Greedy Algorithms

授课老师: 栗师
南京大学计算机科学与技术系
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Enumerate all valid solutions, compare them and output the best one.
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- convention: polynomial time = efficient
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**Goals of algorithm design**
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1. Design efficient algorithms to solve problems
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Goals of algorithm design

1. Design efficient algorithms to solve problems
2. Design more efficient algorithms to solve problems
Common Paradigms for Algorithm Design

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming
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Greedy algorithms are often for optimization problems.
Greedy Algorithms
Divide and Conquer
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Greedy algorithms are often for optimization problems.
They often run in polynomial time due to their simplicity.
Greedy Algorithm

- Build up the solutions in steps
- At each step, make an *irrevocable* decision using a “reasonable” strategy
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A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is “safe”
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem
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- Prove that the reasonable strategy is “safe” *(key)*
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem *(usually easy)*
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- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem **(usually easy)**

**Def.** A strategy is safe: there is always an optimum solution that agrees with the decision made according to the strategy.
1. Toy Example: Box Packing
2. Interval Scheduling
3. Scheduling to minimize lateness
4. Weighted Completion Time Scheduling
5. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
6. Data Compression and Huffman Code
7. Summary
Box Packing

**Input:** \( n \) boxes of capacities \( c_1, c_2, \cdots, c_n \)
\( m \) items of sizes \( s_1, s_2, \cdots, s_m \)
Can put at most 1 item in a box
Item \( j \) can be put into box \( i \) if \( s_j \leq c_i \)

**Output:** A way to put as many items as possible in the boxes.
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Can put at most 1 item in a box
Item \( j \) can be put into box \( i \) if \( s_j \leq c_i \)

**Output:** A way to put as many items as possible in the boxes.

**Example:**
- Box capacities: 60, 40, 25, 15, 12
- Item sizes: 45, 42, 20, 19, 16
- Can put 3 items in boxes: 45 \( \rightarrow \) 60, 20 \( \rightarrow \) 40, 19 \( \rightarrow \) 25
Greedy Algorithm

- Build up the solutions in steps
- At each step, make an **irrevocable** decision using a “reasonable” strategy

Q: Take box 1. Which item should we put in box 1?
A: The item of the largest size that can be put into the box.
Greedy Algorithm

- Build up the solutions in steps
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Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?
Greedy Algorithm

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- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

**Lemma** The strategy that put into box 1 the largest item it can hold is "safe": There is an optimum solution in which box 1 contains the largest item it can hold.
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Lemma  The strategy that put into box 1 the largest item it can hold is “safe”: There is an optimum solution in which box 1 contains the largest item it can hold.

- Intuition: putting the item gives us the easiest residual problem.
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Lemma
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formal proof via exchanging argument:
**Lemma**  There is an optimum solution in which box 1 contains the largest item it can hold.
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Proof.

- Let $j =$ largest item that box 1 can hold.
**Lemma**  There is an optimum solution in which box 1 contains the largest item it can hold.

**Proof.**

- Let $j =$ largest item that box 1 can hold.
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.
Lemma There is an optimum solution in which box 1 contains the largest item it can hold.

Proof.

- Let $j =$ largest item that box 1 can hold.
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.
- Otherwise, assume this is what happens in $S$:

$$S: \quad \text{box 1} \quad \text{item } j \quad \text{......}$$
**Lemma** There is an optimum solution in which box 1 contains the largest item it can hold.

**Proof.**

- Let $j = \text{largest item that box 1 can hold}.$
- Take any optimum solution $S.$ If $j$ is put into Box 1 in $S$, done.
- Otherwise, assume this is what happens in $S'$:

![Diagram](#)

- $s_{j'} \leq s_j$, and swapping gives another solution $S'$. 

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![Diagram](image)

- $s_{j'} \leq s_j$, and swapping gives another solution $S'$
- $S'$ is also an optimum solution. In $S'$, $j$ is put into Box 1. 

□
Notice that the exchanging operation is only for the sake of analysis; it is not a part of the algorithm.
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- Trivial: we decided to put Item $j$ into Box 1, and the remaining instance is obtained by removing Item $j$ and Box 1.
Generic Greedy Algorithm

1. while the instance is non-trivial do
2. make the choice using the greedy strategy
3. reduce the instance

Greedy Algorithm for Box Packing

1. \( T \leftarrow \{1, 2, 3, \ldots, m\} \)
2. for \( i \leftarrow 1 \) to \( n \) do
3. if some item in \( T \) can be put into box \( i \) then
4. \( j \leftarrow \) the largest item in \( T \) that can be put into box \( i \)
5. print(“put item \( j \) in box \( i \)”)
6. \( T \leftarrow T \setminus \{j\} \)
Exchange argument: Proof of Safety of a Strategy

- let $S$ be an arbitrary optimum solution.
- if $S$ is consistent with the greedy choice, done.
- otherwise, show that it can be modified to another optimum solution $S'$ that is consistent with the choice.
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- The procedure is not a part of the algorithm.
Outline

1 Toy Example: Box Packing
2 Interval Scheduling
3 Scheduling to minimize lateness
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Interval Scheduling

**Input:** $n$ jobs, job $i$ with start time $s_i$ and finish time $f_i$

$i$ and $j$ are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

**Output:** A maximum-size subset of mutually compatible jobs
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Greedy Algorithm for Interval Scheduling

- Which of the following strategies are safe?
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Schedule the job with the smallest size?
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- Schedule the job with the smallest size? **No!**
Greedy Algorithm for Interval Scheduling

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Greedy Algorithm for Interval Scheduling

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- Schedule the job with the earliest finish time?
Greedy Algorithm for Interval Scheduling

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![Diagram of intervals on a timeline]

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- Which of the following strategies are safe?
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Lemma  It is safe to schedule the job $j$ with the earliest finish time: There is an optimum solution where the job $j$ with the earliest finish time is scheduled.

Proof.
**Lemma** It is safe to schedule the job $j$ with the earliest finish time: There is an optimum solution where the job $j$ with the earliest finish time is scheduled.

**Proof.**

- Take an arbitrary optimum solution $S$

$$S: \quad \text{[Diagram]}$$
Greedy Algorithm for Interval Scheduling

**Lemma** It is safe to schedule the job $j$ with the earliest finish time: There is an optimum solution where the job $j$ with the earliest finish time is scheduled.

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Greedy Algorithm for Interval Scheduling

**Lemma** It is safe to schedule the job $j$ with the earliest finish time: There is an optimum solution where the job $j$ with the earliest finish time is scheduled.

- What is the remaining task after we decided to schedule $j$?
- Is it another instance of interval scheduling problem?
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Greedy Algorithm for Interval Scheduling

**Schedule**($s$, $f$, $n$)

1: $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
2: **while** $A \neq \emptyset$ **do**
3: \hspace{1em} $j \leftarrow \text{arg min}_{j' \in A} f_{j'}$
4: \hspace{1em} $S \leftarrow S \cup \{j\}; A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
5: **return** $S$

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Greedy Algorithm for Interval Scheduling

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\text{Schedule}(s, f, n)
\]

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Running time of algorithm?
Greedy Algorithm for Interval Scheduling

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Running time of algorithm?

- Naive implementation: $O(n^2)$ time
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1: $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
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5: **return** $S$

Running time of algorithm?

- **Naive implementation:** $O(n^2)$ time
- **Clever implementation:** $O(n \lg n)$ time
Clever Implementation of Greedy Algorithm

**Schedule**\((s, f, n)\)

1: sort jobs according to \(f\) values
2: \(t \leftarrow 0, S \leftarrow \emptyset\)
3: **for** every \(j \in [n]\) according to non-decreasing order of \(f_j\) **do**
4: \(\text{if } s_j \geq t \text{ then}\)
5: \(S \leftarrow S \cup \{j\}\)
6: \(t \leftarrow f_j\)
7: **return** \(S\)
Clever Implementation of Greedy Algorithm

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4:   if \(s_j \geq t\) then
5:     \(S \leftarrow S \cup \{j\}\)
6:     \(t \leftarrow f_j\)
7: return \(S\)
Clever Implementation of Greedy Algorithm

**Schedule**$(s, f, n)$

1. sort jobs according to $f$ values
2. $t ← 0$, $S ← ∅$
3. for every $j ∈ [n]$ according to non-decreasing order of $f_j$ do
4. if $s_j ≥ t$ then
5. $S ← S ∪ \{j\}$
6. $t ← f_j$
7. return $S$

![Diagram of Schedule algorithm](image)
Clever Implementation of Greedy Algorithm

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2: $t \leftarrow 0$, $S \leftarrow \emptyset$
3: for every $j \in [n]$ according to non-decreasing order of $f_j$ do
4:  if $s_j \geq t$ then
5:  $S \leftarrow S \cup \{j\}$
6:  $t \leftarrow f_j$
7: return $S$
Clever Implementation of Greedy Algorithm

Schedule\((s, f, n)\)

1: sort jobs according to \(f\) values
2: \(t \leftarrow 0, S \leftarrow \emptyset\)
3: for every \(j \in [n]\) according to non-decreasing order of \(f_j\) do
4: \hspace{1em} if \(s_j \geq t\) then
5: \hspace{2em} \(S \leftarrow S \cup \{j\}\)
6: \hspace{2em} \(t \leftarrow f_j\)
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Clever Implementation of Greedy Algorithm

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Clever Implementation of Greedy Algorithm

Schedule($s, f, n$)

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3: for every $j \in [n]$ according to non-decreasing order of $f_j$ do
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6: \hspace{2em} $t \leftarrow f_j$
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Clever Implementation of Greedy Algorithm

\textbf{Schedule}(s, f, n)

1: sort jobs according to \( f \) values
2: \( t \leftarrow 0, S \leftarrow \emptyset \)
3: \textbf{for} every \( j \in [n] \) according to non-decreasing order of \( f_j \) \textbf{do}
4: \hspace{1em} \textbf{if} \( s_j \geq t \) \textbf{then}
5: \hspace{2em} \( S \leftarrow S \cup \{j\} \)
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**Schedule**\((s, f, n)\)

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2: \(t \leftarrow 0, S \leftarrow \emptyset\)
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4: \(\textbf{if} \ s_j \geq t \textbf{ then}\)
5: \(S \leftarrow S \cup \{j\}\)
6: \(t \leftarrow f_j\)
7: **return** \(S\)
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1. Toy Example: Box Packing
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Scheduling to minimize lateness

**Input:** \( n \) jobs, each job \( j \in [n] \) with a processing time \( p_j \) and deadline \( d_j \)

**Output:** schedule jobs on 1 machine, to minimize the max. lateness

\[ C_j : \text{completion time of } j \quad \text{lateness} \quad l_j := \max\{C_j - d_j, 0\} \]

**Example input:**

<table>
<thead>
<tr>
<th>( j )</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_j )</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( d_j )</td>
<td>5</td>
<td>7</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

**Solution 1**

\[ C_j : \text{completion time of } j \quad \text{lateness} \quad l_j := \max\{C_j - d_j, 0\} \]

| a | b | c | d |

**Solution 2**

| c | a | b | d |

**Solution 1:** max lateness = \( \max\{0, 3 - 5, 6 - 7, 8 - 4, 9 - 8\} = 4 \)

**Solution 2:** max lateness = \( \max\{0, 2 - 4, 5 - 5, 8 - 7, 9 - 8\} = 1 \)

**Solution 2** is better
Candidate algorithms

Schedule the jobs in some natural order. Which order should we choose?

A. Ascending order of processing times $p_j$
B. Ascending order of slackness $d_j - p_j$
C. Ascending order of deadline $d_j$. 

Lemma

The ascending order of deadlines $d_j$ (the Earliest Deadline First order or the EDF order) is the optimum schedule.
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Lemma The ascending order of deadlines $d_j$ (the Earliest Deadline First order or the EDF order) is the optimum schedule.
maximum lateness = \( \max \left\{ 0, \max_{j \in [n]} \{ C_j - d_j \} \right\} \).
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\[
d_j > d_{j'}
\]

before: 
\[
\max \left\{ t + p_j - d_j, t + p_{j'} - d_{j'} \right\}
\]

after: 
\[
\max \left\{ t + p_{j'} - d_{j'}, t + p_{j} - d_{j} \right\}
\]

before swapping, the maximum of the two terms strictly decreases.
**maximum lateness** = \( \max \left\{ 0, \max_{j \in [n]} \{ C_j - d_j \} \right\}. \)

\[ d_j > d_{j'} \]

\[ \begin{array}{c}
  \cdot \cdot \cdot \cdot \cdot \cdot \\
  t \\
  \cdot \cdot \cdot \cdot \cdot \cdot \\
  j \\
  \cdot \cdot \cdot \cdot \cdot \cdot \\
  j' \\
  \cdot \cdot \cdot \cdot \cdot \cdot \\
  j' \\
  \cdot \cdot \cdot \cdot \cdot \cdot \\
  j \\
\end{array} \]

**before:** \( \max\{t + p_j - d_j, t + p_j + p_{j'} - d_{j'}\} = t + p_j + p_{j'} - d_{j'} \)
• maximum lateness = \( \max \left\{ 0, \max_{j \in [n]} \{ C_j - d_j \} \right\} \).

\[ d_j > d_j' \]

\[
\begin{array}{cccccccc}
\text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} \\
& & j & | & j' & | & \text{\ldots} & \text{\ldots} \\
& & j' & | & j & | & \text{\ldots} & \text{\ldots} \\
\end{array}
\]

• before: \( \max\{t + p_j - d_j, t + p_j + p_j' - d_j'\} = t + p_j + p_j' - d_j' \)

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\( p_{j'} - d_{j'} < p_j + p_{j'} - d_{j'} \) and \( p_j + p_{j'} - d_j < p_j + p_{j'} - d_{j'} \)
• maximum lateness = \( \max \left\{ 0, \max_{j \in [n]} \{ C_j - d_j \} \right\} \).

\[
d_j > d_{j'}
\]

\[
\begin{array}{c}
\cdots \cdots \cdots \cdots \cdots \\
t \\
\cdots \cdots \cdots \cdots \cdots \\
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
j \\
j'
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j \\
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\end{array}
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• \( p_{j'} - d_{j'} < p_j + p_{j'} - d_{j'} \text{ and } p_j + p_{j'} - d_j < p_j + p_{j'} - d_{j'} \)
• \( \max\{t + p_{j'} - d_{j'}, t + p_j + p_{j'} - d_j\} < t + p_j + p_{j'} - d_{j'} \)
- maximum lateness = max \{0, \max_{j \in [n]} \{C_j - d_j\}\}.

- before: max\{t + p_j - d_j, t + p_j + p_{j'} - d_{j'}\} = t + p_j + p_{j'} - d_{j'}
- after: max\{t + p_{j'} - d_{j'}, t + p_j + p_{j'} - d_j\}
- \(p_{j'} - d_{j'} < p_j + p_{j'} - d_{j'}\) and \(p_j + p_{j'} - d_j < p_j + p_{j'} - d_{j'}\)
- max\{t + p_{j'} - d_{j'}, t + p_j + p_{j'} - d_j\} < t + p_j + p_{j'} - d_{j'}
- after swapping, the maximum of the two terms strictly decreases
Repeated Swapping (for Analysis Only)

1: let $S$ be any schedule (i.e, a permutation of $[n]$)
2: while there are two adjacent jobs $j$ and $j'$ in $S$, with $j$ before $j'$ and $d_j > d_{j'}$ do
3: swap $j$ and $j'$ in $S$
Repeated Swapping (for Analysis Only)

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Q: Does the algorithm terminate?
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Q: Does the algorithm terminate?

A: Yes. Number of inversions go down!

- $(j, j')$ is an inversion in $S$ if $j$ appears before $j'$ and $d_j > d_{j'}$. 
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- So the algorithm converges to an EDF order.
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Q: What if there are multiple EDF orders, i.e., some jobs have the same deadline?
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- So the algorithm converges to an EDF order.

Q: What if there are multiple EDF orders, i.e., some jobs have the same deadline?

A: All EDF orders have the same maximum lateness.
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Scheduling to Minimize Weighted Completion Time

**Input:** A set of \( n \) jobs \([n] := \{1, 2, 3, \cdots, n\}\)

each job \( j \) has a weight \( w_j \) and processing time \( p_j \)

**Output:** an ordering of jobs so as to minimize the total weighted completion time of jobs

\[
\begin{align*}
  p_a &= 1 & w_1 &= 2 \\
  p_b &= 2 & w_2 &= 5 \\
  p_c &= 3 & w_3 &= 7 \\
\end{align*}
\]

\[
\begin{array}{cccc}
  a & b & c \\
  w_1 &= 2 & w_2 &= 5 & w_3 &= 7 \\
\end{array}
\]

\[
\begin{array}{cccc}
  w_a &= 2 & w_b &= 5 & w_c &= 7 \\
  a & b & c \\
  0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

\[
\begin{array}{cccc}
  w_b &= 5 & w_c &= 7 & w_a &= 2 \\
  b & c & a \\
  0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

\[
\begin{align*}
  \text{cost} &= 2 \times 1 + 5 \times 3 + 7 \times 6 = 59 \\
  \text{cost} &= 5 \times 2 + 7 \times 5 + 2 \times 6 = 57
\end{align*}
\]
Candidate algorithms

Schedule the jobs in some natural order. Which order should we choose?

A. Ascending order of processing times $p_j$
B. Descending order of slackness $w_j$
C. Ascending order of $p_j - w_j$
D. Ascending order of $p_j / w_j$
Candidate algorithms

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C. Ascending order of $p_j - w_j$
D. Ascending order of $p_j/w_j$

Def. The Smith ratio of a job is $w_j/p_j$.

Lemma The descending order of Smith ratios (the Smith rule) is optimum.
A schedule $S$, $j$ is right before $j'$. 

\[ w_{j'} w_j = \Rightarrow w_{j} w_{j'} \]

Therefore, swapping decrease the weighted completion time if $p_{j'} w_{j'} < p_j w_j$.

Using the same argument as for the maximum lateness problem: ascending order of $p_j / w_j$ is optimum.

Indeed, optimum weighted completion time is

\[ X_{j \in [n]} w_j p_j + X_{1 \leq j < j' \leq n} \min\{w_j p_{j'}, w_{j'} p_j\} \]
A schedule $S$, $j$ is right before $j'$.

**Q:** How does the total weighted completion time change if we swap $j$ and $j'$?

$(\cdots, j, j', \cdots) \implies (\cdots, j', j, \cdots)$
A schedule $S$, $j$ is right before $j'$.

Q: How does the total weighted completion time change if we swap $j$ and $j'$? $(\cdots, j, j', \cdots) \implies (\cdots, j', j, \cdots)$

A: $w_{j'} p_j \implies w_j p_{j'}$

Therefore, swapping decreases the total weighted completion time if $p_{j'} w_{j'} < p_j w_j$.

Using the same argument as for the maximum lateness problem: ascending order of $p_j / w_j$ is optimum.

Indeed, the optimum weighted completion time is:

$$X_{j \in [n]} w_j p_j + \sum_{j < j'} \min\{w_j p_{j'}, w_{j'} p_j\}$$
A schedule $S$, $j$ is right before $j'$.

**Q:** How does the total weighted completion time change if we swap $j$ and $j'$?

$(\cdots, j, j', \cdots) \Rightarrow (\cdots, j', j, \cdots)$

**A:** $w_jp_j \Rightarrow w_j'p_j'$

Therefore, swapping decrease the weighted completion time if

$$\frac{p_j}{w_j} < \frac{p_j'}{w_j'}.$$
A schedule $S$, $j$ is right before $j'$.  

**Q:** How does the total weighted completion time change if we swap $j$ and $j'$?  

$$(\cdots, j, j', \cdots) \implies (\cdots, j', j, \cdots)$$

**A:** $w_{j'}p_j \implies w_jp_{j'}$

Therefore, swapping decrease the weighted completion time if $\frac{p_{j'}}{w_{j'}} < \frac{p_j}{w_j}$.

Using the same argument as for the maximum lateness problem: ascending order of $p_j/w_j$ is optimum.
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Using the same argument as for the maximum lateness problem: ascending order of $p_j/w_j$ is optimum.

Indeed, optimum weighted completion time is

$$\sum_{j \in [n]} w_j p_j + \sum_{1 \leq j < j' \leq n} \min\{w_j p_{j'}, w_{j'} p_j\}.$$
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Offline Caching

- Cache that can store \( k \) pages
- Sequence of page requests

Cache miss happens if requested page not in cache. We need to bring the page into cache, and evict some existing page if necessary. Cache hit happens if requested page already in cache. Goal: minimize the number of cache misses.
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests

```
page sequence
1
5
4
2
5
3
2
1
```

```
cache

```
**Offline Caching**

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<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>✗</td>
</tr>
<tr>
<td>5</td>
<td>✗</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3</td>
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Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests
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Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.

```
page sequence:
1 5 4 2 5 3 2 1

cache:
- 1
- 1 5
- 1 2 4
- 1 2 5
```
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
Offline Caching

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<table>
<thead>
<tr>
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<th>cache</th>
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<tbody>
<tr>
<td>1</td>
<td>✗ 1</td>
</tr>
<tr>
<td>5</td>
<td>✗ 1</td>
</tr>
<tr>
<td>4</td>
<td>✗ 1</td>
</tr>
<tr>
<td>2</td>
<td>✗ 1, 5</td>
</tr>
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<td>✗ 1</td>
</tr>
<tr>
<td>3</td>
<td>✗ 1</td>
</tr>
<tr>
<td>2</td>
<td>✗ 1</td>
</tr>
<tr>
<td>1</td>
<td>✔</td>
</tr>
</tbody>
</table>
Offline Caching

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<tbody>
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<tr>
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</tr>
<tr>
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<tr>
<td>2</td>
<td>✔ 1 2 3</td>
</tr>
<tr>
<td>1</td>
<td>✔ 1 2 3</td>
</tr>
</tbody>
</table>
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<table>
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<td>✔ 1 2 3</td>
</tr>
<tr>
<td>1</td>
<td>✔ 1 2 3</td>
</tr>
</tbody>
</table>

misses = 6
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
- Cache hit happens if requested page already in cache.
- Goal: minimize the number of cache misses.

<table>
<thead>
<tr>
<th>Page Sequence</th>
<th>Cache</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>✗ 1</td>
</tr>
<tr>
<td>5</td>
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<td>✓ 1, 2, 3</td>
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<tr>
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misses = 6
A Better Solution for Example

<table>
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<th>Cache</th>
<th>Cache</th>
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<tr>
<td>2</td>
<td>✔ 1 2 3</td>
<td>✔ 1 3 2</td>
</tr>
<tr>
<td>1</td>
<td>✔ 1 2 3</td>
<td>✔ 1 3 2</td>
</tr>
</tbody>
</table>

misses = 6
misses = 5
Offline Caching Problem

**Input:**
- $k$: the size of cache
- $n$: number of pages
- $\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

**Output:**
- $i_1, i_2, i_3, \cdots, i_T \in \{\text{hit}, \text{empty}\} \cup [n]$: indices of pages to evict ("hit" means evicting no page, "empty" means evicting empty page)

We use $[n]$ for $\{1, 2, 3, \cdots, n\}$. 

Offline Caching: we know the whole sequence ahead of time.

Online Caching: we have to make decisions on the fly, before seeing future requests.

Q: Which one is more realistic?

A: Online caching
### Offline Caching Problem

**Input:**  
- \( k \): the size of cache  
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Q: Which one is more realistic?
A: Online caching

Q: Why do we study the offline caching problem?
- Offline Caching: we know the whole sequence ahead of time.
- Online Caching: we have to make decisions on the fly, before seeing future requests.

**Q:** Which one is more realistic?

**A:** Online caching

**Q:** Why do we study the offline caching problem?

**A:** Use the offline solution as a benchmark to measure the “competitive ratio” of online algorithms
Offline Caching: Potential Greedy Algorithms

- FIFO (First-In-First-Out): always evict the first page in cache

- LRU (Least-Recently-Used): evict page whose most recent access was earliest

- LFU (Least-Frequently-Used): evict page that was least frequently requested

All the above algorithms are not optimum! Indeed all the algorithms are "online", i.e., the decisions can be made without knowing future requests. Online algorithms cannot be optimum.
Offline Caching: Potential Greedy Algorithms

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- All the above algorithms are not optimum!
  - Indeed all the algorithms are “online”, i.e., the decisions can be made without knowing future requests. Online algorithms can not be optimum.
FIFO is not optimum

requests

FIFO

1
2
3
4
1
2
3
4
1
FIFO is not optimum
FIFO is not optimum

requests

1
2
3
4

FIFO

1

38/89
FIFO is not optimum

requests

1
2
3
4
1
2
3
4
1
FIFO

× 1

×
FIFO is not optimum
FIFO is not optimum

requests

1
2
3
4

FIFO

1
2
3
4
1
FIFO is not optimum

requests

1
2
3
4
1
2
3
4
1
2
3

FIFO

\[ \begin{array}{ccc}
  & & \\
  & & \\
  & & \\
 \end{array} \]

\[ \begin{array}{ccc}
  \times & 1 & \\
  & 1 & \\
  & 1 & 2 \\
  & 1 & 2 & 3 \\
 \end{array} \]
FIFO is not optimum

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FIFO is not optimum

<table>
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FIFO

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<td></td>
</tr>
<tr>
<td>4</td>
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</tr>
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</table>
FIFO is not optimum

requests

FIFO

1
2
3
4
1
2
3
4
1
2
3
4
1
2
3
4
1
2
3
4
1
2
3
4
1
2
3
4
1
2
3
4
FIFO is not optimum

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</tr>
<tr>
<td>1</td>
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</tr>
</tbody>
</table>

This diagram illustrates the FIFO (First-In, First-Out) method for handling requests. Each request is added to the queue and served in the order they arrive. The diagram shows how requests are processed over time, with each row representing a new request and the boxes indicating the status of each request in the queue.
FIFO is not optimum

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
<th>misses = 5</th>
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<tbody>
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</tr>
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</table>
FIFO is not optimum

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
<th>Furthest-in-Future</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>✗ 1</td>
<td>✗ 1</td>
</tr>
<tr>
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</table>

misses = 5

misses = 4
Optimum Offline Caching

Furthest-in-Future (FF)

- Algorithm: every time, evict the page that is not requested until furthest in the future, if we need to evict one.
- The algorithm is not an online algorithm, since the decision at a step depends on the request sequence in the future.
Furthest-in-Future (FF)

<table>
<thead>
<tr>
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<th>FIFO</th>
<th>Furthest-in-Future</th>
</tr>
</thead>
<tbody>
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<td>✗ 1</td>
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<tr>
<td>2</td>
<td>✗ 1 2</td>
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<tr>
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<td>✗ 1 2 3</td>
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<td>4</td>
<td>✗ 4 2 3</td>
<td>✗ 1 4 3</td>
</tr>
<tr>
<td>1</td>
<td>✗ 4 1 3</td>
<td>✓ 1 4 3</td>
</tr>
</tbody>
</table>

misses = 5

misses = 4
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

1 5 1 5 4 3
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3
Example

requests

\[
\begin{array}{ccccccc}
1 & 5 & 4 & 2 & 5 & 3 & 2 \\
\hline
x & x & x & x & x & & \\
1 & 1 & 1 & 1 & 2 & & \\
5 & 5 & 5 & 5 & & & \\
4 & 4 & & & & &
\end{array}
\]
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

x x x x

1 1 1 2

5 5 5

4 4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

×  ×  ×  ×  ✓

☐  1  1  1  2  2

☐  ☐  5  5  5  5

☐  ☐  ☐  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

\(\times\) \(\times\) \(\times\) \(\times\) \(\checkmark\)

\(\square\) \(1\) \(1\) \(1\) \(2\) \(2\)

\(\square\) \(\square\) \(5\) \(5\) \(5\) \(5\)

\(\square\) \(\square\) \(\square\) \(4\) \(4\) \(4\)
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

x x x x ✔️ x

☐ 1 1 1 2 2 2
☐ ☐ 5 5 5 5 3
☐ ☐ ☐ 4 4 4 4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

1  2  3  4  5  

1  1  1  2  2  2

5  5  5  5  3

4  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

X  X  X  X  ✓  X  ✓  ✓

☐  1  1  1  2  2  2  2

☐  ☐  5  5  5  5  3  3

☐  ☐  ☐  4  4  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

X  X  X  X  ✓  X  ✓  ✓  ✓

1  1  1  2  2  2  2  2

5  5  5  5  3  3  3

4  4  4  4  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

× × × × ✔ × ✔ ✔ ✔

1  1  1  2  2  2  2  2  2

5  5  5  5  5  3  3  3  3

4  4  4  4  4  4  4  4  4
Example

requests

- 1  5  4  2  5  3  2  4  3  1  5  3
- X X X X ✓ X ✓ ✓ ✓
- 1  1  1  2  2  2  2  2  2
- 5  5  5  5  3  3  3  3
- 4  4  4  4  4  4  4
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

- - - - - - - - - - - -

x x x x v x x x x

- - - - - - - - - - - -

1 1 1 2 2 2 2 2 2 1

- - - - - - - - - - - -

5 5 5 5 3 3 3 3 3

- - - - - - - - - - - -

4 4 4 4 4 4 4 4 4
Example

requests

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□ □ □ 5 5 5 5 5 3 3 3 3 3 3

□ □ □ □ 4 4 4 4 4 4 4 4 4 4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3  

X  X  X  X  ✓  X  ✓  ✓  ✓  X  X  ✓  

☐  1  1  1  2  2  2  2  2  2  1  5  5  

☐  ☐  5  5  5  5  3  3  3  3  3  3  3  

☐  ☐  ☐  4  4  4  4  4  4  4  4  4  4  4
Recall: Designing and Analyzing Greedy Algorithms

**Greedy Algorithm**
- Build up the solutions in steps
- At each step, make an **irrevocable** decision using a “reasonable” strategy

**A Common Way to Analyze Greedy Algorithms**
- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
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Offline Caching Problem

**Input:**  
\( k \) : the size of cache  
\( n \) : number of pages  
\( \rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n] \): sequence of requests

**Output:**  
\( i_1, i_2, i_3, \cdots, i_t \in \{ \text{hit, empty} \} \cup [n] \)  
- empty stands for an empty page  
- “hit” means evicting no pages
Offline Caching Problem

**Input:**
- $k$: the size of cache
- $n$: number of pages
- $\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests
- $p_1, p_2, \cdots, p_k \in \{\text{empty}\} \cup [n]$: initial set of pages in cache

**Output:**
- $i_1, i_2, i_3, \cdots, i_t \in \{\text{hit, empty}\} \cup [n]$
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  - “hit” means evicting no pages
A Common Way to Analyze Greedy Algorithms

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A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. **It is safe to evict $p^*$ at time 1.**
A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

**Lemma**  Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. There is an optimum solution in which $p^*$ is evicted at time 1.
Proof.
1. $S$: any optimum solution
2. $p^*$: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$. 

<table>
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<td>$S$:</td>
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1. $S$: any optimum solution
2. $p^*$: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$.
3. Assume $S$ evicts some $p' \neq p^*$ at time 1; otherwise done.
   - In the example, $p' = 2$. 
Proof.

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   - In the example, \( p^* = 3 \).
3. Assume \( S \) evicts some \( p' \neq p^* \) at time 1; otherwise done.
   - In the example, \( p' = 2 \).
Proof.

Create $S'$.

$S'$ evicts $p^*$ (=3) instead of $p'$ (=2) at time 1.

After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p'$ (=2) and $S$ contains $p^*$ (=3).

From now on, $S'$ will "copy" $S$.
Proof.

Create $S'$. $S'$ evicts $p^* (=3)$ instead of $p' (=2)$ at time 1.
**Proof.**

Create $S'$. $S'$ evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.
Proof.


5. After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p'(=2)$ and $S$ contains $p^*(=3)$. 
Proof.

4 Create $S'$. $S'$ evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.

5 After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p'(=2)$ and $S$ contains $p^*(=3)$.

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\[\begin{array}{|c|c|}
\hline
4 & 5 \\
\hline
1 & \times \\
\hline
2 & \times \\
\hline
3 & \times \\
\hline
\end{array}\]

\[\begin{array}{|c|c|}
\hline
2 & 3 \\
\hline
1 & \times \\
\hline
4 & \times \\
\hline
3 & \times \\
\hline
\end{array}\]

\[\begin{array}{|c|c|}
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\hline
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\hline
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\hline
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6. From now on, $S'$ will “copy” $S$. 
Proof.

If $S$ evicted the page $p^*$, $S'$ will evict the page $p'$, then the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.

Assume $S$ did not evict $p^*$ (=3) before we see $p'$ (=2).
Proof.

If $S$ evicted the page $p^*$, $S'$ will evict the page $p'$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.
Proof.

7 If \( S \) evicted the page \( p^* \), \( S' \) will evict the page \( p' \). Then, the cache status of \( S \) and that of \( S' \) will be the same. \( S \) and \( S' \) will be exactly the same from now on.

8 Assume \( S \) did not evict \( p^*(=3) \) before we see \( p'(=2) \).
Proof.

7 If $S$ evicted the page $p^*$, $S'$ will evict the page $p'$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.

8 Assume $S$ did not evict $p^*(=3)$ before we see $p'(=2)$.
Proof.

If \( S \) evicts \( p^* (=3) \) for \( p^' (=2) \), then \( S \) won't be optimum. Assume otherwise.

So far, \( S' \) has 1 less page-miss than \( S \) does.

The status of \( S' \) and that of \( S \) only differ by 1 page.
**Proof.**

If \( S \) evicts \( p^* = 3 \) for \( p' = 2 \), then \( S \) won't be optimum. Assume otherwise.

So far, \( S' \) has 1 less page-miss than \( S \) does.

The status of \( S' \) and that of \( S \) only differ by 1 page.
Proof.

If $S$ evicts $p^*$ (3) for $p'$ (2), then $S$ won't be optimum. Assume otherwise.

So far, $S'$ has 1 less page-miss than $S$ does.

The status of $S'$ and that of $S$ only differ by 1 page.
Proof.

9 If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won’t be optimum. Assume otherwise.
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---

$\begin{array}{ccccc}
1 & 1 & 5 & 5 & \cdots \\
2 & 4 & 4 & 4 & \cdots \\
3 & 3 & 3 & 3 & \cdots \\
\end{array}$

$\begin{array}{ccccc}
1 & 1 & 5 & 5 & \cdots \\
2 & 4 & 4 & 4 & \cdots \\
3 & 2 & 2 & 2 & \cdots \\
\end{array}$

$\begin{array}{ccccc}
4 & 5 & 4 & \cdots & 2 \\
\times & \times & \checkmark & & \times \\
1 & 1 & 5 & 5 & \cdots \\
2 & 4 & 4 & 4 & \cdots \\
3 & 3 & 3 & 3 & \cdots \\
\end{array}$

$\begin{array}{ccccc}
1 & 1 & 5 & 5 & \cdots \\
2 & 4 & 4 & 4 & \cdots \\
3 & 2 & 2 & 2 & \cdots \\
\checkmark & \checkmark & & & \\
\end{array}$

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3 & 3 & 3 & 3 & \cdots \\
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1 & 1 & 5 & 5 & \cdots \\
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\checkmark & \checkmark & & & \\
\end{array}$
Proof.

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10 So far, $S'$ has 1 less page-miss than $S$ does.
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10 So far, $S'$ has 1 less page-miss than $S$ does.

11 The status of $S'$ and that of $S$ only differ by 1 page.
Proof.

We can then guarantee that $S'$ make at most the same number of page-misses as $S$ does.

Idea: if $S$ has a page-hit and $S'$ has a page-miss, we use the opportunity to make the status of $S'$ the same as that of $S$.
Proof.

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□
Thus, we have shown how to create another solution $S'$ with the same number of page-misses as that of the optimum solution $S$. Thus, we proved

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. There is an optimum solution in which $p^*$ is evicted at time 1.
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**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. It is safe to evict $p^*$ at time 1.
Thus, we have shown how to create another solution $S'$ with the same number of page-misses as that of the optimum solution $S$. Thus, we proved

**Lemma**  Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. It is safe to evict $p^*$ at time 1.

**Theorem**  The furthest-in-future strategy is optimum.
1: \textbf{for } t \leftarrow 1 \textbf{ to } T \textbf{ do }
2: \qquad \textbf{if } \rho_t \textbf{ is in cache } \textbf{ then } \textbf{ do nothing }
3: \qquad \textbf{else if } \textbf{ there is an empty page in cache } \textbf{ then }
4: \qquad \quad \textbf{evict the empty page and load } \rho_t \textbf{ in cache }
5: \qquad \textbf{else }
6: \qquad \quad p^* \leftarrow \text{page in cache that is not used furthest in the future }
7: \qquad \quad \text{evict } p^* \text{ and load } \rho_t \text{ in cache }
Q: How can we make the algorithm as fast as possible?

A: The running time can be made to be $O(n + T \log k)$. For each page $p$, use a linked list (or an array with dynamic size) to store the time steps in which $p$ is requested. We can find the next time a page is requested easily. Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.
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- We can find the next time a page is requested easily.
- Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.
The diagram illustrates a simulation of time and pages for five processes, P1 through P5, over a period of 12 time units. Each process has a set of pages and associated priority values.

**Time and Pages**: The table shows the time on the x-axis and pages on the y-axis, with processes P1 through P5 listed.

**Priority Queue**: The priority queue contains pages and priority values for each process. The priority values are used to determine the order in which pages are served.

- **P1**: Pages 1, 10
- **P2**: Pages 4, 7
- **P3**: Pages 6, 9, 12
- **P4**: Pages 3, 8
- **P5**: Pages 2, 5, 11

The priority values are likely used to prioritize service, ensuring that pages with higher values are served before those with lower values.
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**Priority Queue**

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**Process Pages**

- **P1:** 1 10
- **P2:** 4 7
- **P3:** 6 9 12
- **P4:** 3 8
- **P5:** 2 5 11
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priority queue

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<tr>
<td>P5</td>
<td>5</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>----</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>10</td>
</tr>
<tr>
<td>P5</td>
<td>5</td>
</tr>
</tbody>
</table>

**Pages**

- P1: 10
- P2: 7
- P3: 12
- P4: 8
- P5: 11
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>10</td>
</tr>
<tr>
<td>P5</td>
<td>5</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

- **P1:** 1 10
- **P2:** 4 7
- **P3:** 6 9 12
- **P4:** 3 8
- **P5:** 2 5 11

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P5</td>
</tr>
<tr>
<td></td>
<td>P4</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**priority queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P5</td>
<td>5</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P5</td>
<td>11</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>
The diagram illustrates the behavior of a priority queue with pages and their priority values. The time and pages are laid out in a table with corresponding values.

- **Pages**: P1, P2, P3, P4, P5
- **Time**: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
- **Priority Queue**:
  - **Pages**: P2, P3, P4
  - **Priority Values**: 7, 9, 8

The diagram shows the pages and their priority values at different times, with a checkmark indicating a correct operation or state.
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

P1: 1 10

P2: 4 7

P3: 6 9 12

P4: 3 8

P5: 2 5 11

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>
The diagram illustrates a simulation of page replacement using the Priority Queue algorithm. The table on the right shows the priority queue with the following entries:

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>

The page states are:

- **P1**: [1, 10]
- **P2**: [4, 7]
- **P3**: [6, 9, 12]
- **P4**: [3, 8]
- **P5**: [2, 5, 11]
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

- **P1:** 1 10
- **P2:** 4 7
- **P3:** 6 9 12
- **P4:** 3 8
- **P5:** 2 5 11

### Priority Queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>$\infty$</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>
The diagram illustrates a priority queue and its corresponding pages over a period of time.

**Priority Queue:**

<table>
<thead>
<tr>
<th>Pages</th>
<th>Priority Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

**Pages Over Time:**

<table>
<thead>
<tr>
<th>Time</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>P1, P5, P4, P2, P5, P3, P2, P4, P3, P1, P5, P3</td>
</tr>
</tbody>
</table>

- **P1:** 1, 10
- **P2:** 4, 7
- **P3:** 6, 9, 12
- **P4:** 3, 8
- **P5:** 2, 5, 11
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

### Priority Queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>12</td>
</tr>
<tr>
<td>P3</td>
<td>∞</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

**Pages:**
- P1: 1 10
- P2: 4 7
- P3: 6 9 12
- P4: 3 8
- P5: 2 5 11
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

- **P1:** 1 10
- **P2:** 4 7
- **P3:** 6 9 12
- **P4:** 3 8
- **P5:** 2 5 11

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

This diagram illustrates the allocation of pages to processes over time, with a priority queue for managing page requests.
The diagram illustrates the operation of a priority queue with pages and their corresponding file numbers. The time is listed on the left, with pages marked from 0 to 12. The priority queue contains the following entries:

- **P1:** pages 1 and 10
- **P2:** pages 4 and 7
- **P3:** pages 6, 9, and 12
- **P4:** pages 3 and 8
- **P5:** pages 2, 5, and 11

The priority values are as follows:

- **P1:** ∞
- **P3:** 12
- **P4:** ∞
The table below illustrates the priority queue used in a specific scheduling algorithm. The table tracks the allocation of pages for each process over time, with processes labeled P1 to P5. The priority queue is updated whenever a page is needed or released, and the highest priority process is served first. The table includes the time at which each page is allocated and the process number. The priority queue is also shown with the current priorities for each process.

### Time Table

<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

### Priority Queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

- **P1:**
  - 1
  - 10

- **P2:**
  - 4
  - 7

- **P3:**
  - 6
  - 9
  - 12

- **P4:**
  - 3
  - 8

- **P5:**
  - 2
  - 5
  - 11

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>
The image shows a table representing the pages and time for different processes. The table has columns for time, pages, and a priority queue. The priority queue includes columns for pages and priority values.

- **Processes:** P1, P2, P3, P4, P5
- **Pages:** P1, P2, P3, P4, P5
- **Time:** 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
- **Priority Queue:**
  - P5: ∞
  - P3: 12
  - P4: ∞

The table illustrates the priority queue for different processes at various times.
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

### Priority Queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P5</td>
<td>$\infty$</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>$\infty$</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

P1:  

<table>
<thead>
<tr>
<th>time</th>
<th>10</th>
</tr>
</thead>
</table>

P2:  

<table>
<thead>
<tr>
<th>time</th>
<th>4</th>
<th>7</th>
</tr>
</thead>
</table>

P3:  

<table>
<thead>
<tr>
<th>time</th>
<th>6</th>
<th>9</th>
<th>12</th>
</tr>
</thead>
</table>

P4:  

<table>
<thead>
<tr>
<th>time</th>
<th>3</th>
<th>8</th>
</tr>
</thead>
</table>

P5:  

<table>
<thead>
<tr>
<th>time</th>
<th>2</th>
<th>5</th>
<th>11</th>
</tr>
</thead>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P5</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>∞</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

Marked pages:
- P1 at time 10
- P2 at time 4
- P3 at time 12
- P4 at time 8
- P5 at time 11

Verification:
- Pages 1 and 10 are marked.
- Pages 4, 7, 6, 9, 12, 3, 8, 2, 5, and 11 are not marked.
1: for every $p \leftarrow 1$ to $n$ do
2: \hspace{1em} times[p] \leftarrow$ array of times in which $p$ is requested, in
\hspace{2em} increasing order $\triangleright$ put $\infty$ at the end of array
3: \hspace{1em} pointer[p] \leftarrow 1
4: $Q \leftarrow$ empty priority queue
5: for every $t \leftarrow 1$ to $T$ do
6: \hspace{1em} pointer[$\rho_t$] \leftarrow pointer[$\rho_t$] + 1
7: \hspace{1em} if $\rho_t \in Q$ then
8: \hspace{3em} $Q$.increase-key($\rho_t$, times[$\rho_t$, pointer[$\rho_t$]]), print “hit”,
\hspace{2em} continue
9: \hspace{1em} else
10: \hspace{2em} if $Q$.size() < $k$ then
11: \hspace{4em} print “load $\rho_t$ to an empty page”
12: \hspace{2em} else
13: \hspace{4em} $p \leftarrow Q$.extract-max(), print “evict $p$ and load $\rho_t$”
14: \hspace{4em} $Q$.insert($\rho_t$, times[$\rho_t$, pointer[$\rho_t$]]) $\triangleright$ add $\rho_t$ to $Q$ with key
\hspace{5em} value times[$\rho_t$, pointer[$\rho_t$]]
Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Scheduling to minimize lateness
4. Weighted Completion Time Scheduling
5. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
6. Data Compression and Huffman Code
7. Summary
Let $V$ be a ground set of size $n$.

**Def.** A priority queue is an abstract data structure that maintains a set $U \subseteq V$ of elements, each with an associated key value, and supports the following operations:

- $\text{insert}(v, \text{key\_value})$: insert an element $v \in V \setminus U$, with associated key value $\text{key\_value}$.
- $\text{decrease\_key}(v, \text{new\_key\_value})$: decrease the key value of an element $v \in U$ to $\text{new\_key\_value}$
- $\text{extract\_min}()$: return and remove the element in $U$ with the smallest key value
- …
Simple Implementations for Priority Queue

- $n =$ size of ground set $V$

<table>
<thead>
<tr>
<th>data structures</th>
<th>insert</th>
<th>extract_min</th>
<th>decrease_key</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sorted array</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Simple Implementations for Priority Queue**

- \( n = \text{size of ground set } V \)

<table>
<thead>
<tr>
<th>data structures</th>
<th>insert</th>
<th>extract_min</th>
<th>decrease_key</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td>( O(1) )</td>
<td>( O(n) )</td>
<td>( O(1) )</td>
</tr>
<tr>
<td>sorted array</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Simple Implementations for Priority Queue

- \( n = \text{size of ground set } V \)

<table>
<thead>
<tr>
<th>data structures</th>
<th>insert</th>
<th>extract_min</th>
<th>decrease_key</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td>( O(1) )</td>
<td>( O(n) )</td>
<td>( O(1) )</td>
</tr>
<tr>
<td>sorted array</td>
<td>( O(n) )</td>
<td>( O(1) )</td>
<td>( O(n) )</td>
</tr>
</tbody>
</table>
$n = \text{size of ground set } V$

<table>
<thead>
<tr>
<th>data structures</th>
<th>insert</th>
<th>extract_min</th>
<th>decrease_key</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>sorted array</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>heap</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
</tr>
</tbody>
</table>
Heap

The elements in a heap is organized using a complete binary tree:

- Nodes are indexed as \( \{1, 2, 3, \ldots, s\} \)
- Parent of node \( i \): \( \lfloor i / 2 \rfloor \)
- Left child of node \( i \): \( 2i \)
- Right child of node \( i \): \( 2i + 1 \)
A heap $H$ contains the following fields:

- $s$: size of $U$ (number of elements in the heap)
- $A[i], 1 \leq i \leq s$: the element at node $i$ of the tree
- $p[v], v \in U$: the index of node containing $v$
- $key[v], v \in U$: the key value of element $v$

**Diagram:**

```
    1
   / \  \
  f   g
   \ / \  /  \
  2  3  e  b
     \ /  |
      4   5
```

- $s = 5$
- $A = (\text{'f'}, \text{'g'}, \text{'c'}, \text{'e'}, \text{'b'})$
- $p[\text{'f'}] = 1, p[\text{'g'}] = 2, p[\text{'c'}] = 3, p[\text{'e'}] = 4, p[\text{'b'}] = 5$
Heap

The following heap property is satisfied:
- for any two nodes $i$, $j$ such that $i$ is the parent of $j$, we have $key[A[i]] \leq key[A[j]]$.

A heap. Numbers in the circles denote key values of elements.
\texttt{\textbf{insert}}(v, \textit{key\_value})
$\text{insert}(v, key_{\text{value}})$
\textit{insert}(v, key\_value)
\texttt{insert}(v, \textit{key\_value})
\text{insert}(v, \text{key\_value})
**insert**(v, key_value)

1: \( s \leftarrow s + 1 \)  
2: \( A[s] \leftarrow v \)  
3: \( p[v] \leftarrow s \)  
4: \( key[v] \leftarrow key_value \)  
5: heapify_up(s)

**heapify-up**(i)

1: while \( i > 1 \) do  
2: \( j \leftarrow \lfloor i/2 \rfloor \)  
3: if \( key[A[i]] < key[A[j]] \) then  
4: swap A[i] and A[j]  
5: \( p[A[i]] \leftarrow i, p[A[j]] \leftarrow j \)  
6: \( i \leftarrow j \)  
7: else break
extract_min()
extract_min()
extract_min()
extract\_min()
extract_min()
extract_min()
extract_min()

1: ret ← A[1]
3: p[A[1]] ← 1
4: s ← s − 1
5: if s ≥ 1 then
6:    heapify_down(1)
7: return ret

heapify-down(i)

1: while 2i ≤ s do
2:    if 2i = s or
3:        key[A[2i]] ≤ key[A[2i + 1]] then
4:        j ← 2i
5:    else
6:        j ← 2i + 1
7:    if key[A[j]] < key[A[i]] then
8:        swap A[i] and A[j]
10:    i ← j
11: else break

decrease_key(v, key_value)

1: key[v] ← key_value
2: heapify-up(p[v])
Running time of heapify\_up and heapify\_down: $O(\lg n)$
• Running time of heapify_up and heapify_down: $O(\lg n)$
• Running time of insert, exact_min and decrease_key: $O(\lg n)$
- Running time of heapify\_up and heapify\_down: $O(\lg n)$
- Running time of insert, exact\_min and decrease\_key: $O(\lg n)$

<table>
<thead>
<tr>
<th>data structures</th>
<th>insert</th>
<th>extract_min</th>
<th>decrease_key</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
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Two Definitions Needed to Prove that the Procedures Maintain Heap Property

**Def.** We say that $H$ is almost a heap except that $key[A[i]]$ is too small if we can increase $key[A[i]]$ to make $H$ a heap.

**Def.** We say that $H$ is almost a heap except that $key[A[i]]$ is too big if we can decrease $key[A[i]]$ to make $H$ a heap.
Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Scheduling to minimize lateness
4. Weighted Completion Time Scheduling
5. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
6. Data Compression and Huffman Code
7. Summary
Encoding Letters Using Bits

- 8 letters \( a, b, c, d, e, f, g, h \) in a language
- need to encode a message using bits
- idea: use 3 bits per letter

\[
\begin{array}{cccccccc}
  a & b & c & d & e & f & g & h \\
  000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
\end{array}
\]

\( deacf g \rightarrow 011100000010101110 \)

- Q: Can we have a better encoding scheme?
  - Seems unlikely: must use 3 bits per letter

- Q: What if some letters appear more frequently than the others?
Q: If some letters appear more frequently than the others, can we have a better encoding scheme?

A: Using variable-length encoding scheme might be more efficient.

Idea

- using fewer bits for letters that are more frequently used, and more bits for letters that are less frequently used.
Q: What is the issue with the following encoding scheme?

- $a$: 0
- $b$: 1
- $c$: 00

Solution: Use prefix codes to guarantee a unique decoding.
Q: What is the issue with the following encoding scheme?
n: 0  b: 1  c: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to aa or c.
Q: What is the issue with the following encoding scheme?

- $a$: 0
- $b$: 1
- $c$: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to $aa$ or $c$.

Solution

Use prefix codes to guarantee a unique decoding.
Prefix Codes

**Def.** A prefix code for a set $S$ of letters is a function $\gamma : S \rightarrow \{0, 1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.
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<th>$d$</th>
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<tbody>
<tr>
<td></td>
<td>001</td>
<td>0000</td>
<td>0001</td>
<td>100</td>
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<tr>
<td>$e$</td>
<td>11</td>
<td>1010</td>
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<td>01</td>
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<tr>
<td>$f$</td>
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<td>$g$</td>
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</tr>
<tr>
<td>$h$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Diagram:

```
    0 1
   / \ /
  0 1 1
 /   /
0    1
|    |
|    |
b    c
   / \
  a   d
 /   /
0 1 0
|   |   |
|   |   |
b   f
   / \
 a   g
```
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

<table>
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<tr>
<th></th>
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</tbody>
</table>

- 0001001100000001011110100001001
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

\[
\begin{array}{cccc}
  a & b & c & d \\
  001 & 0000 & 0001 & 100 \\
  e & f & g & h \\
  11 & 1010 & 1011 & 01 \\
\end{array}
\]

- 0001/001100000001011110100001001
- c
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

<p>| | | | |</p>
<table>
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- 0001/001/100000001011110100001001
- ca
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

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- 0001/001/100/000001011110100001001
- cad
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

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- $0001/001/100/0000/01011110100001001$
- cadb
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

\[
\begin{array}{cccc}
  a & b & c & d \\
  001 & 0000 & 0001 & 100 \\
  e & f & g & h \\
  11 & 1010 & 1011 & 01 \\
\end{array}
\]

- 0001/001/100/0000/01/011110100001001
- cadbh
Prefix Codes Guarantee Unique Decoding

Reason: there is only one way to cut the first code.

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- 0001/001/100/0000/01/01/1110100001001
- cadbh
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

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- 0001/001/100/0000/01/01/11/10100001001
- cadbhhe
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

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</tbody>
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- 0001/001/100/0000/01/01/11/1010/0001001
- cadbhhef
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

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<thead>
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<th></th>
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- $0001/001/100/0000/01/01/11/1010/0001/001$
- cadbhhefc
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- 0001/001/100/0000/01/01/11/1010/0001/001/
- cadbhhefca
Properties of Encoding Tree

Rooted binary tree
Left edges labelled 0 and right edges labelled 1
A leaf corresponds to a code for some letter
If coding scheme is not wasteful: a non-leaf has exactly two children

Best Prefix Codes
Input: frequencies of letters in a message
Output: prefix coding scheme with the shortest encoding for the message
Properties of Encoding Tree

- Rooted binary tree

Diagram:

```
          0
         / \        1
        0 1          0
       /   \       / 1
      h   e       0 1
     /   / \\   /   /
    0 1  a  0 1  f  g
   /   / \   /   / \  
  b   c   d  e   f   g
```
Properties of Encoding Tree

- Rooted binary tree
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Best Prefix Codes

**Input:** frequencies of letters in a message

**Output:** prefix coding scheme with the shortest encoding for the message
### Example

<table>
<thead>
<tr>
<th>Letters</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequencies</td>
<td>18</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

| Scheme 1 Length | 2 | 3 | 3 | 2 | 2 |
| Total | 89 |

| Scheme 2 Length | 1 | 3 | 3 | 3 | 3 |
| Total | 87 |

| Scheme 3 Length | 1 | 4 | 4 | 3 | 2 |
| Total | 84 |

![Scheme 1 Diagram](image1)

![Scheme 2 Diagram](image2)

![Scheme 3 Diagram](image3)
### Example

<table>
<thead>
<tr>
<th>letters</th>
<th>a</th>
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<th>d</th>
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<td>6</td>
<td>10</td>
</tr>
<tr>
<td>scheme 1 length</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>scheme 2 length</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>scheme 3 length</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

#### Diagrams

- **Scheme 1**
  - Tree structure with labels `a`, `b`, `c`, `d`, `e`.
  - `a` is at the top, followed by `b`, `c`, `d`, and `e`.

- **Scheme 2**
  - Tree structure with labels `a`, `b`, `c`, `d`, `e`.
  - `a` is at the top, followed by `b`, `c`, `d`, and `e`.

- **Scheme 3**
  - Tree structure with labels `a`, `b`, `c`, `d`, `e`.
  - `a` is at the top, followed by `b`, `c`, `d`, and `e`.

The scheme 1 length totals 89, scheme 2 length totals 87, and scheme 3 length totals 84.
Example Input: \((a: 18, b: 3, c: 4, d: 6, e: 10)\)
Example Input: \((a: 18, b: 3, c: 4, d: 6, e: 10)\)

**Q:** What types of decisions should we make?
Example Input: \((a: 18, b: 3, c: 4, d: 6, e: 10)\)

Q: What types of decisions should we make?

- Can we directly give a code for some letter?
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- Can we directly give a code for some letter?
- Hard to design a strategy; residual problem is complicated.
- Can we partition the letters into left and right sub-trees?
- Not clear how to design the greedy algorithm

**A:** We can choose two letters and make them brothers in the tree.
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
- There are two deepest leaves that are brothers
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
- There are two deepest leaves that are brothers

best to put the two least frequent symbols here!
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
- There are two deepest leaves that are brothers

**Lemma** It is safe to make the two least frequent letters brothers.
Lemma There is an optimum encoding tree, where the two least frequent letters are brothers.
Lemma  There is an optimum encoding tree, where the two least frequent letters are brothers.

- So we can irrevocably decide to make the two least frequent letters brothers.
Lemma There is an optimum encoding tree, where the two least frequent letters are brothers.

So we can irrevocably decide to make the two least frequent letters brothers.

Q: Is the residual problem another instance of the best prefix codes problem?
**Lemma**  There is an optimum encoding tree, where the two least frequent letters are brothers.

- So we can irrevocably decide to make the two least frequent letters brothers.

**Q:** Is the residual problem another instance of the best prefix codes problem?

**A:** Yes, though it is not immediate to see why.
- $f_x$: the frequency of the letter $x$ in the support.
- $x_1$ and $x_2$: the two letters we decided to put together.
- $d_x$ the depth of letter $x$ in our output encoding tree.

\[
\sum_{x \in S} f_x d_x
\]

\[
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2}
\]

\[
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}
\]
- $f_x$: the frequency of the letter $x$ in the support.
- $x_1$ and $x_2$: the two letters we decided to put together.
- $d_x$ the depth of letter $x$ in our output encoding tree.

$$
\sum_{x \in S} f_x d_x \\
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}
$$
- $f_x$: the frequency of the letter $x$ in the support.
- $x_1$ and $x_2$: the two letters we decided to put together.
- $d_x$: the depth of letter $x$ in our output encoding tree.

**Def:** $f_{x'} = f_{x_1} + f_{x_2}$

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&= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1} \\
&= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x'} (d_{x'} + 1)
\end{align*}
\]

Def: \( f_{x'} = f_{x_1} + f_{x_2} \)
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= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x'} (d_{x'} + 1)
= \sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x + f_{x'}
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- \( f_x \): the frequency of the letter \( x \) in the support.
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\]

Def: \( f_{x'} = f_{x_1} + f_{x_2} \)
In order to minimize
\[
\sum_{x \in S} f_x d_x,
\]
we need to minimize
\[
\sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x,
\]
subject to that \( d \) is the depth function for an encoding tree of \( S \setminus \{x_1, x_2\} \).

- This is exactly the best prefix codes problem, with letters \( S \setminus \{x_1, x_2\} \cup \{x'\} \) and frequency vector \( f \)!
Example

A 27  B 15  C 11  D 9  E 8  F 5
Example
Example
Example
Example
Example
Example
Example

A: 00
B: 10
C: 010
D: 011
E: 110
F: 111
Def. The codes given the greedy algorithm is called the Huffman codes.
Def. The codes given the greedy algorithm is called the **Huffman codes**.

**Huffman**(\(S, f\))

1: while \(|S| > 1\) do
2: let \(x_1, x_2\) be the two letters with the smallest \(f\) values
3: introduce a new letter \(x'\) and let \(f_{x'} = f_{x_1} + f_{x_2}\)
4: let \(x_1\) and \(x_2\) be the two children of \(x'\)
5: \(S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}\)
6: return the tree constructed
Algorithm using Priority Queue

Huffman($S, f$)

1. $Q \leftarrow \text{build-priority-queue}(S)$
2. while $Q$.size $> 1$ do
3.     $x_1 \leftarrow Q$.extract-min()
4.     $x_2 \leftarrow Q$.extract-min()
5.     introduce a new letter $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
6.     let $x_1$ and $x_2$ be the two children of $x'$
7.     $Q$.insert($x'$, $f_{x'}$)
8. return the tree constructed
Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Scheduling to minimize lateness
4. Weighted Completion Time Scheduling
5. Offline Caching
   • Heap: Concrete Data Structure for Priority Queue
6. Data Compression and Huffman Code
7. Summary
Summary for Greedy Algorithms

**Greedy Algorithm**

- Build up the solutions in steps
- At each step, make an **irrevocable** decision using a “reasonable” strategy
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- Build up the solutions in steps
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- Offline Caching: evict the page that is used furthest in the future
Greedy Algorithm

- Build up the solutions in steps
- At each step, make an **irrevocable** decision using a “reasonable” strategy

- Interval scheduling problem: schedule the job $j^*$ with the earliest deadline
- Offline Caching: evict the page that is used furthest in the future
- Huffman codes: make the two least frequent letters brothers
A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
Summary for Greedy Algorithms

A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

Def. A strategy is “safe” if there is always an optimum solution that “agrees with” the decision made according to the strategy.
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
- If $S$ agrees with the decision made according to the strategy, done
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
- If $S$ agrees with the decision made according to the strategy, done
- So assume $S$ does not agree with decision
Proving a Strategy is Safe

- Take an arbitrary optimum solution \( S \)
- If \( S \) agrees with the decision made according to the strategy, done
- So assume \( S \) does not agree with decision
- Change \( S \) slightly to another optimum solution \( S' \) that agrees with the decision
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- Change $S$ slightly to another optimum solution $S'$ that agrees with the decision
- Interval scheduling problem: exchange $j^*$ with the first job in an optimal solution
Proving a Strategy is Safe

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- Change $S$ slightly to another optimum solution $S'$ that agrees with the decision
  - Interval scheduling problem: exchange $j^*$ with the first job in an optimal solution
  - Offline caching: a complicated “copying” algorithm
  - Huffman codes: move the two least frequent letters to the deepest leaves.
Summary for Greedy Algorithms

A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is “safe” (key)
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Summary for Greedy Algorithms

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- Prove that the reasonable strategy is “safe” (key)
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- Interval scheduling problem: remove $j^*$ and the jobs it conflicts with
Summary for Greedy Algorithms

### A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is “safe” *(key)*
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

- Interval scheduling problem: remove $j^*$ and the jobs it conflicts with
- Offline caching: trivial
Summary for Greedy Algorithms

A Common Way to Analyze Greedy Algorithms

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

- Interval scheduling problem: remove $j^*$ and the jobs it conflicts with
- Offline caching: trivial
- Huffman codes: merge two letters into one