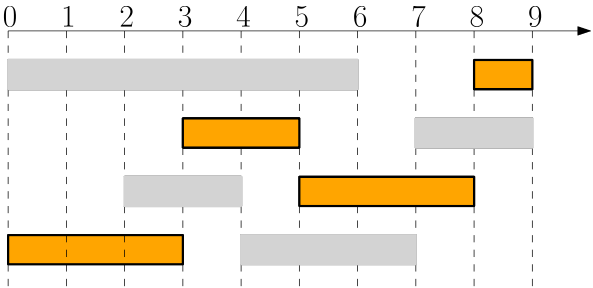

Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

1: $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
2: **while** $A \neq \emptyset$ **do**
3: $\quad j \leftarrow \arg\min_{j' \in A} f_{j'}$
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Running time of algorithm?

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Running time of algorithm?

- Naive implementation: $O(n^2)$ time

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Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time

Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

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sort jobs according to f values
t ← 0, S ← ∅
for every j ∈ [n] according to non-decreasing order of f_j do
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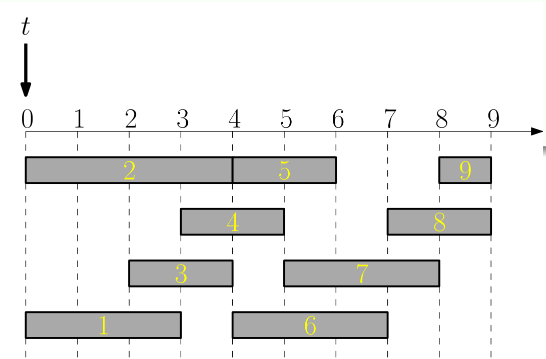

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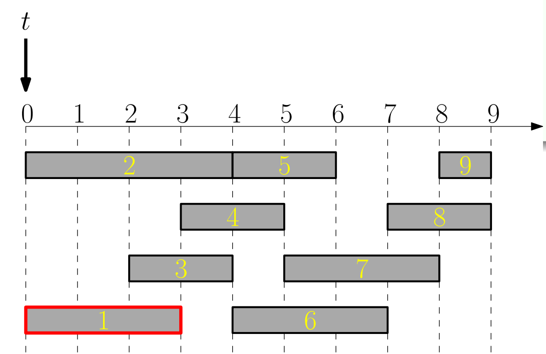

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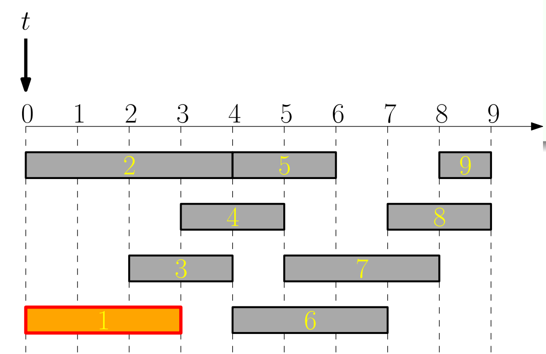

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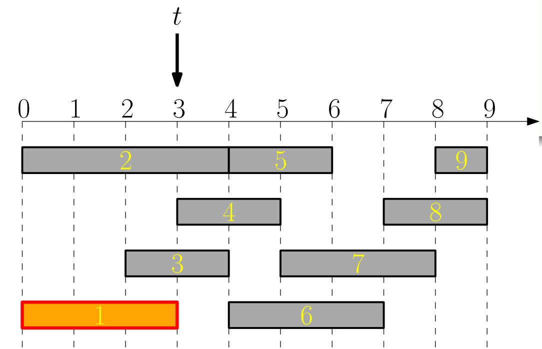

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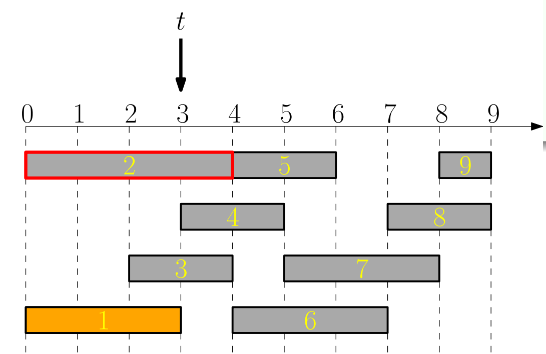

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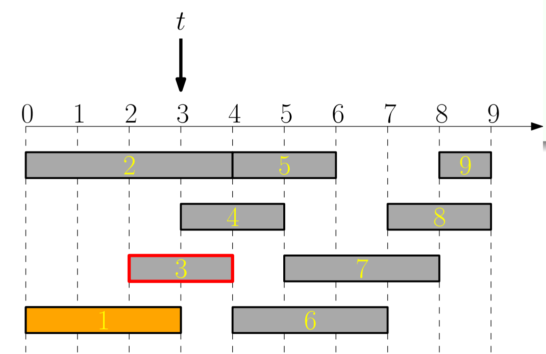

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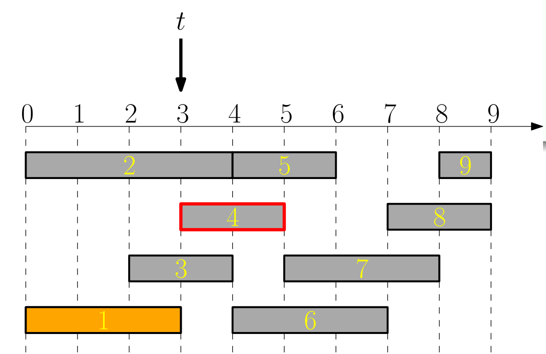

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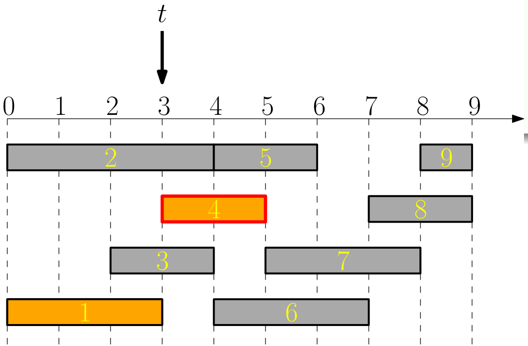

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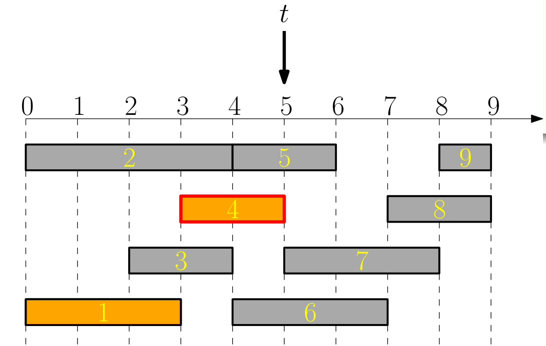

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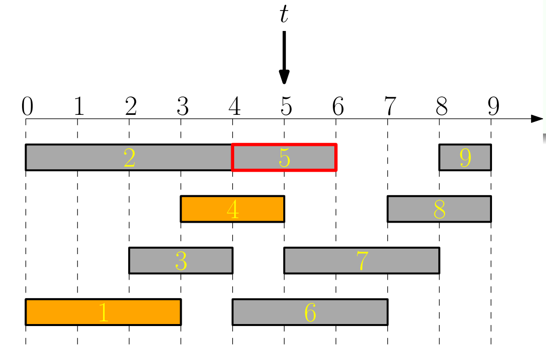

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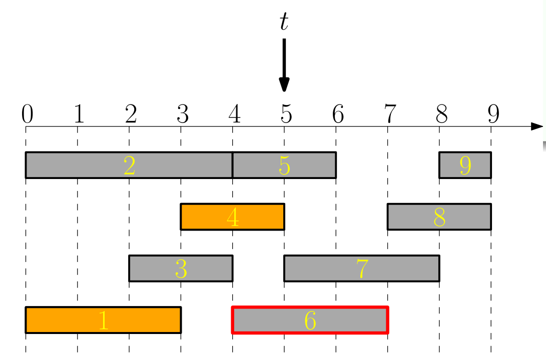

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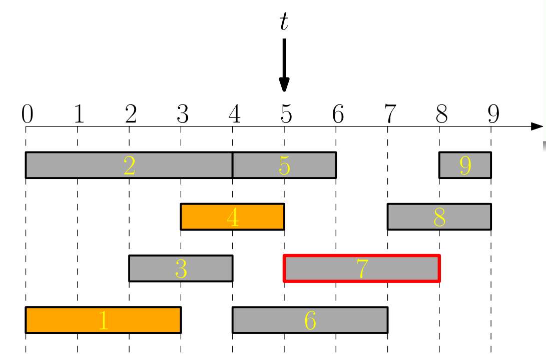

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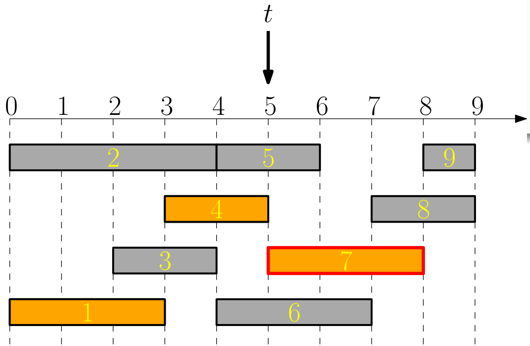

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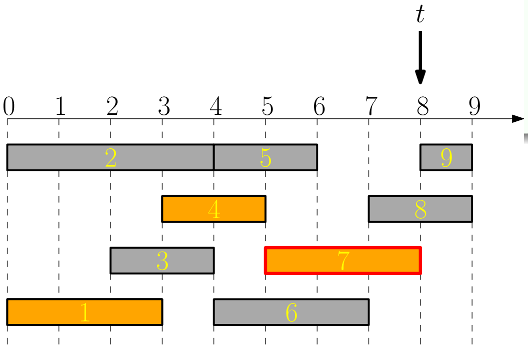

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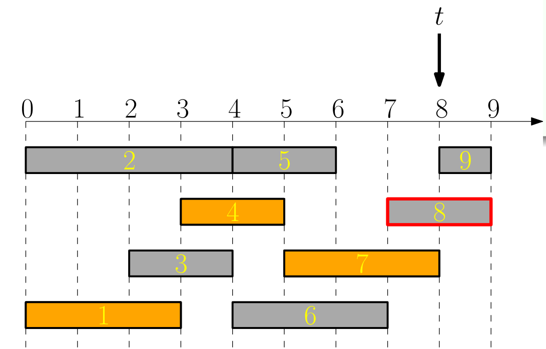

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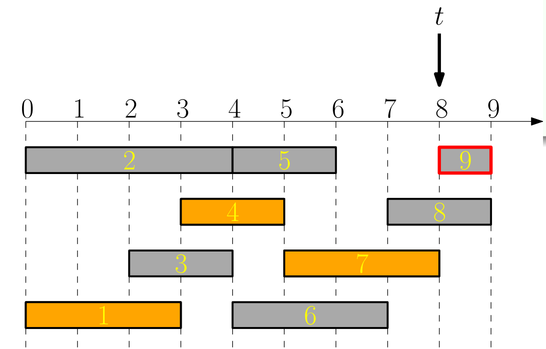

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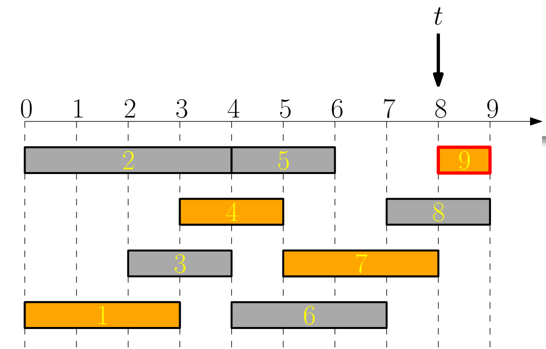

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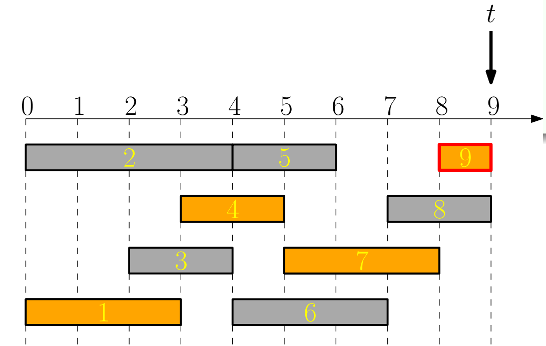


Outline

Scheduling to minimize lateness

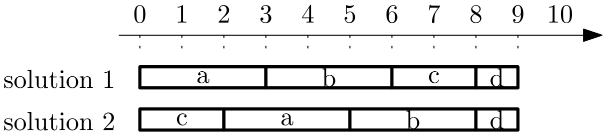
Input: n jobs, each job $j \in [n]$ with a processing time p_j and deadline d_j

Output: schedule jobs on 1 machine, to minimize the max. lateness

C_j: completion time of j lateness $l_j := \max\{C_j - d_j, 0\}$

- Example input:

j	a	b	c	d
p_j	3	3	2	1
d_j	5	7	4	8



- solution 1: max lateness $= \max\{0, 3-5, 6-7, 8-4, 9-8\} = 4$
- solution 2: max lateness $= \max\{0, 2-4, 5-5, 8-7, 9-8\} = 1$
- solution 2 is better

Candidate algorithms

Schedule the jobs in some natural order. Which order should we choose?

A. Ascending order of processing times p_j
B. Ascending order of slackness $d_j - p_j$
C. Ascending order of deadline d_j.

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Lemma The ascending order of deadlines d_j (the Earliest Deadline First order or the EDF order) is the optimum schedule.

- maximum lateness $= \max\left\{0, \max_{j\in[n]}\{C_j - d_j\}\right\}$.

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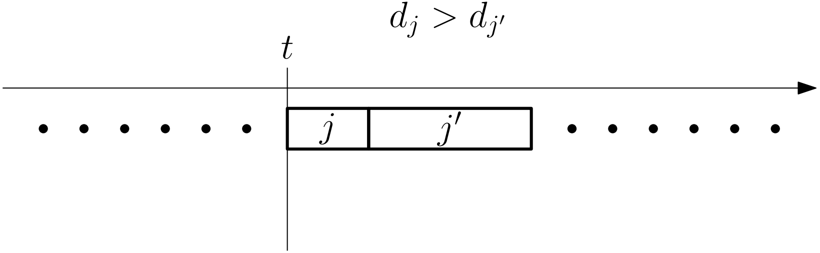

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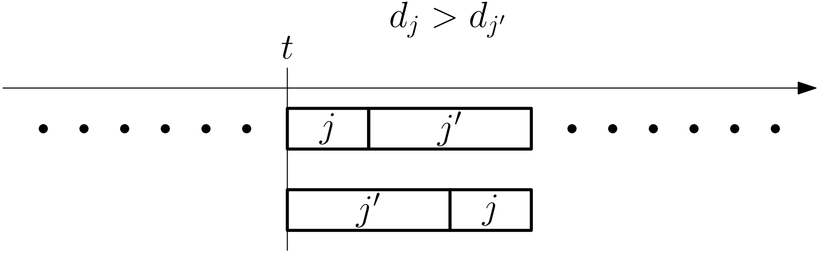

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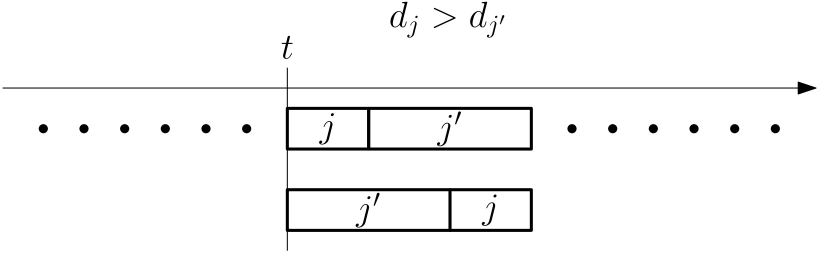

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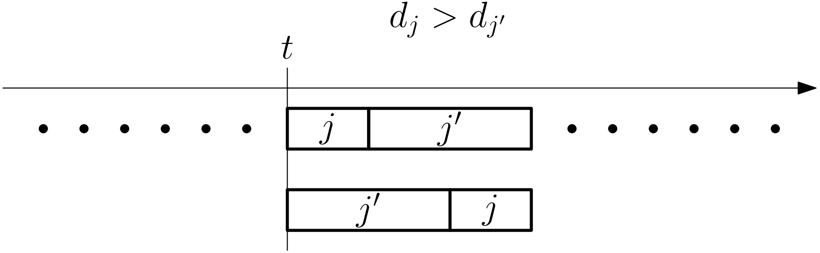

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- after: $\max\{t + p_{j'} - d_{j'}, t + p_j + p_{j'} - d_j\}$

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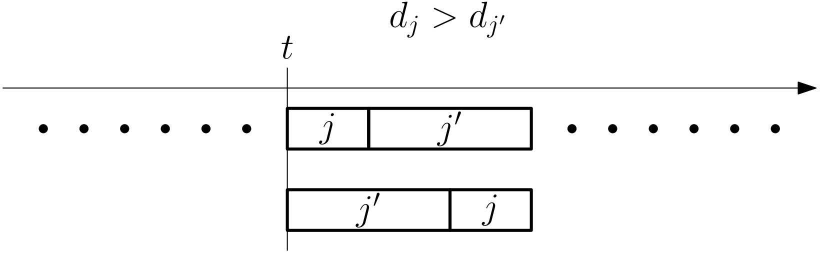

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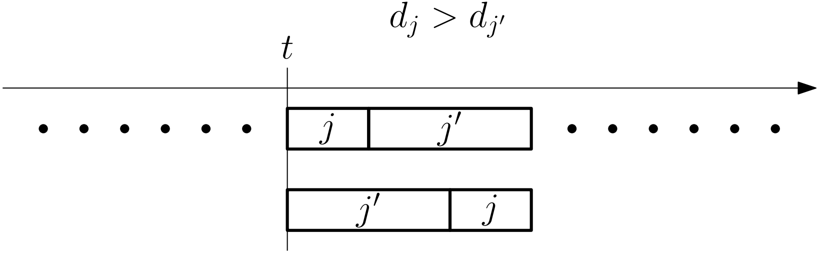

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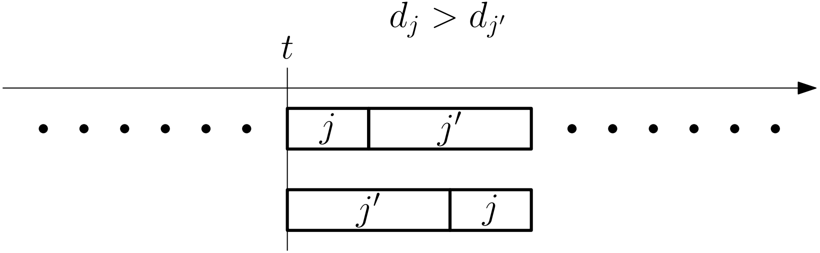

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- $\max\{t + p_{j'} - d_{j'}, t + p_j + p_{j'} - d_j\} < t + p_j + p_{j'} - d_{j'}$
- after swapping, the maximum of the two terms strictly decreases

Repeated Swapping (for Analysis Only)

let S be any schedule (i.e, a permutation of $[n]$)
while there are two adjacent jobs j and j' in S, with j before j' and $d_j > d_{j'}$ **do**
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A: All EDF orders have the same maximum lateness.

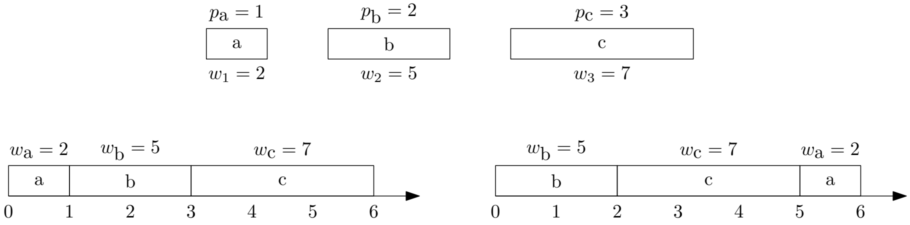
Outline

Scheduling to Minimize Weighted Completion Time

Input: A set of n jobs $[n] := \{1, 2, 3, \cdots, n\}$

each job j has a weight w_j and processing time p_j

Output: an ordering of jobs so as to minimize the total weighted completion time of jobs



$\text{cost} = 2 \times 1 + 5 \times 3 + 7 \times 6 = 59$

$\text{cost} = 5 \times 2 + 7 \times 5 + 2 \times 6 = 57$

Candidate algorithms

Schedule the jobs in some natural order. Which order should we choose?

- A. Ascending order of processing times p_j
- B. Descending order of slackness w_j
- C. Ascending order of $p_j - w_j$
- D. Ascending order of p_j / w_j

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Def. The Smith ratio of a job is w_j/p_j.

Lemma The descending order of Smith ratios (the Smith rule) is optimum.

- A schedule S, j is right before j'.

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Q: How does the total weighted completion time change if we swap j and j'? $(\cdots, j, j', \cdots) \Longrightarrow (\cdots, j', j, \cdots)$

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A: $w_{j'}p_j \quad \Longrightarrow \quad w_j p_{j'}$

- Therefore, swapping decrease the weighted completion time if $\frac{p_{j'}}{w_{j'}} < \frac{p_j}{w_j}$.

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- Indeed, optimum weighted completion time is

$$\sum_{j \in [n]} w_j p_j + \sum_{1 \leq j < j' \leq n} \min\{w_j p_{j'}, w_{j'} p_j\}.$$

Outline

Offline Caching

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- Sequence of page requests

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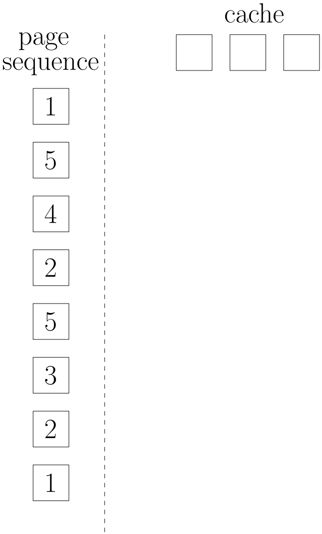

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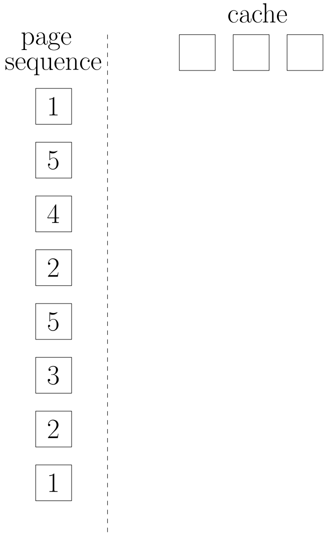

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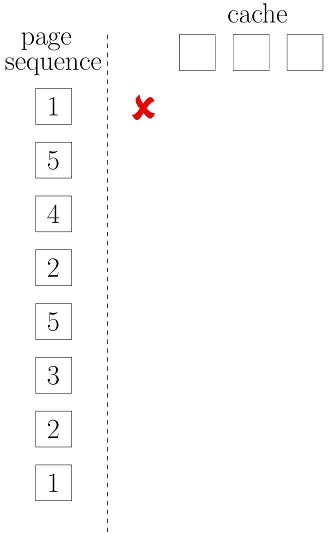

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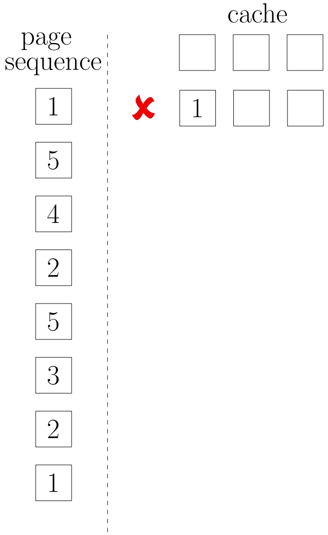

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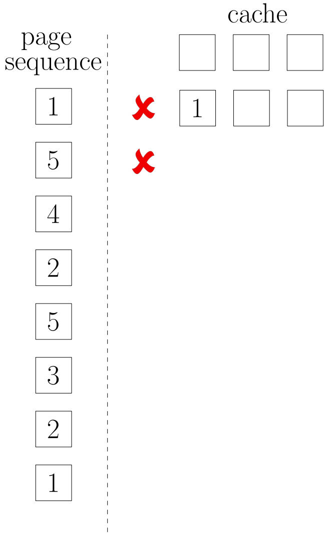

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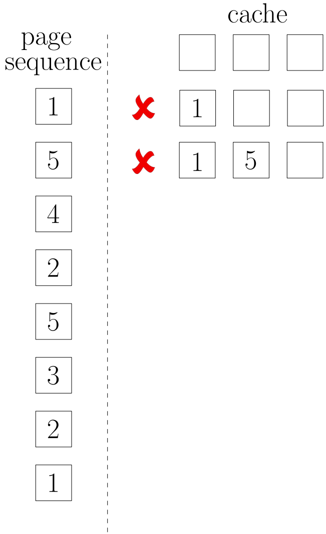

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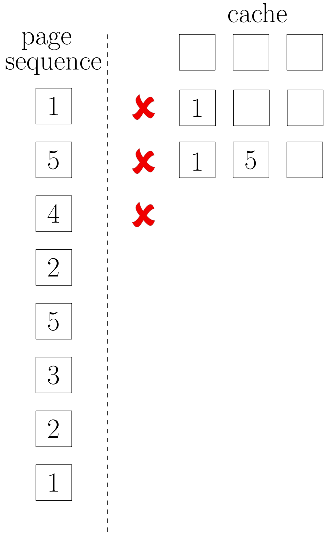

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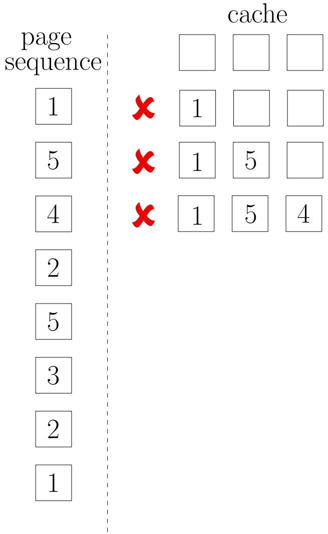

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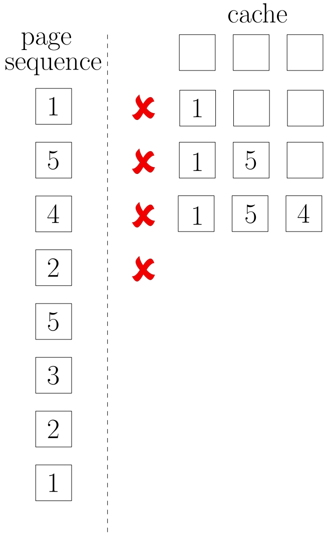

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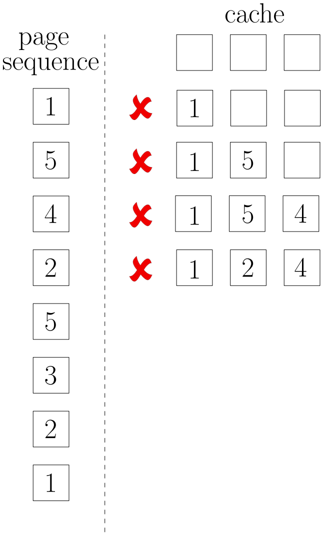

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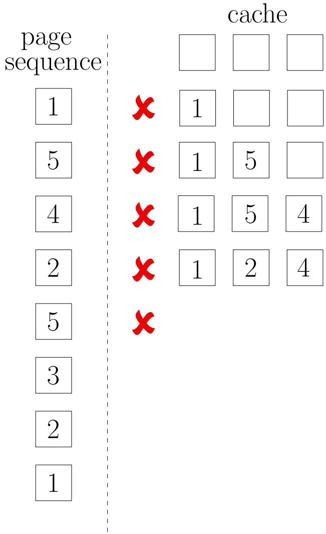

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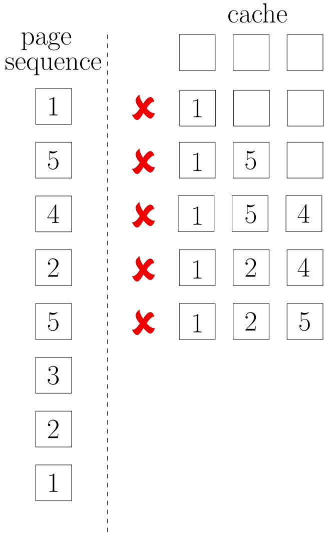

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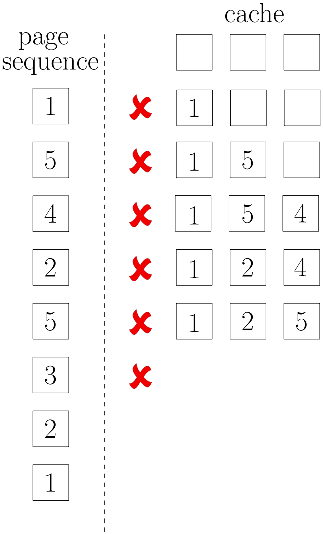

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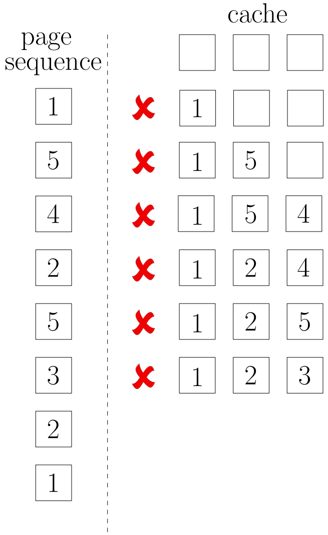

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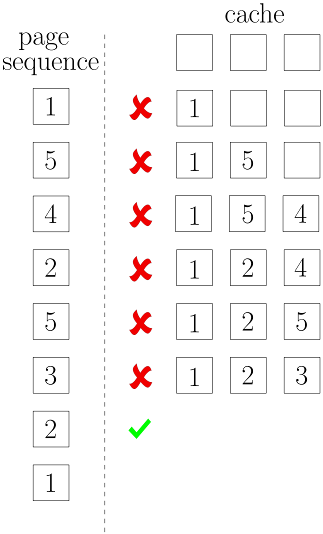

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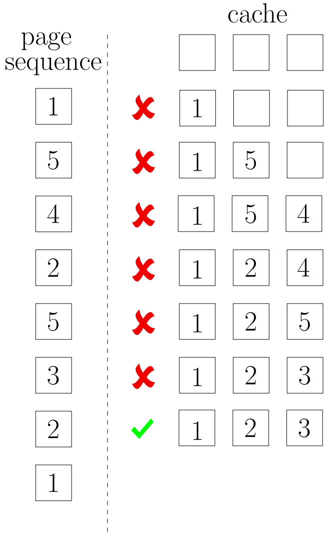

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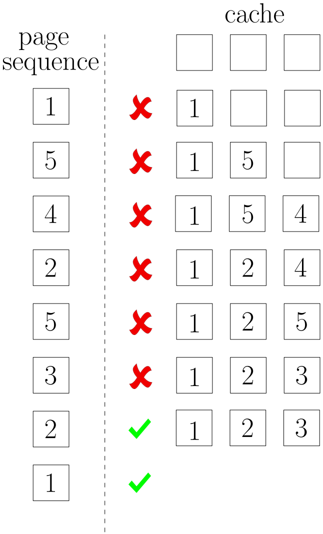

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page sequence		cache		
1	✗	1		
5	✗	1	5	
4	✗	1	5	4
2	✗	1	2	4
5	✗	1	2	5
3	✗	1	2	3
2	✓	1	2	3
1	✓	1	2	3

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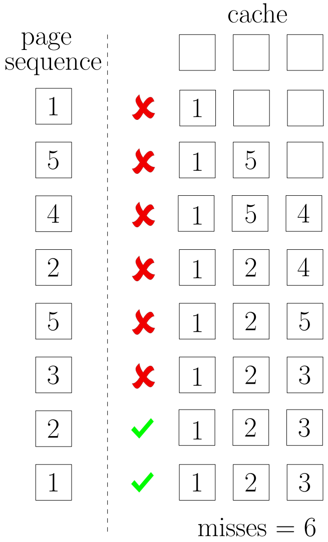

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- Goal: minimize the number of cache misses.

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4	✘	1	5	4
2	✘	1	2	4
5	✘	1	2	5
3	✘	1	2	3
2	✓	1	2	3
1	✓	1	2	3

misses $= 6$

A Better Solution for Example

page sequence		cache				cache		
1	✗	1			✗	1		
5	✗	1	5		✗	1	5	
4	✗	1	5	4	✗	1	5	4
2	✗	1	2	4	✗	1	5	2
5	✗	1	2	5	✓	1	5	2
3	✗	1	2	3	✗	1	3	2
2	✓	1	2	3	✓	1	3	2
1	✓	1	2	3	✓	1	3	2
		misses = 6				misses = 5		

Offline Caching Problem

Input: k : the size of cache

n : number of pages

We use $[n]$ for $\{1, 2, 3, \cdots, n\}$.

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

Output: $i_1, i_2, i_3, \cdots, i_T \in \{\text{hit}, \text{empty}\} \cup [n]$: indices of pages to evict ("hit" means evicting no page, "empty" means evicting empty page)

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- Offline Caching: we know the whole sequence ahead of time.
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Q: Why do we study the offline caching problem?

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A: Online caching

Q: Why do we study the offline caching problem?

A: Use the offline solution as a benchmark to measure the "competitive ratio" of online algorithms

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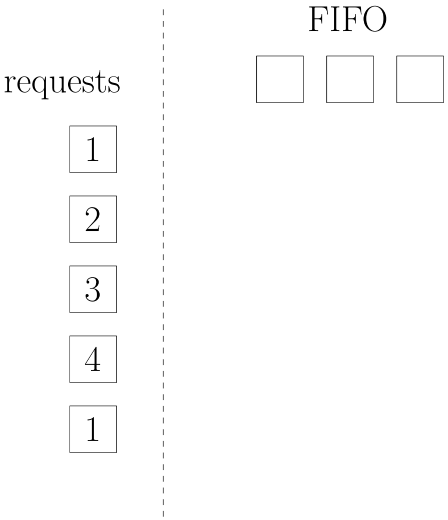
- All the above algorithms are not optimum!
- Indeed all the algorithms are "online", i.e, the decisions can be made without knowing future requests. Online algorithms can not be optimum.

FIFO is not optimum



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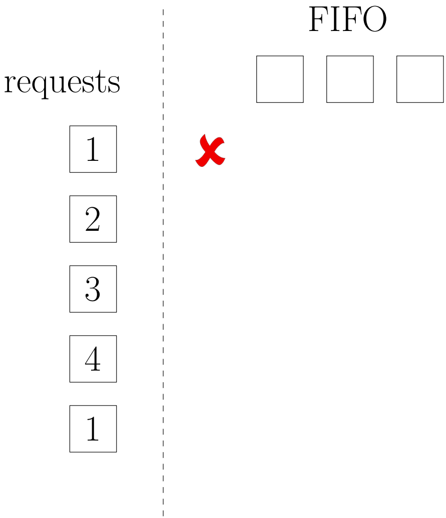

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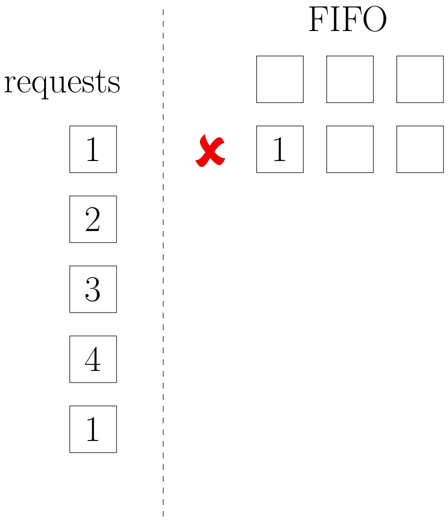

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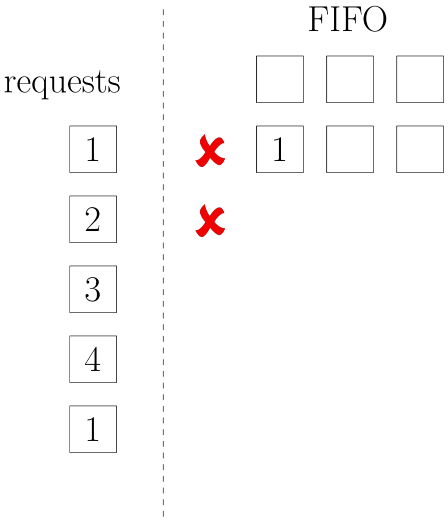

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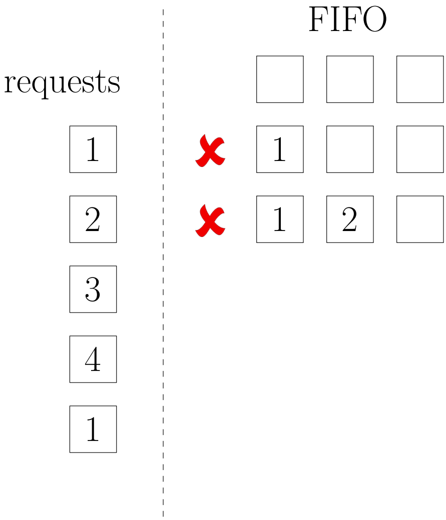

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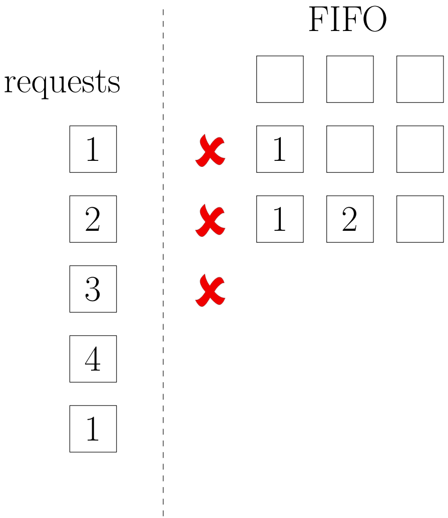

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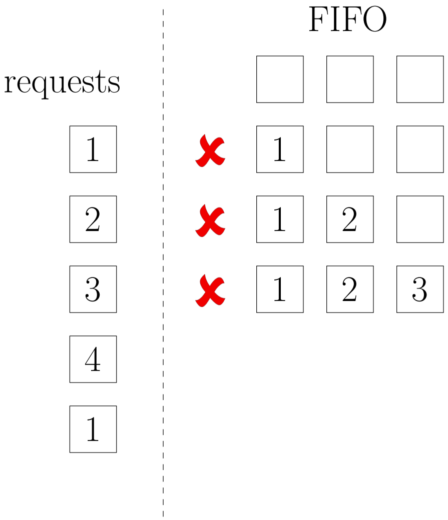

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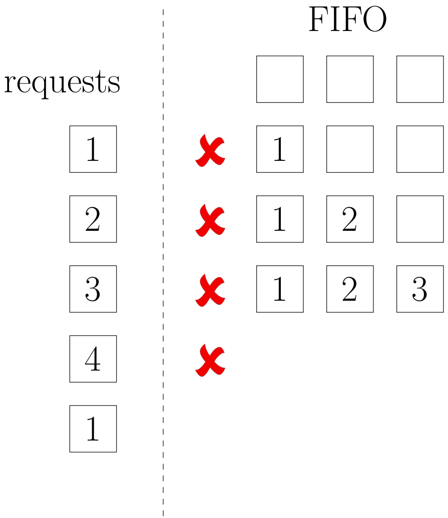

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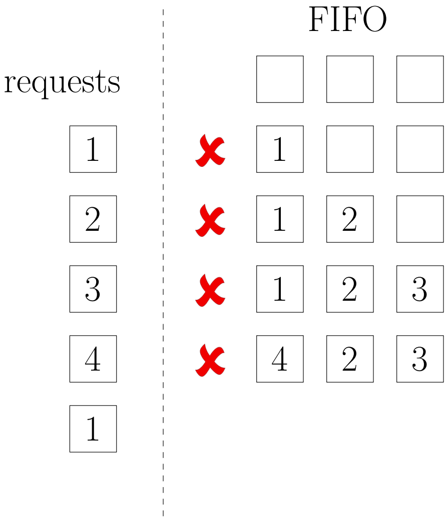

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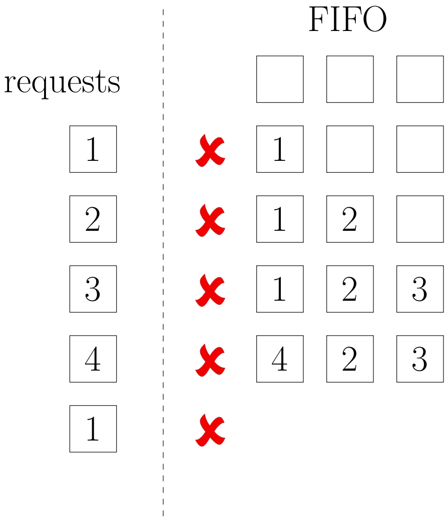

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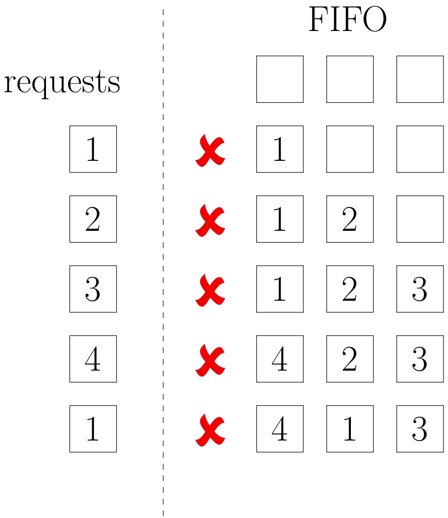

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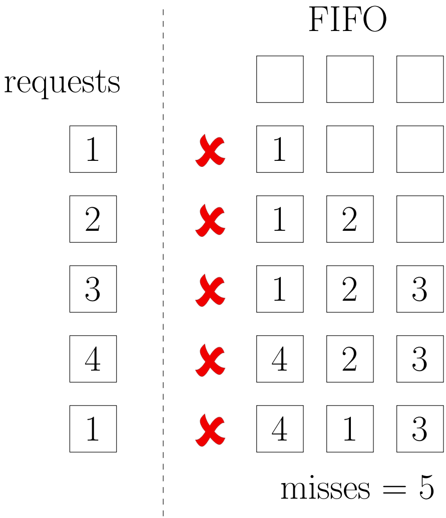

FIFO is not optimum

requests		FIFO				Furthest-in-Future		
1	✗	1			✗	1		
2	✗	1	2		✗	1	2	
3	✗	1	2	3	✗	1	2	3
4	✗	4	2	3	✗	1	4	3
1	✗	4	1	3	✓	1	4	3
		misses = 5				misses = 4		

Optimum Offline Caching

Furthest-in-Future (FF)

- Algorithm: every time, evict the page that is not requested until furthest in the future, if we need to evict one.
- The algorithm is not an online algorithm, since the decision at a step depends on the request sequence in the future.

Furthest-in-Future (FF)

requests		FIFO				Furthest-in-Future		
1	✗	1			✗	1		
2	✗	1	2		✗	1	2	
3	✗	1	2	3	✗	1	2	3
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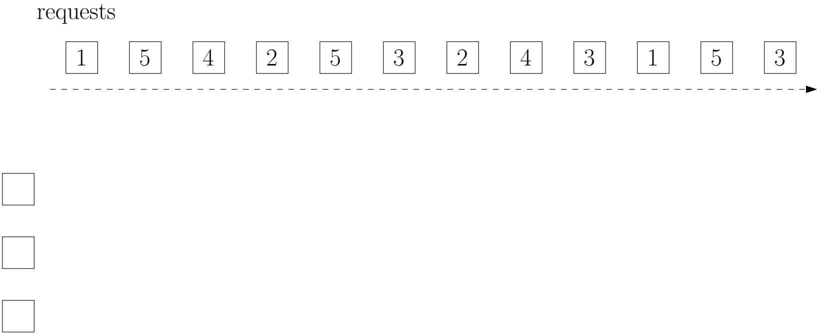
Example



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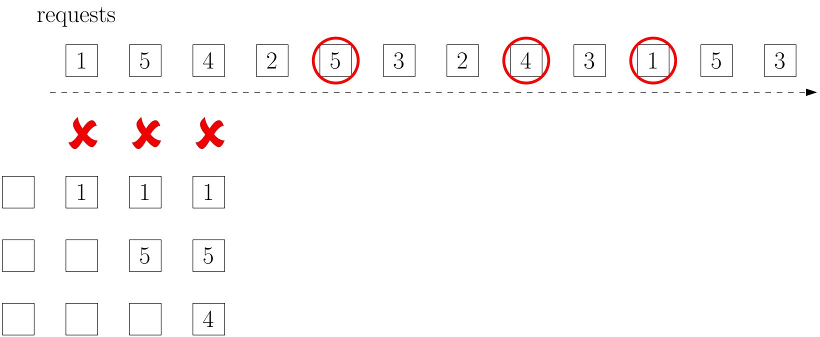

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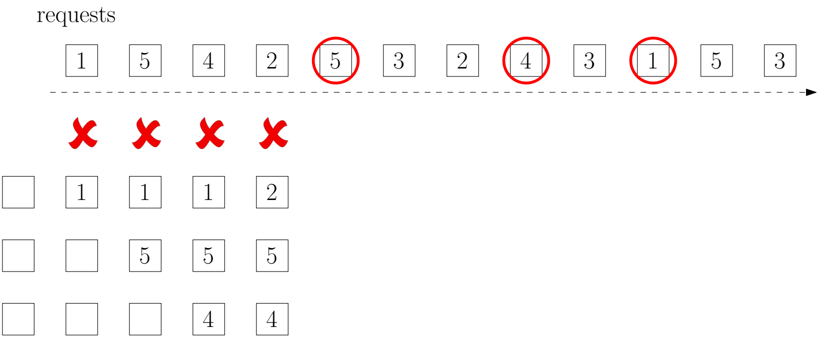

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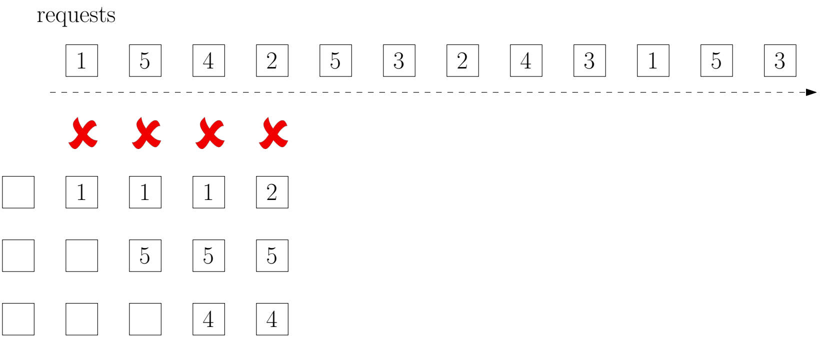

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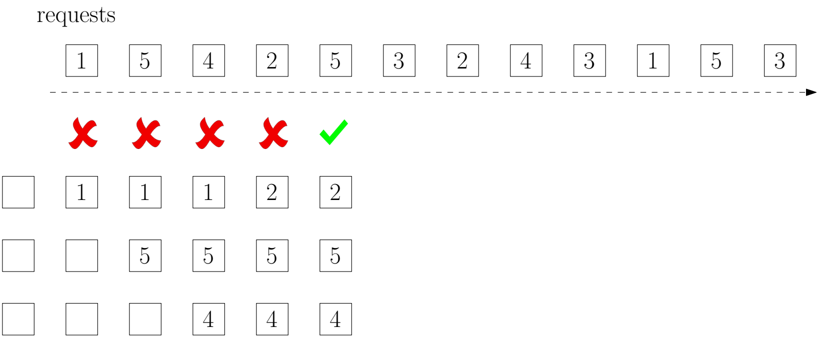

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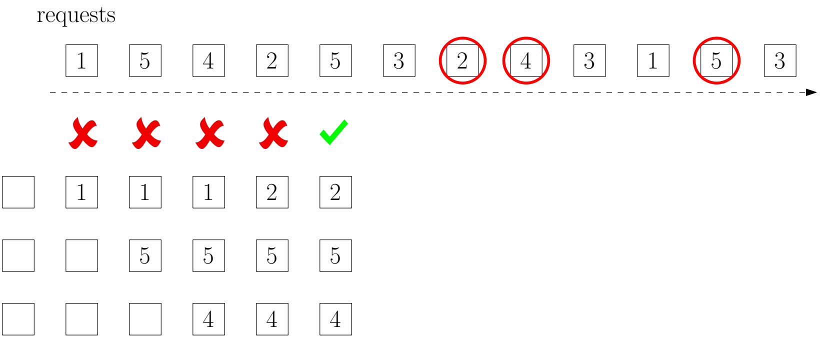

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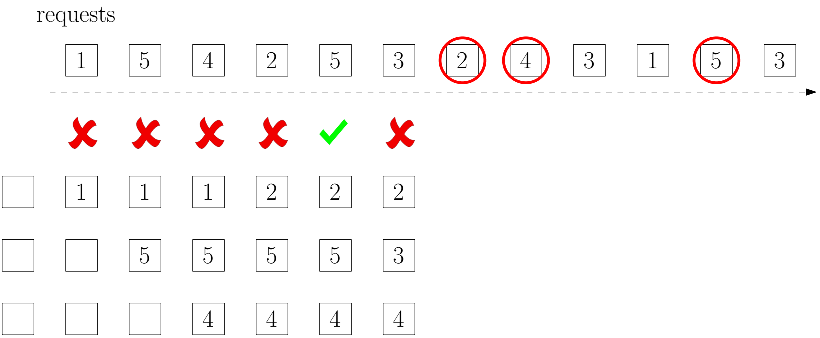

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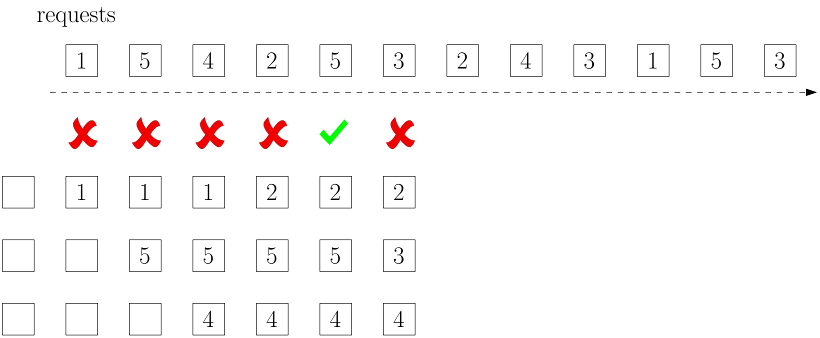

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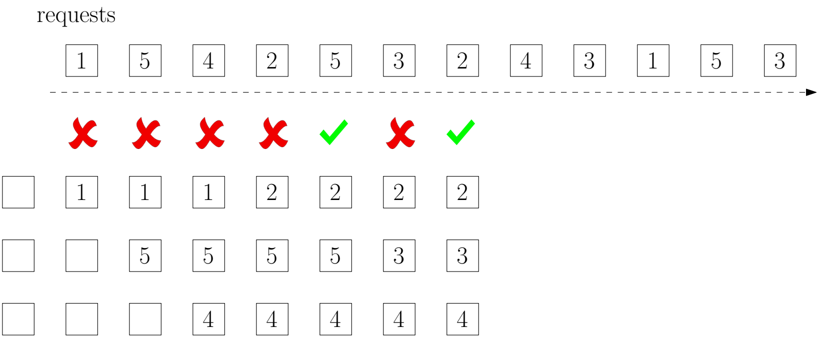

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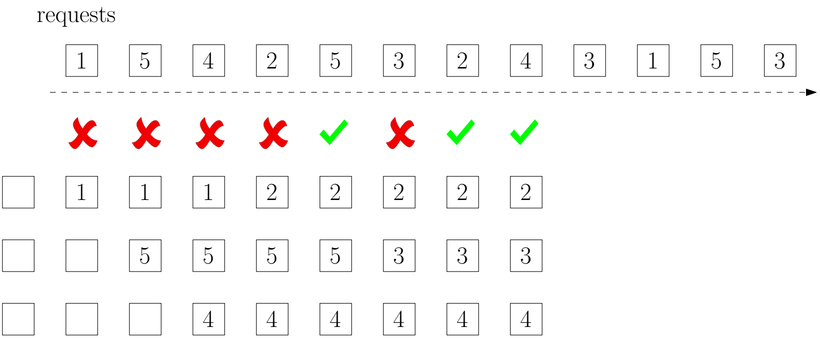

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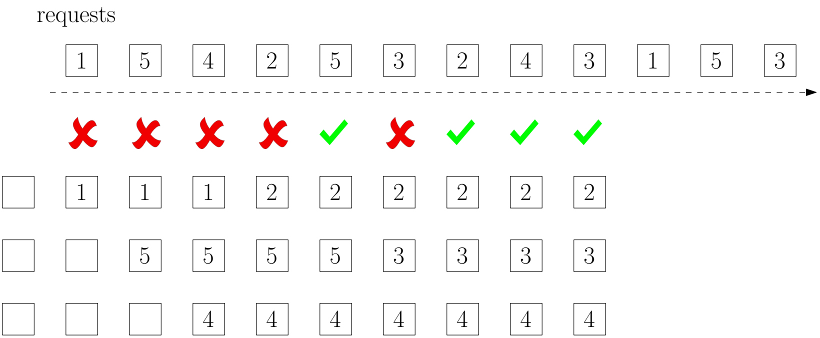

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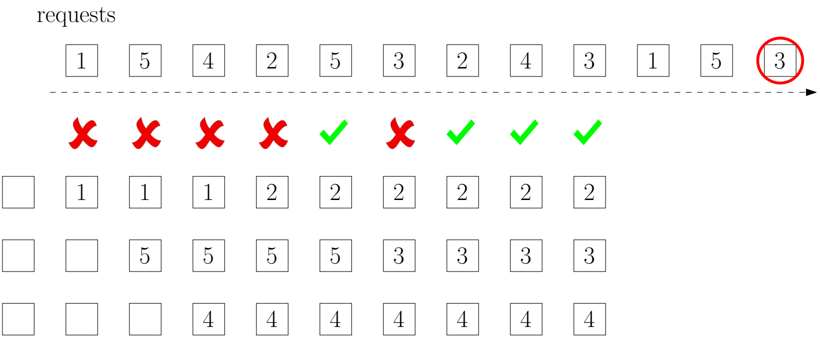

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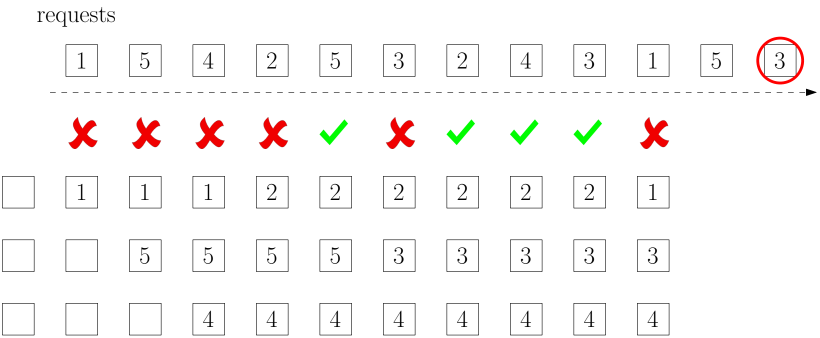

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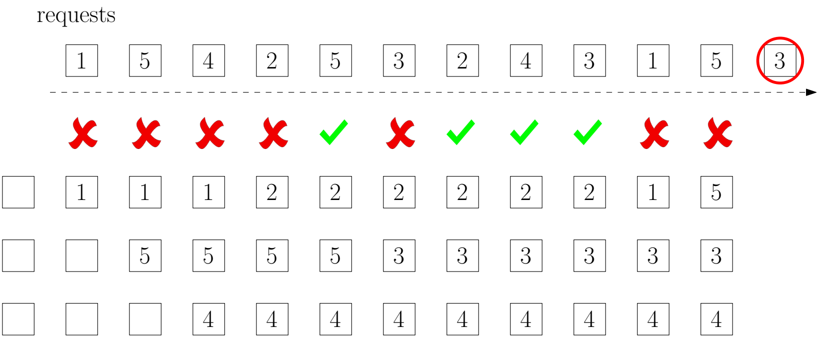
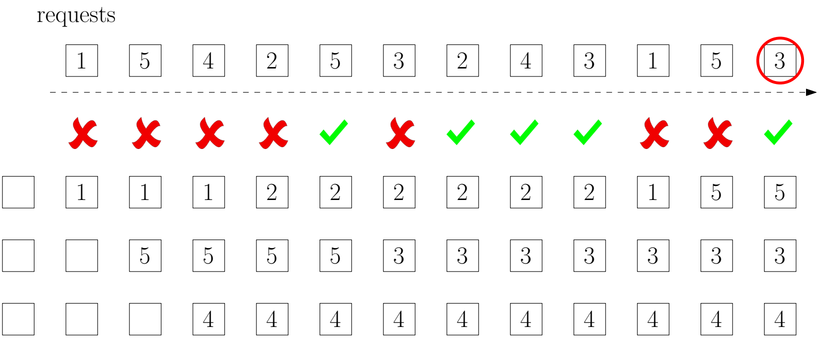

requests
1 5 4 2 5 3 2 4 3 1 5 3
1 1 1 2 2 2 2 2 2 1 5
5 5 5 5 3 3 3 3 3 3
4 4 4 4 4 4 4 4 4

Example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Recall: Designing and Analyzing Greedy Algorithms

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a “reasonable” strategy

A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

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Offline Caching Problem

Input: k : the size of cache

n : number of pages

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

Output: $i_1, i_2, i_3, \cdots, i_t \in \{\text{hit}, \text{empty}\} \cup [n]$

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Input: k : the size of cache

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$p_1, p_2, \cdots, p_k \in \{\text{empty}\} \cup [n]$: initial set of pages in cache

Output: $i_1, i_2, i_3, \cdots, i_t \in \{\text{hit}, \text{empty}\} \cup [n]$

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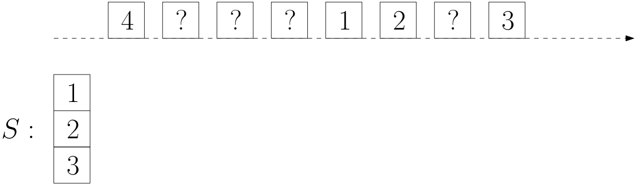
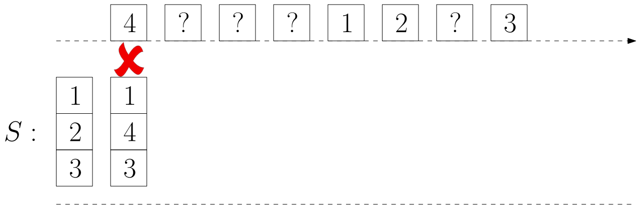
Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. There is an optimum solution in which p^* is evicted at time 1.



Proof.

1. S: any optimum solution
2. p^*: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$.



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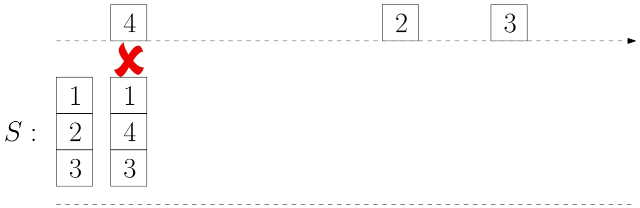

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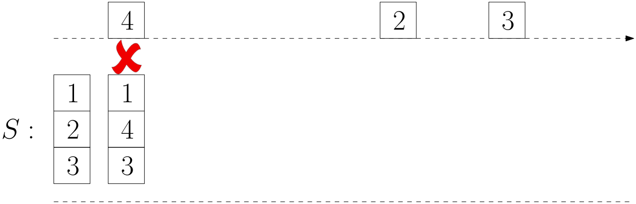

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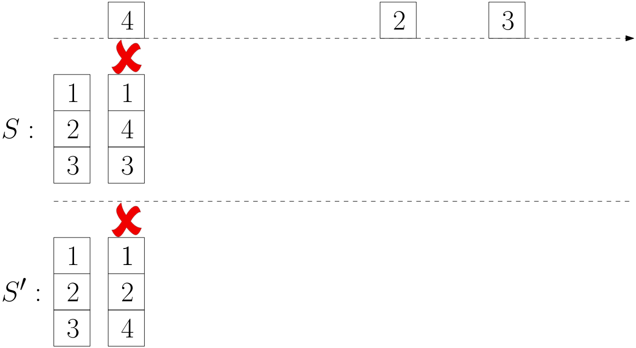

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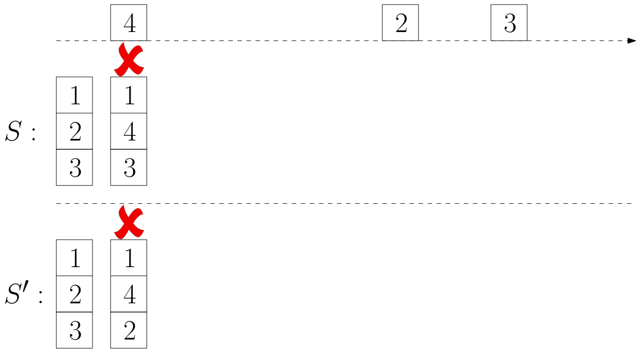

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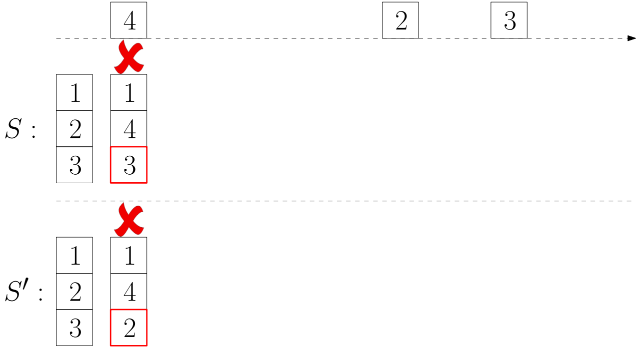

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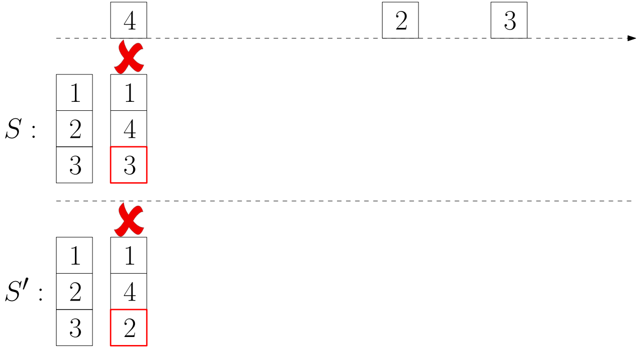

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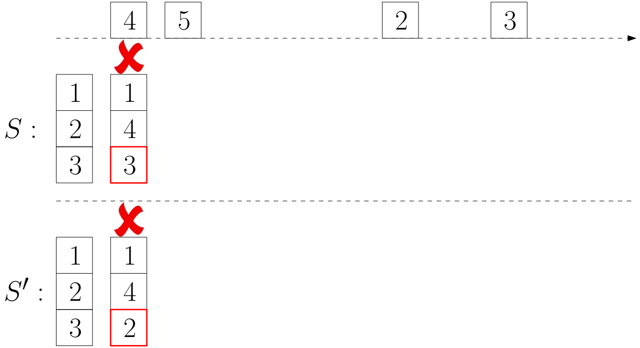

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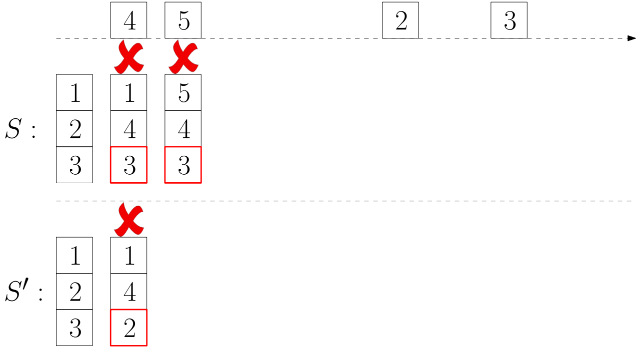

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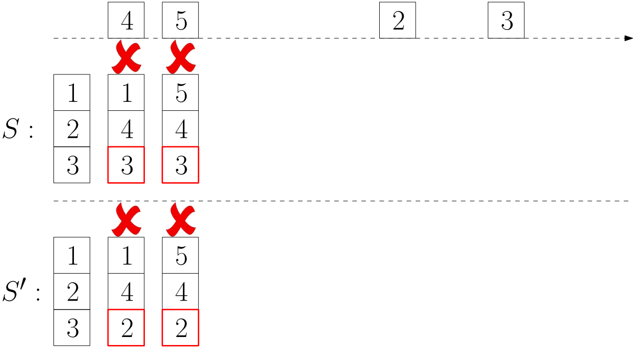

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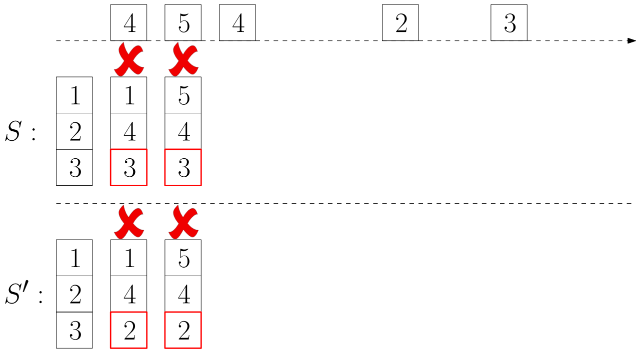

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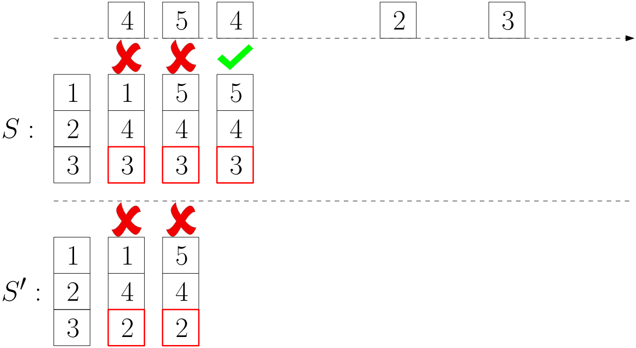

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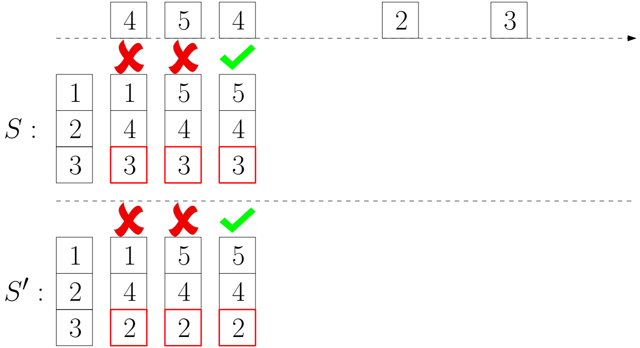

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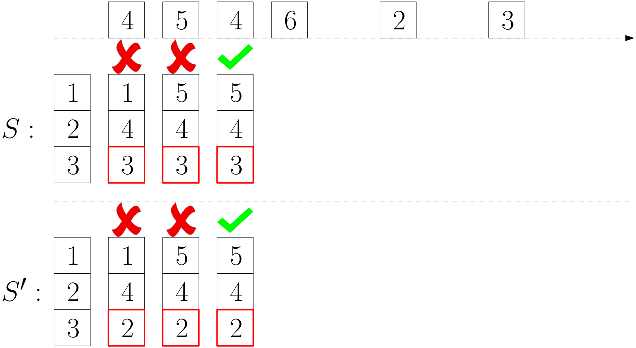

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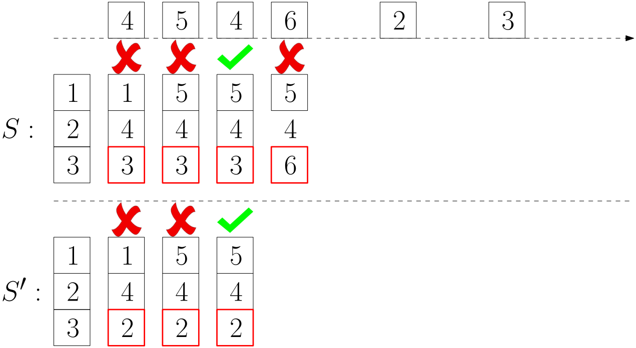

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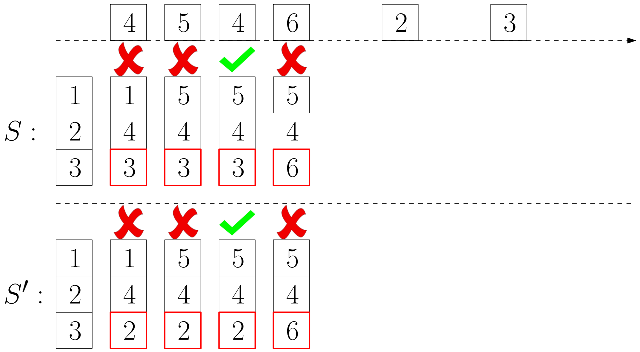

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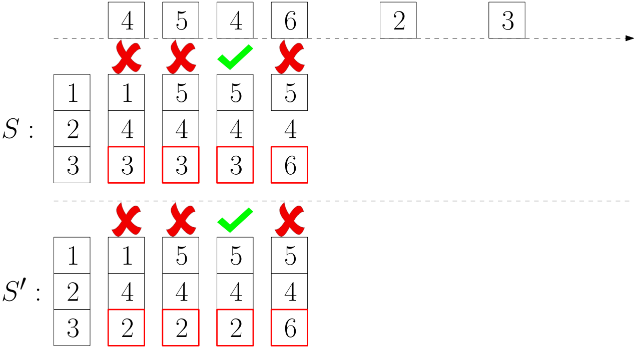

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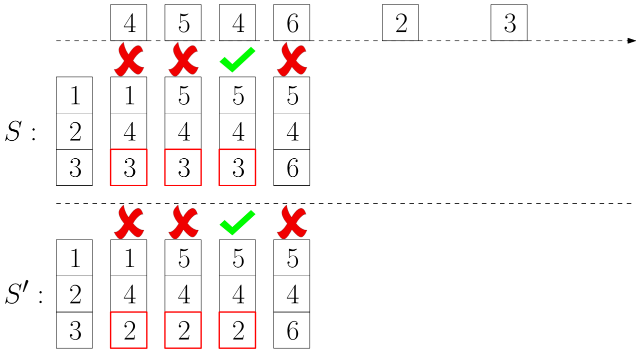

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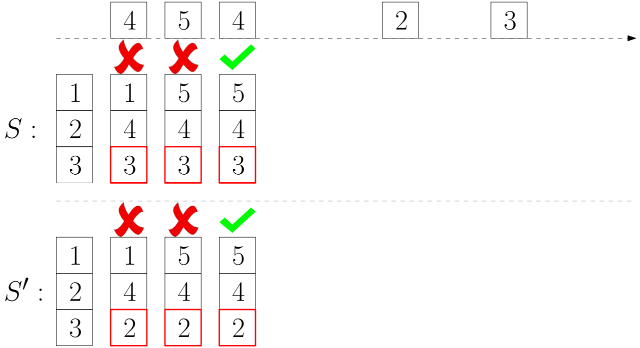

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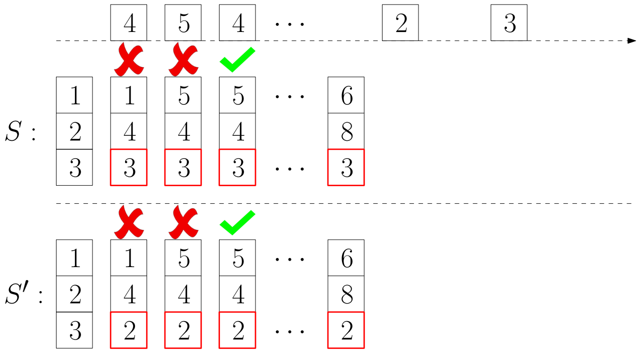

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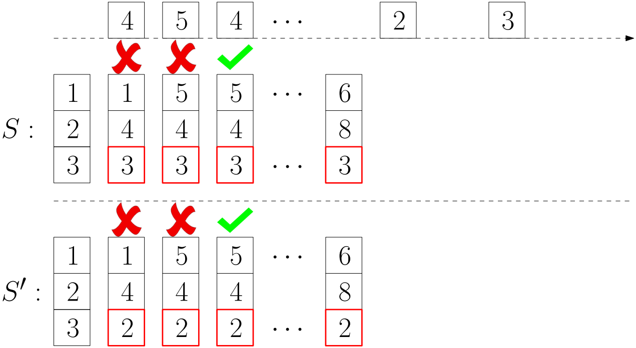

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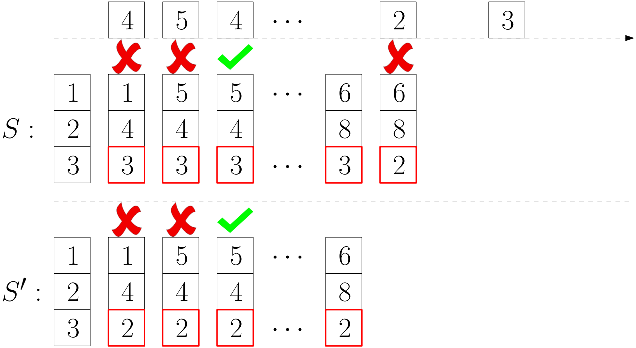

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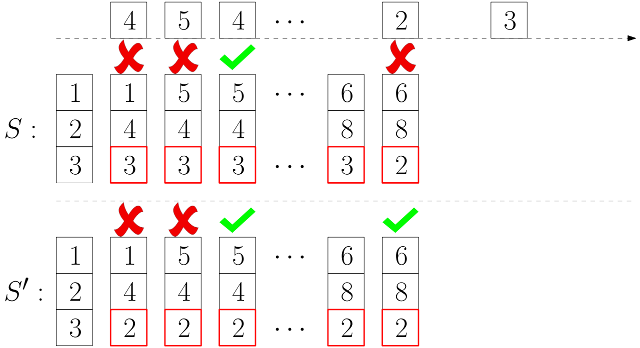

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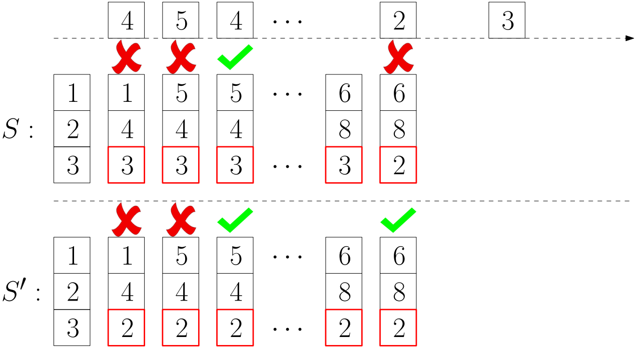

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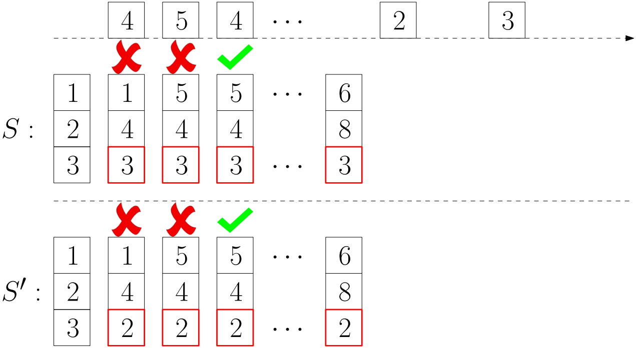

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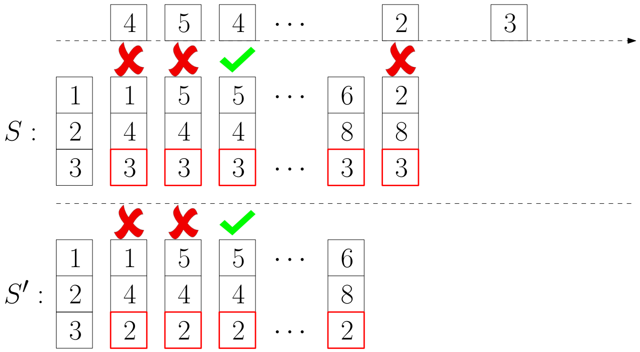

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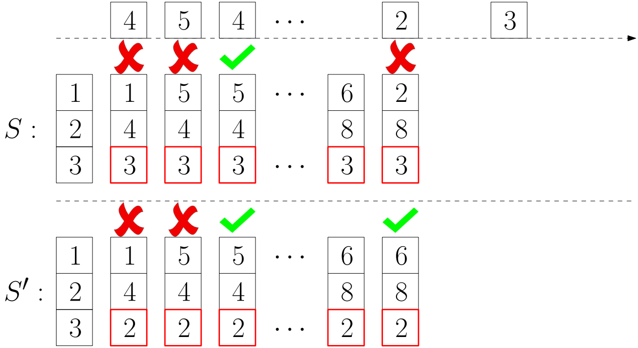

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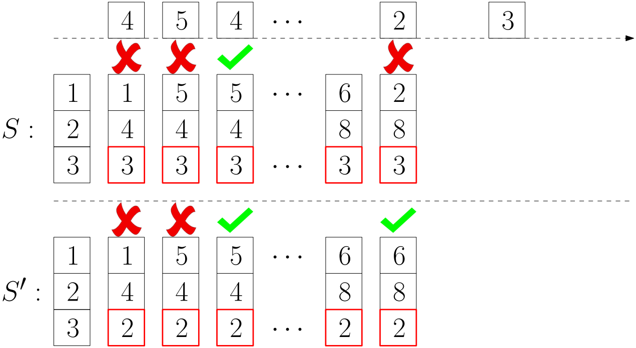

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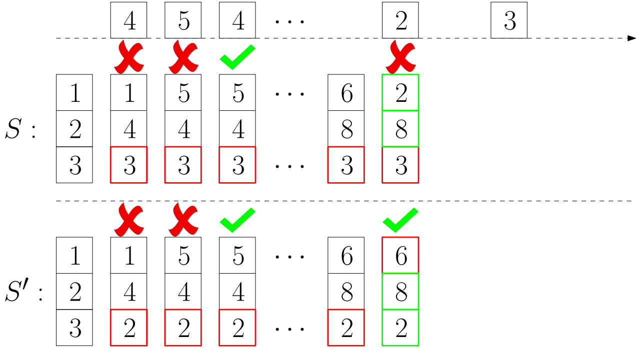

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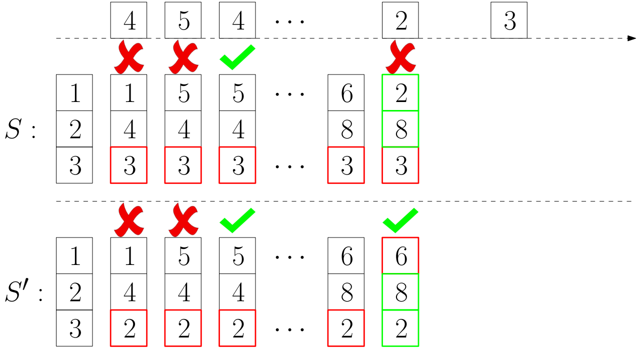

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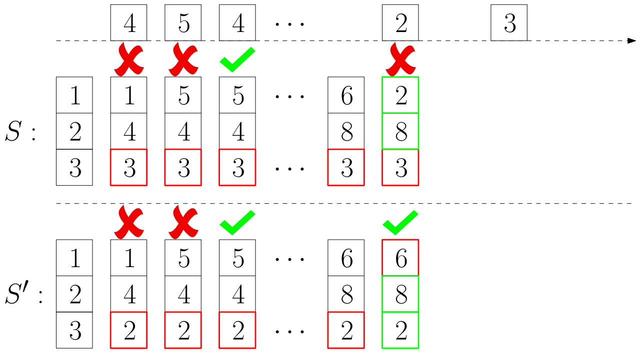

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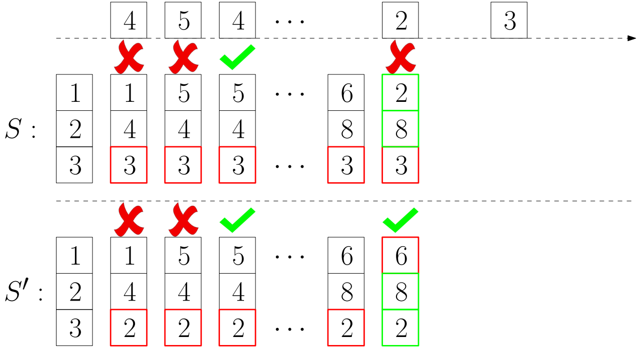

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12. We can then guarantee that S' make at most the same number of page-misses as S does.
   - Idea: if S has a page-hit and S' has a page-miss, we use the opportunity to make the status of S' the same as that of S. □

- Thus, we have shown how to create another solution S' with the same number of page-misses as that of the optimum solution S. Thus, we proved

Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. There is an optimum solution in which p^* is evicted at time 1.

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Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. It is safe to evict p^* at time 1.

Theorem The furthest-in-future strategy is optimum.

```
for t ← 1 to T do
    if ρ_t is in cache then do nothing
    else if there is an empty page in cache then
        evict the empty page and load ρ_t in cache
    else
        p* ← page in cache that is not used furthest in the future
        evict p* and load ρ_t in cache
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- For each page p, use a linked list (or an array with dynamic size) to store the time steps in which p is requested.
  - We can find the next time a page is requested easily.
- Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values

time	0	1	2	3	4	5	6	7	8	9	10	11	12
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priority queue

pages	priority values
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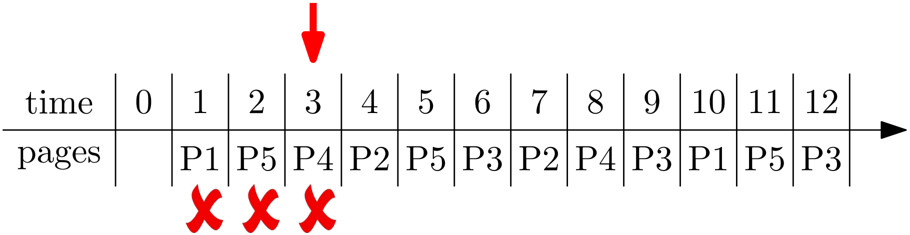
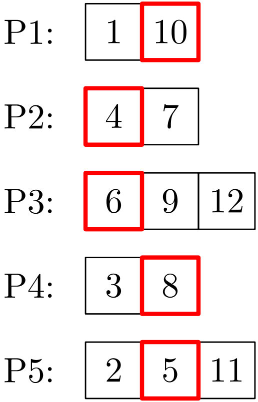
P5	5





priority queue

pages	priority values
P1	10
P5	5
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P1	10
P5	5
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P5	5
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P2	7
P5	5
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P2	7
P5	5
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓							

P1:	1	**10**	
P2:	4	**7**	
P3:	**6**	9	12
P4:	3	**8**	
P5:	2	5	**11**

priority queue

pages	priority values
P2	7
P5	11
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3

P1: 1, **10**

P2: 4, **7**

P3: **6**, 9, 12

P4: 3, **8**

P5: 2, 5, **11**

priority queue

pages	priority values
P2	7
P5	11
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓							

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P2	7
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗						

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P2	7
P3	9
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗						

P1: 1, 10

P2: 4, 7

P3: 6, 9, 12

P4: 3, 8

P5: 2, 5, 11

priority queue

pages	priority values
P2	7
P3	9
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓					

P1: 1 | **10**

P2: 4 | 7 | **(empty)**

P3: 6 | **9** | 12

P4: 3 | **8**

P5: 2 | 5 | **11**

priority queue

pages	priority values
P2	∞
P3	9
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓					

P1: 1, 10

P2: 4, 7,

P3: 6, 9, 12

P4: 3, 8

P5: 2, 5, 11

priority queue

pages	priority values
P2	∞
P3	9
P4	8

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓				

P1: 1, **10**

P2: 4, 7, **[ ]**

P3: 6, **9**, 12

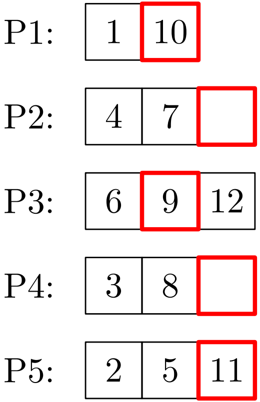
P4: 3, 8, **[ ]**

P5: 2, 5, **11**

priority queue

pages	priority values
P2	∞
P3	9
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓				



priority queue

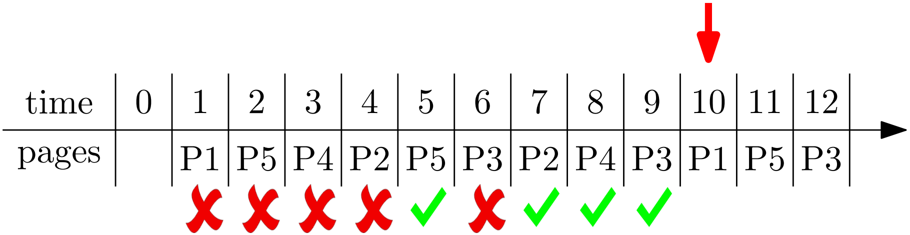
pages	priority values
P2	∞
P3	9
P4	∞

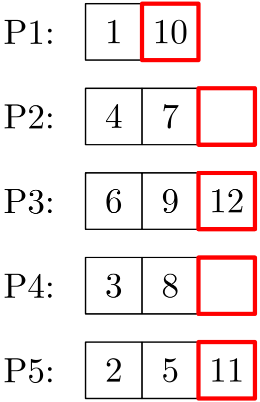
time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓			

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

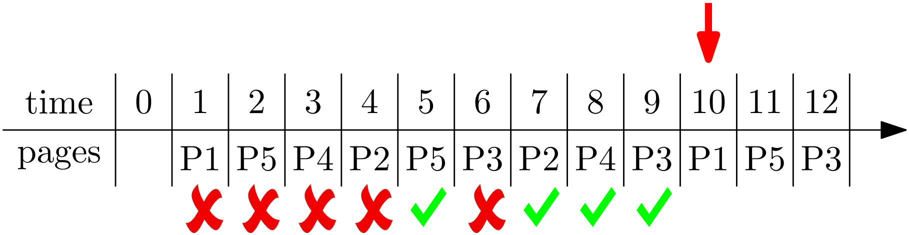
pages	priority values
P2	∞
P3	12
P4	∞





priority queue

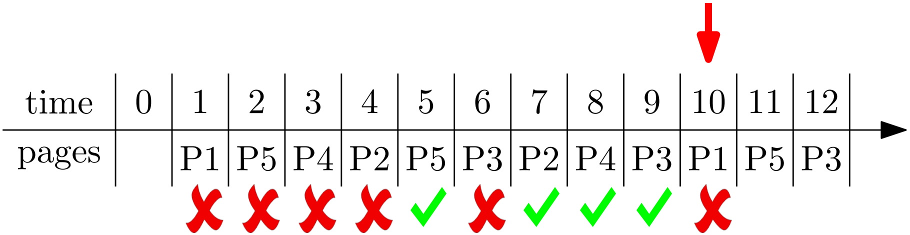
pages	priority values
P2	∞
P3	12
P4	∞



P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P3	12
P4	∞



P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P1	∞
P3	12
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓	✗		

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

pages	priority values
P1	∞
P3	12
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓	✗		

P1:	1	10	
P2:	4	7	
P3:	6	9	12
P4:	3	8	
P5:	2	5	11

priority queue

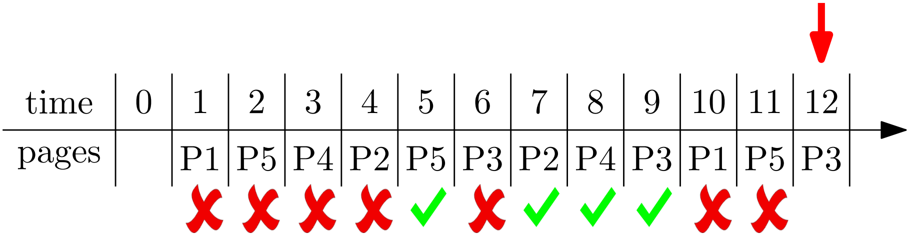
pages	priority values
P3	12
P4	∞

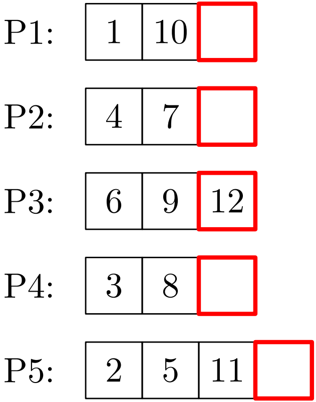
time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓	✗	✗	

P1:	1	10		
P2:	4	7		
P3:	6	9	12	
P4:	3	8		
P5:	2	5	11	

priority queue

pages	priority values
P5	∞
P3	12
P4	∞





priority queue

pages	priority values
P5	∞
P3	12
P4	∞

time	0	1	2	3	4	5	6	7	8	9	10	11	12
pages		P1	P5	P4	P2	P5	P3	P2	P4	P3	P1	P5	P3
		✗	✗	✗	✗	✓	✗	✓	✓	✓	✗	✗	✓

P1:	1	10		
P2:	4	7		
P3:	6	9	12	
P4:	3	8		
P5:	2	5	11	

priority queue

pages	priority values
P5	∞
P3	∞
P4	∞

for every $p \leftarrow 1$ to n **do**
  $times[p] \leftarrow$ array of times in which p is requested, in increasing order ▷ put ∞ at the end of array
  $pointer[p] \leftarrow 1$
$Q \leftarrow$ empty priority queue
for every $t \leftarrow 1$ to T **do**
  $pointer[\rho_t] \leftarrow pointer[\rho_t] + 1$
  if $\rho_t \in Q$ **then**
    Q.increase-key$(\rho_t, times[\rho_t, pointer[\rho_t]])$, **print** "hit", **continue**
  if $Q.size() < k$ **then**
    print "load ρ_t to an empty page"
  else
    $p \leftarrow Q$.extract-max(), **print** "evict p and load ρ_t"
  Q.insert$(\rho_t, times[\rho_t, pointer[\rho_t]])$ ▷ add ρ_t to Q with key value $times[\rho_t, pointer[\rho_t]]$

Outline

- Let V be a ground set of size n.

Def. A priority queue is an abstract data structure that maintains a set $U \subseteq V$ of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element $v \in V \setminus U$, with associated key value key_value.
- decrease_key(v, new_key_value): decrease the key value of an element $v \in U$ to new_key_value
- extract_min$()$: return and remove the element in U with the smallest key value
- $\cdots$

Simple Implementations for Priority Queue

- n = size of ground set V

data structures	**insert**	**extract_min**	**decrease_key**
array			
sorted array			

Simple Implementations for Priority Queue

- n = size of ground set V

data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array			

Simple Implementations for Priority Queue

- n = size of ground set V

data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$

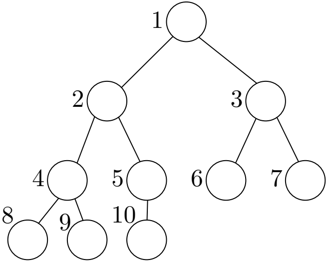
Simple Implementations for Priority Queue

- n = size of ground set V

data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Heap

The elements in a heap is organized using a complete binary tree:

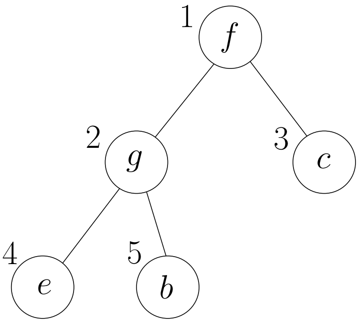


- Nodes are indexed as $\{1, 2, 3, \cdots, s\}$
- Parent of node i: $\lfloor i/2 \rfloor$
- Left child of node i: $2i$
- Right child of node i: $2i + 1$

Heap

A heap H contains the following fields

- s: size of U (number of elements in the heap)
- $A[i], 1 \leq i \leq s$: the element at node i of the tree
- $p[v], v \in U$: the index of node containing v
- $key[v], v \in U$: the key value of element v

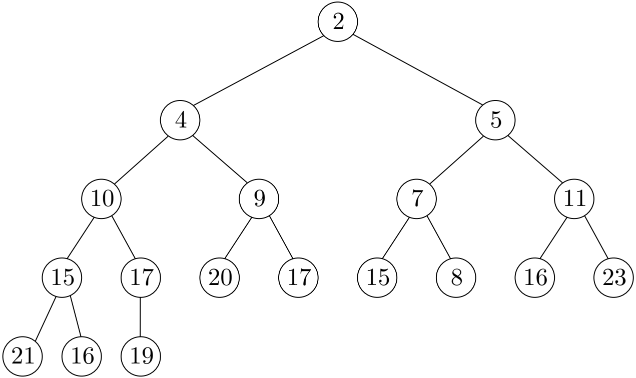


- $s = 5$
- $A = (\text{'}f\text{'}, \text{'}g\text{'}, \text{'}c\text{'}, \text{'}e\text{'}, \text{'}b\text{'})$
- $p[\text{'}f\text{'}] = 1, p[\text{'}g\text{'}] = 2, p[\text{'}c\text{'}] = 3,$
  $p[\text{'}e\text{'}] = 4, p[\text{'}b\text{'}] = 5$

Heap

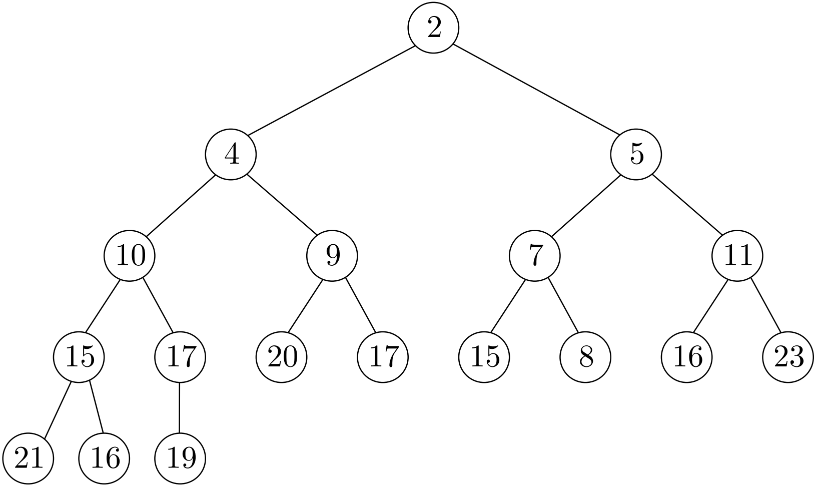
The following heap property is satisfied:

- for any two nodes i, j such that i is the parent of j, we have $key[A[i]] \leq key[A[j]]$.

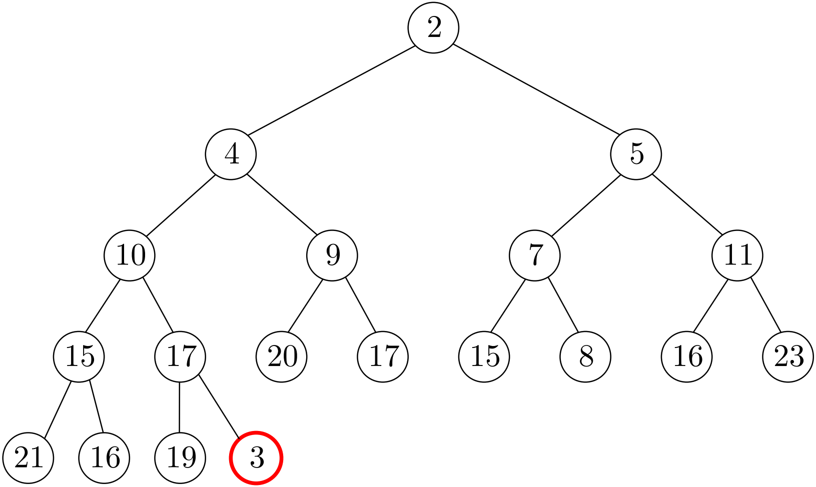


A heap. Numbers in the circles denote key values of elements.

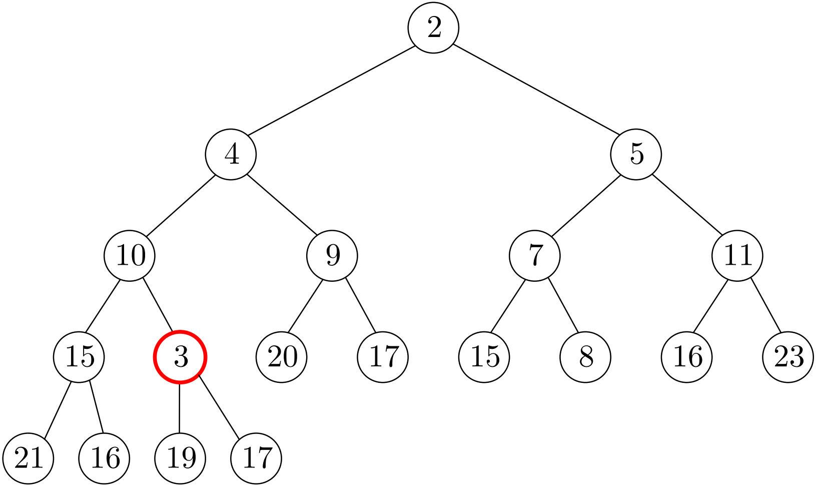
insert(v, key_value)



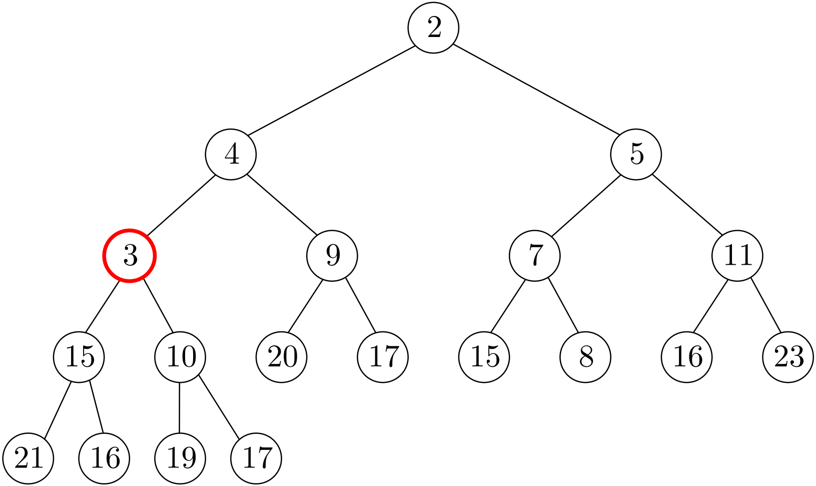
insert(v, key_value)



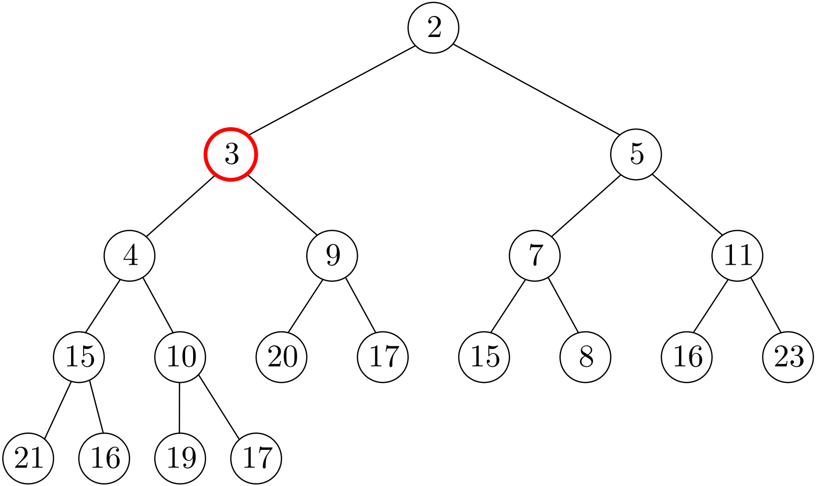
insert(v, key_value)



insert(v, key_value)



insert(v, key_value)



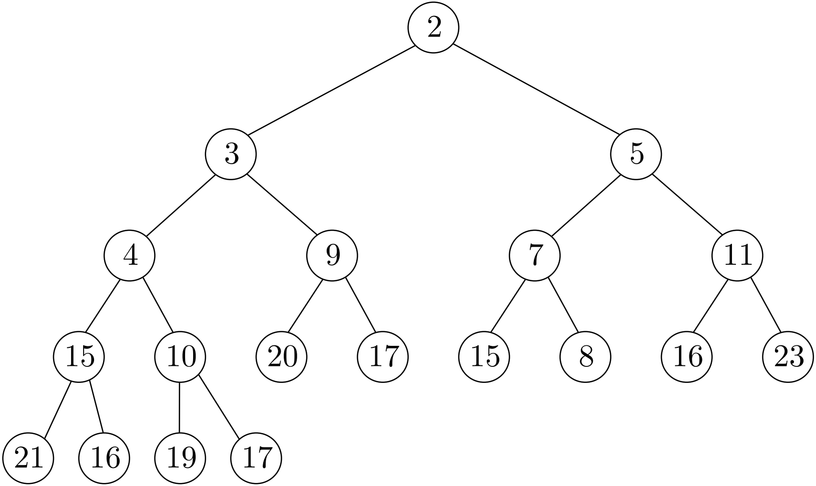
insert(v, key_value)

$s \leftarrow s + 1$
$A[s] \leftarrow v$
$p[v] \leftarrow s$
$key[v] \leftarrow key\_value$
heapify_up(s)

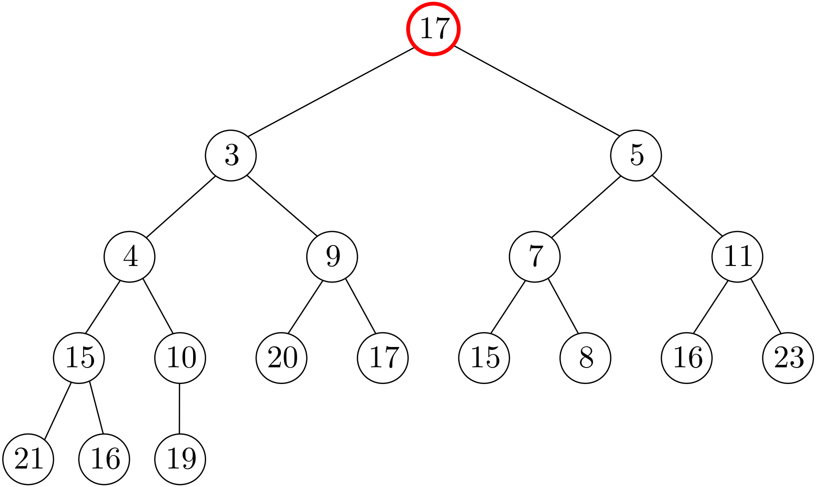
heapify-up(i)

while $i > 1$ **do**
    $j \leftarrow \lfloor i/2 \rfloor$
    if $key[A[i]] < key[A[j]]$ **then**
        swap $A[i]$ and $A[j]$
        $p[A[i]] \leftarrow i$, $p[A[j]] \leftarrow j$
        $i \leftarrow j$
    else break

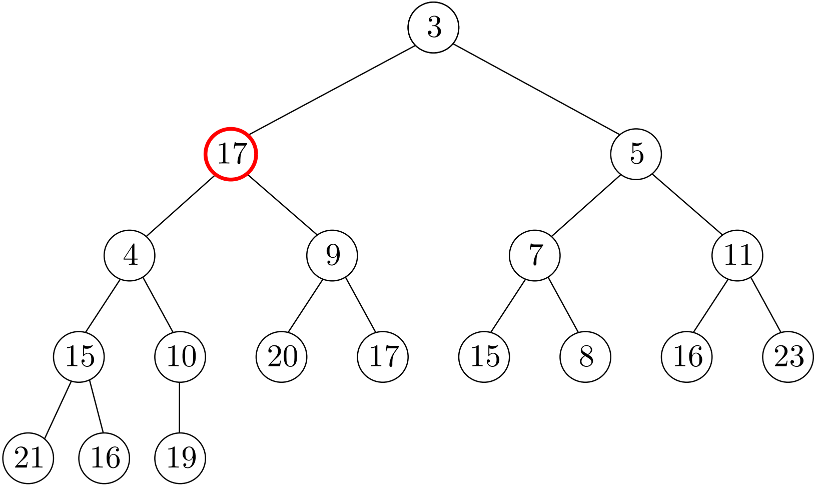
extract_min()



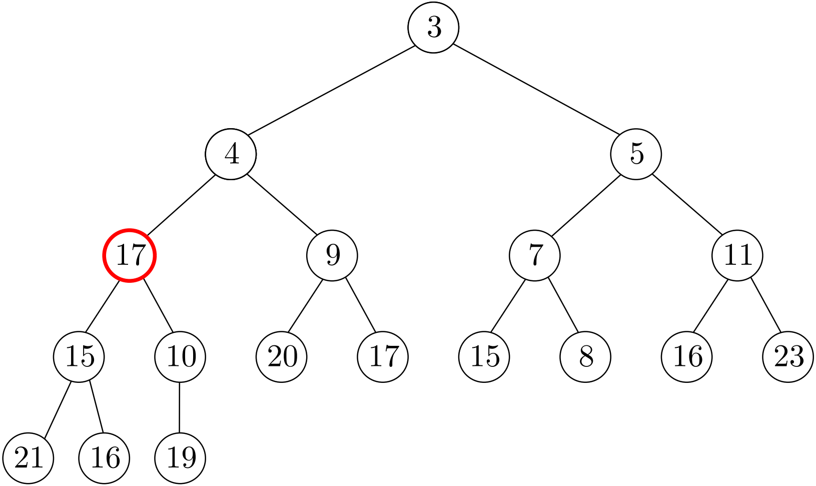
extract_min()



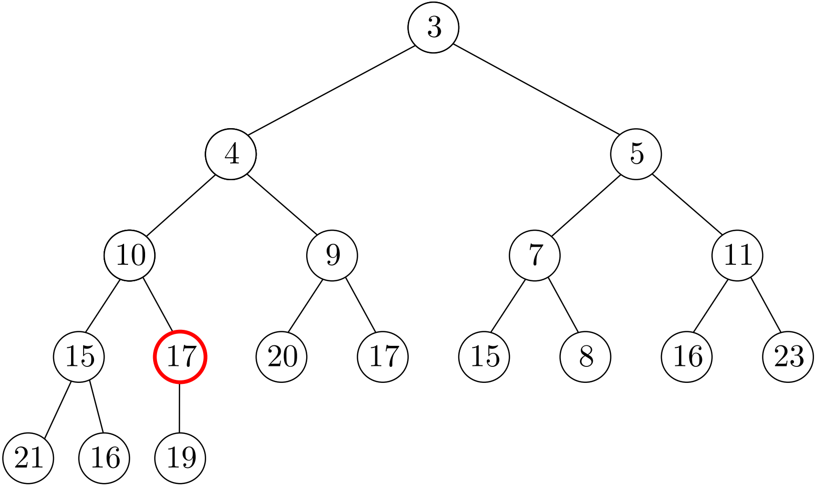
extract_min()



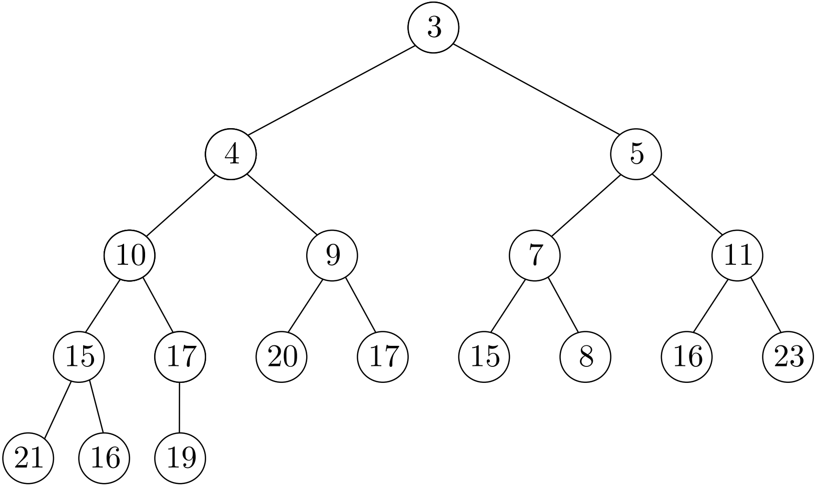
extract_min()



extract_min()



extract_min()



extract_min()

$ret \leftarrow A[1]$
$A[1] \leftarrow A[s]$
$p[A[1]] \leftarrow 1$
$s \leftarrow s - 1$
if $s \geq 1$ **then**
  heapify_down(1)
return ret

decrease_key(v, key_val

$key[v] \leftarrow key\_value$
heapify-up$(p[v])$

heapify-down(i)

while $2i \leq s$ **do**
  if $2i = s$ or $key[A[2i]] \leq key[A[2i+1]]$ **then**
    $j \leftarrow 2i$
  else
    $j \leftarrow 2i + 1$
  if $key[A[j]] < key[A[i]]$ **then**
    swap $A[i]$ and $A[j]$
    $p[A[i]] \leftarrow i$, $p[A[j]] \leftarrow j$
    $i \leftarrow j$
  else break

- Running time of heapify_up and heapify_down: $O(\lg n)$

- Running time of heapify_up and heapify_down: $O(\lg n)$
- Running time of insert, exact_min and decrease_key: $O(\lg n)$

- Running time of heapify_up and heapify_down: $O(\lg n)$
- Running time of insert, exact_min and decrease_key: $O(\lg n)$

data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Two Definitions Needed to Prove that the Procedures Maintain Heap Property

Def. We say that H is almost a heap except that $key[A[i]]$ is too small if we can increase $key[A[i]]$ to make H a heap.

Def. We say that H is almost a heap except that $key[A[i]]$ is too big if we can decrease $key[A[i]]$ to make H a heap.

Outline

Encoding Letters Using Bits

- 8 letters a, b, c, d, e, f, g, h in a language
- need to encode a message using bits
- idea: use 3 bits per letter

a	b	c	d	e	f	g	h
000	001	010	011	100	101	110	111

$$deacfg \to 011100000010101110$$

Q: Can we have a better encoding scheme?

- Seems unlikely: must use 3 bits per letter

Q: What if some letters appear more frequently than the others?

Q: If some letters appear more frequently than the others, can we have a better encoding scheme?

A: Using variable-length encoding scheme might be more efficient.

Idea

- using fewer bits for letters that are more frequently used, and more bits for letters that are less frequently used.

Q: What is the issue with the following encoding scheme?

- a: 0 b: 1 c: 00

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- a: 0 b: 1 c: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to aa or c.

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- a: 0 b: 1 c: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to aa or c.

Solution

Use prefix codes to guarantee a unique decoding.

Prefix Codes

Def. A prefix code for a set S of letters is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

Prefix Codes

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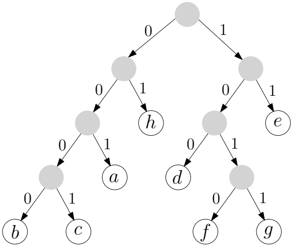
a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



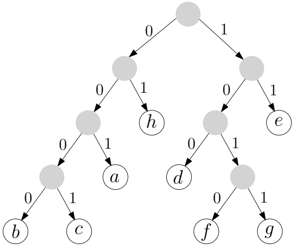
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

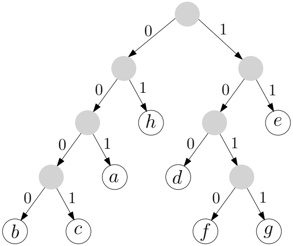
a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



- 0001001100000001011110100001001

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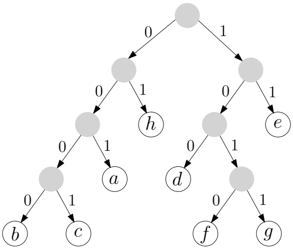

- 0001/0011000000101111010000100 1

- c

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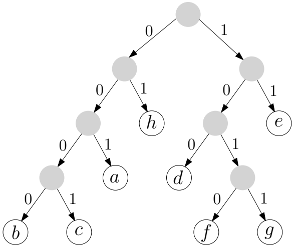

- 0001/001/100000001011110100001001
- ca

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001	0000	0001	100
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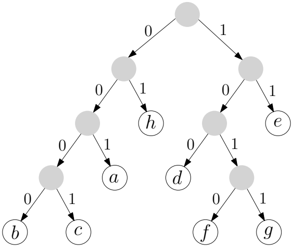

- 0001/001/100/000001011110100001001
- cad

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001	0000	0001	100
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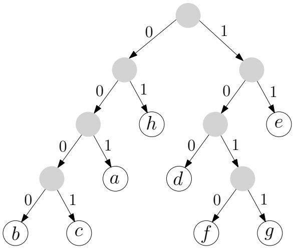

- 0001/001/100/0000/01011110100001001
- cadb

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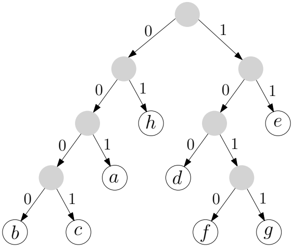

- 0001/001/100/0000/01/0111101000001001
- cadbh

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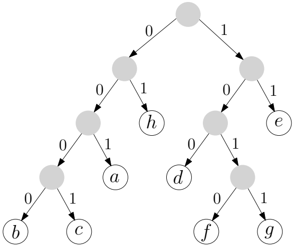

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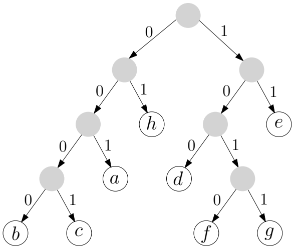

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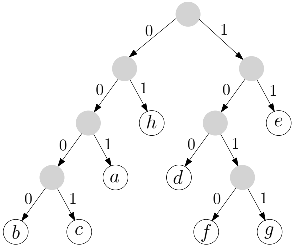

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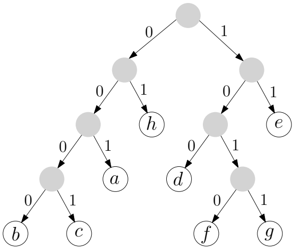

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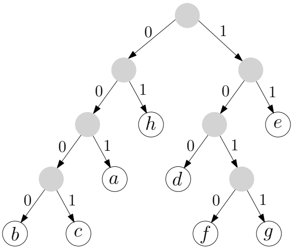

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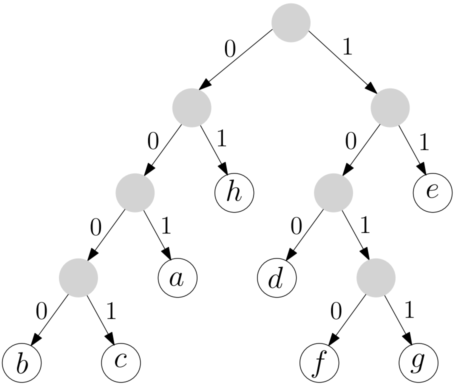
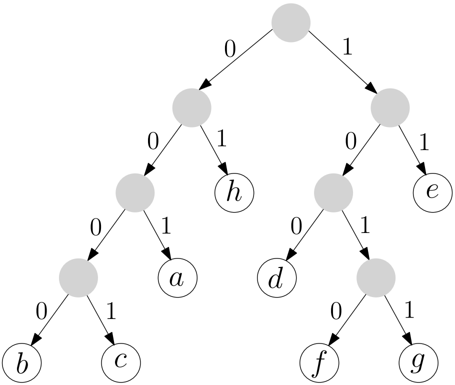
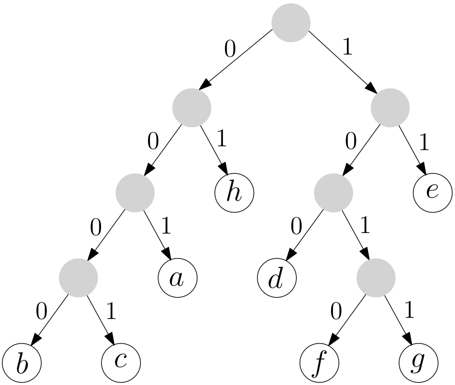


Properties of Encoding Tree



Properties of Encoding Tree

- Rooted binary tree



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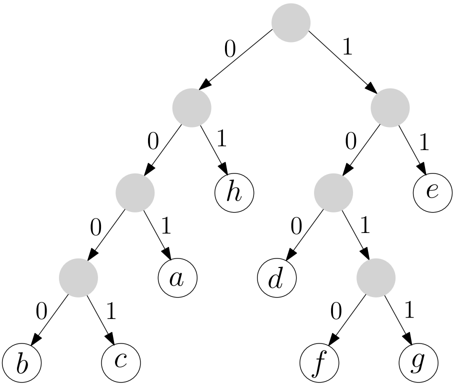

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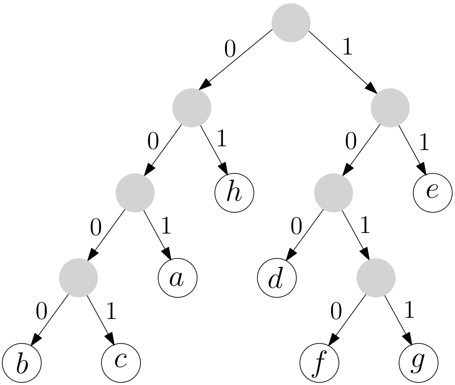

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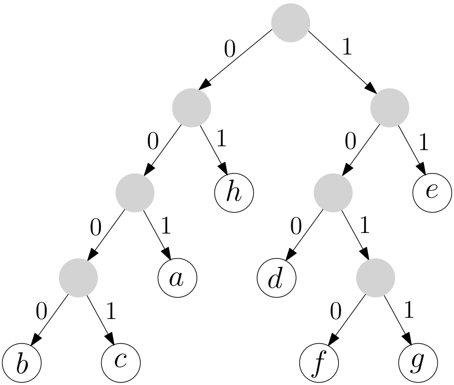

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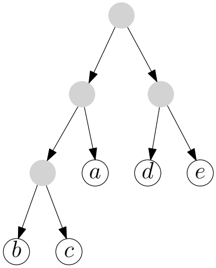
Best Prefix Codes

Input: frequencies of letters in a message

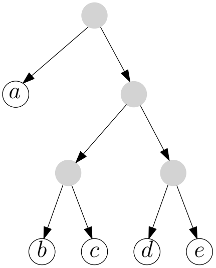
Output: prefix coding scheme with the shortest encoding for the message

example

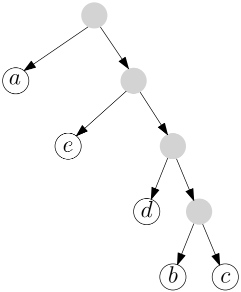
letters	a	b	c	d	e	
frequencies	18	3	4	6	10	



scheme 1



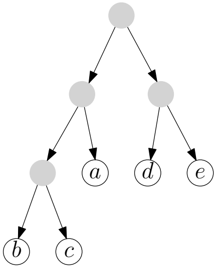
scheme 2



scheme 3

example

letters	a	b	c	d	e	
frequencies	18	3	4	6	10	
scheme 1 length	2	3	3	2	2	total = 89
scheme 2 length	1	3	3	3	3	total = 87
scheme 3 length	1	4	4	3	2	total = 84



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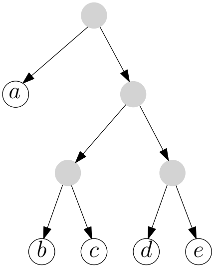
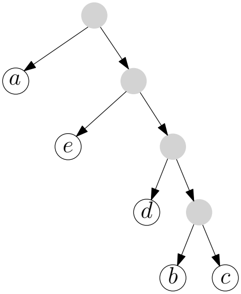


scheme 2



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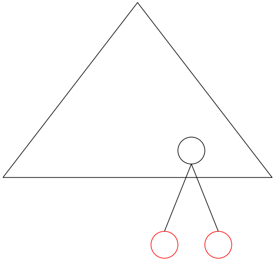
- Not clear how to design the greedy algorithm

A: We can choose two letters and make them brothers in the tree.

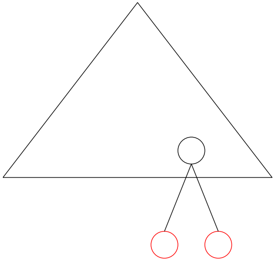
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the "structure" of the optimum encoding tree



Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the "structure" of the optimum encoding tree
- There are two deepest leaves that are brothers



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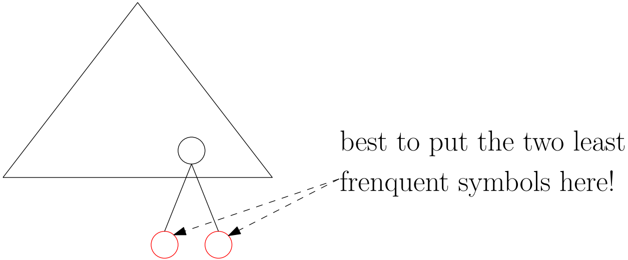

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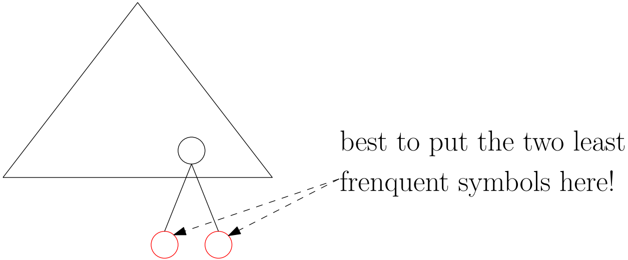

Lemma It is safe to make the two least frequent letters brothers.

Lemma There is an optimum encoding tree, where the two least frequent letters are brothers.

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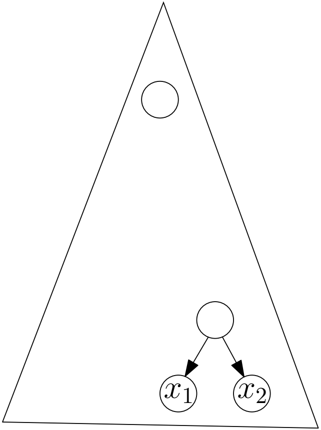
Lemma There is an optimum encoding tree, where the two least frequent letters are brothers.

- So we can irrevocably decide to make the two least frequent letters brothers.

Q: Is the residual problem another instance of the best prefix codes problem?

A: Yes, though it is not immediate to see why.

- f_x: the frequency of the letter x in the support.
- x_1 and x_2: the two letters we decided to put together.
- d_x the depth of letter x in our output encoding tree.



$$
\begin{aligned}
&\sum_{x \in S} f_x d_x \\
= &\sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
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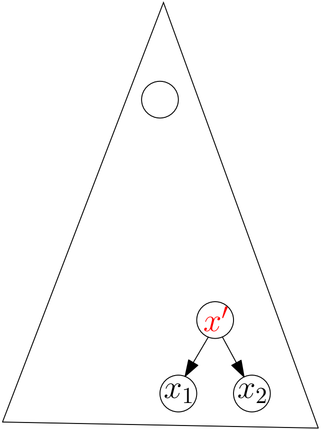

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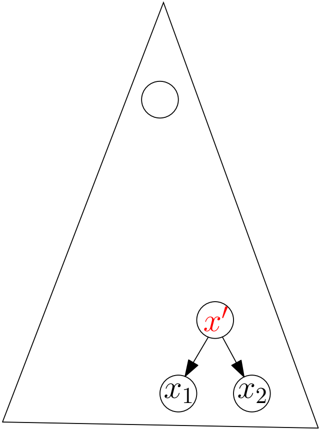

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Def: $f_{x'} = f_{x_1} + f_{x_2}$

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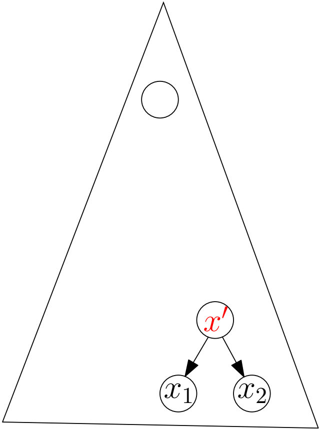

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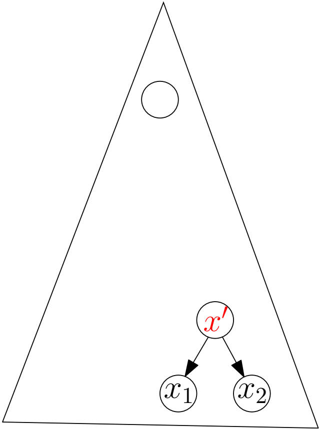

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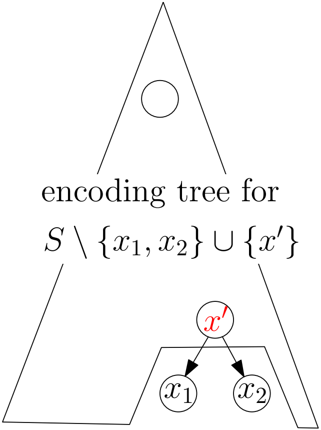

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\end{aligned}$$

In order to minimize

$$\sum_{x \in S} f_x d_x,$$

we need to minimize

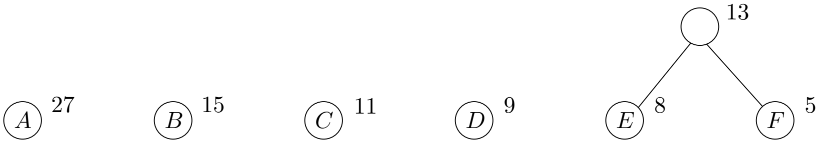
$$\sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x,$$

subject to that d is the depth function for an encoding tree of $S \setminus \{x_1, x_2\}$.

- This is exactly the best prefix codes problem, with letters $S \setminus \{x_1, x_2\} \cup \{x'\}$ and frequency vector f!

Example

Example



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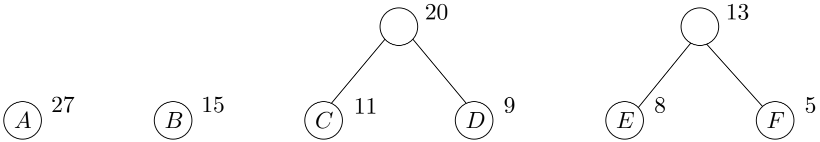

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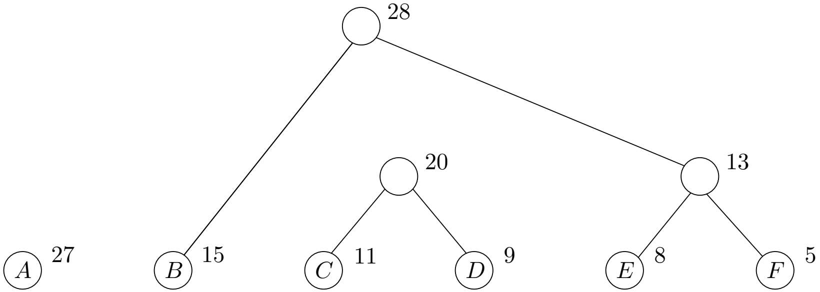
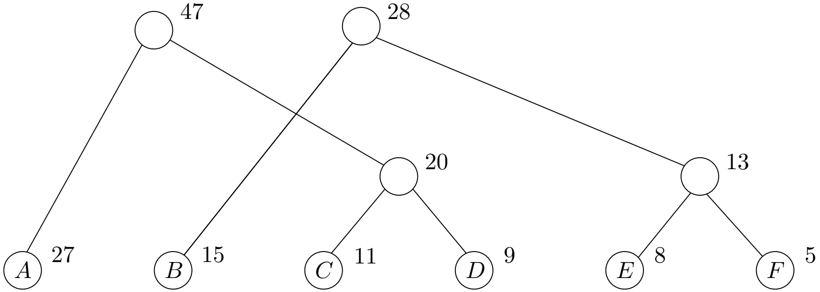


Example



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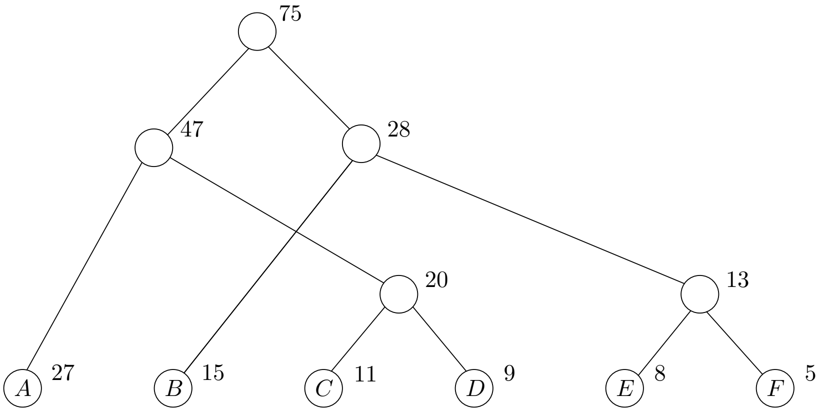

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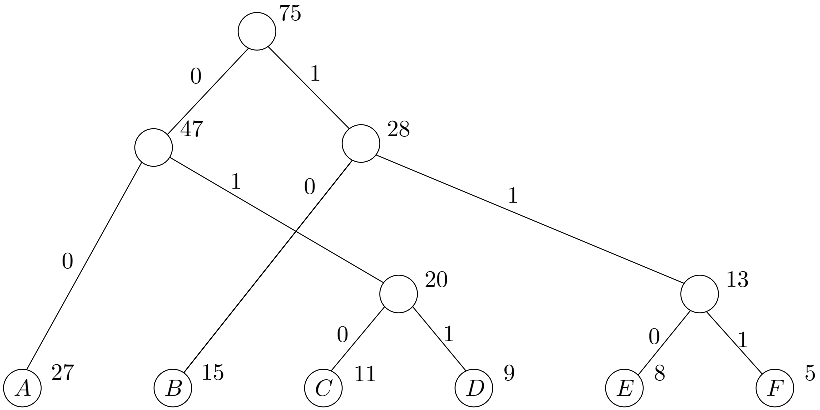

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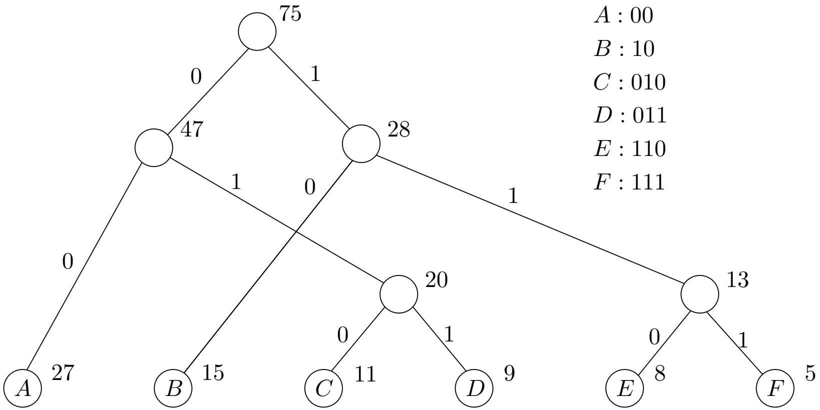

$A : 00$

$B : 10$

$C : 010$

$D : 011$

$E : 110$

$F : 111$

Def. The codes given the greedy algorithm is called the Huffman codes.

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Huffman(S, f)

1: **while** $|S| > 1$ **do**
2: let x_1, x_2 be the two letters with the smallest f values
3: introduce a new letter x' and let $f_{x'} = f_{x_1} + f_{x_2}$
4: let x_1 and x_2 be the two children of x'
5: $S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$
6: **return** the tree constructed

Algorithm using Priority Queue

Huffman(S, f)

```
Q ← build-priority-queue(S)
while Q.size > 1 do
    x_1 ← Q.extract-min()
    x_2 ← Q.extract-min()
    introduce a new letter x' and let f_{x'} = f_{x_1} + f_{x_2}
    let x_1 and x_2 be the two children of x'
    Q.insert(x', f_{x'})
return the tree constructed
```

Outline

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- At each step, make an irrevocable decision using a "reasonable" strategy

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Def. A strategy is "safe" if there is always an optimum solution that "agrees with" the decision made according to the strategy.

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- Two problems that do not fall into the category: lateness, weighted completion time