算法设计与分析(2024年春季学期)
Introduction and Syllabus

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Outline

- Course Webpage:
  https://tcs.nju.edu.cn/shili/courses/2024spring-algo

Outline

Outline

What is an Algorithm?

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.
- Computational problem: specifies the input/output relationship.
- An algorithm solves a computational problem if it produces the correct output for any given input.

Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of a and b

Example:

- Input: 210, 270
- Output: 30

- Algorithm: Euclidean algorithm
- $\gcd(270, 210) = \gcd(210, 270 \bmod 210) = \gcd(210, 60)$
- $(270, 210) \to (210, 60) \to (60, 30) \to (30, 0)$

Examples

Sorting

Input: sequence of n numbers $(a_1, a_2, \cdots, a_n)$

Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \le a'_2 \le \cdots \le a'_n$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

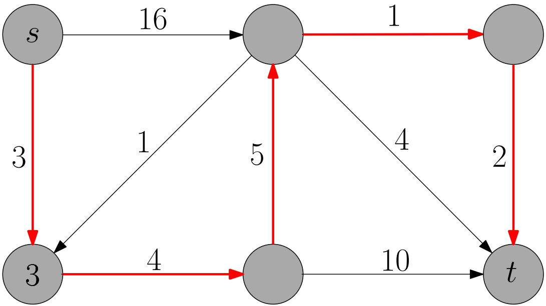
- Algorithms: insertion sort, merge sort, quicksort, ...

Examples

Shortest Path

Input: directed graph $G = (V, E)$, $s, t \in V$

Output: a shortest path from s to t in G



- Algorithm: Dijkstra's algorithm

Algorithm = Computer Program?

- Algorithm: "abstract", can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: "concrete", implementation of algorithm, using a particular programming language

Pseudo-Code

Pseudo-Code:

Euclidean(a, b)

1: **while** $b > 0$ **do**
2: $\quad (a, b) \leftarrow (b, a \bmod b)$
3: **return** a

C++ program:

```
int Euclidean(int a, int b){
    int c;
    while (b > 0){
        c = b;
        b = a % b;
        a = c;
    }
    return a;
}
```

Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
  - extensibility
  - modularity
  - object-oriented model
  - user-friendliness (e.g, GUI)
  - ...
- Why is it important to study the running time (efficiency) of an algorithm?

1. feasible vs. infeasible
2. efficient algorithms: less engineering tricks needed, can use languages aiming for easy programming (e.g, python)
3. fundamental
4. it is fun!

Outline

Sorting Problem

Input: sequence of n numbers $(a_1, a_2, \cdots, a_n)$

Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

Insertion-Sort

- At the end of j-th iteration, the first j numbers are sorted.

iteration 1: $53, 12, 35, 21, 59, 15$

iteration 2: $12, 53, 35, 21, 59, 15$

iteration 3: $12, 35, 53, 21, 59, 15$

iteration 4: $12, 21, 35, 53, 59, 15$

iteration 5: $12, 21, 35, 53, 59, 15$

iteration 6: $12, 15, 21, 35, 53, 59$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for j ← 2 to n do
    key ← A[j]
    i ← j − 1
    while i > 0 and A[i] > key do
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← key
```

- $j = 6$
- $key = 15$

12	15	21	35	53	59
$\uparrow$					
i					

Outline

Analysis of Insertion Sort

- Correctness
- Running time

Correctness of Insertion Sort

- Invariant: after iteration j of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

$$\text{after } j = 1 : 53, 12, 35, 21, 59, 15$$

$$\text{after } j = 2 : 12, 53, 35, 21, 59, 15$$

$$\text{after } j = 3 : 12, 35, 53, 21, 59, 15$$

$$\text{after } j = 4 : 12, 21, 35, 53, 59, 15$$

$$\text{after } j = 5 : 12, 21, 35, 53, 59, 15$$

$$\text{after } j = 6 : 12, 15, 21, 35, 53, 59$$

Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?
- A1: Running time as the function of size
- possible definition of size :
  - Sorting problem: # integers,
  - Greatest common divisor: total length of two integers
  - Shortest path in a graph: # edges in graph
- Q2: Which input?
  - For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.
- A2: Worst-case analysis:
  - Running time for size n = worst running time over all possible arrays of length n

Analyzing Running Time of Insertion Sort

- Q3: How fast is the computer?
- Q4: Programming language?
- A: They do not matter!

Important idea: asymptotic analysis

- Focus on growth of running-time as a function, not any particular value.

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$

- $n^2/100 - 3n + 10 \Rightarrow n^2/100 \Rightarrow n^2$
- $n^2/100 - 3n + 10 = O(n^2)$

Asymptotic Analysis: O-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n^2 + 10 = O(n^2)$

O-notation allows us to ignore

- architecture of computer
- programming language
- how we measure the running time: seconds or # instructions?
- to execute $a \leftarrow b + c$:
  - program 1 requires 10 instructions, or 10^{-8} seconds
  - program 2 requires 2 instructions, or 10^{-9} seconds
  - they only change by a constant in the running time, which will be hidden by the $O(\cdot)$ notation

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$

- Does not tell which algorithm is faster for a specific n!
- Algorithm 2 will eventually beat algorithm 1 as n increases.

- For Algorithm 1: if we increase n by a factor of 2, running time increases by a factor of 4
- For Algorithm 2: if we increase n by a factor of 2, running time increases by a factor of 2

Asymptotic Analysis of Insertion Sort

insertion-sort(A, n)

for $j \leftarrow 2$ to n **do**
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ **do**
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$

- Worst-case running time for iteration j of the outer loop?
  Answer: $O(j)$
- Total running time $= \sum_{j=2}^{n} O(j) = O(\sum_{j=2}^{n} j)$
  $= O(\frac{n(n+1)}{2} - 1) = O(n^2)$

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer n and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.
- What is the precision of real numbers?
  Most of the time, we only consider integers.
- Can we do better than insertion sort asymptotically?
- Yes: merge sort, quicksort and heap sort take $O(n \log n)$ time

Questions?

Outline

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

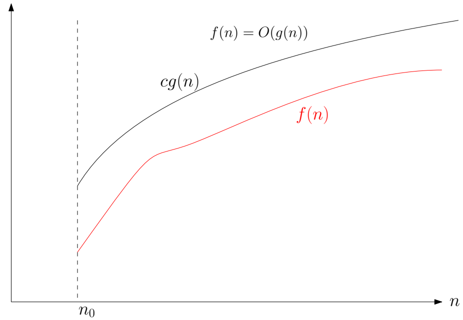
- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes
- $100n - n^2/10 + 50$? No
- We only consider asymptotically positive functions.

O-Notation: Asymptotic Upper Bound

O-Notation For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\}.$$

- In short, $f(n) \in O(g(n))$ if $f(n) \le cg(n)$ for some $c > 0$ and every large enough n.



O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\}.$$

- $3n^2 + 2n \in O(n^2 - 10n)$

Proof.

Let $c = 4$ and $n_0 = 50$, for every $n > n_0 = 50$, we have,

$$3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)$$
$$= -n^2 + 42n \le 0.$$
$$3n^2 + 2n \le c(n^2 - 10n)$$

□

O-Notation For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$
- $n^{100} \in O(2^n)$
- $n^3 \notin O(10n^2)$

Asymptotic Notations	O	Ω	Θ
Comparison Relations	$\leq$		

Conventions

- We use "$f(n) = O(g(n))$" to denote "$f(n) \in O(g(n))$"
- $3n^2 + 2n = O(n^2)$

- "$=$" is asymmetric: we do not write $O(n^2) = 3n^2 + 2n$
- Analogy: Mike is a student. ~~A student is Mike.~~

- We use "$O(g(n)) = O(g'(n))$" to denote "$O(g(n)) \subseteq O(g'(n))$".
- $O(3n^2 + 2n) = O(n^2)$
- Again, "$=$" is asymmetric.
- $O(n^3) = O(3n^2 + 2n)$ makes sense, but is wrong.

- Analogy: All students are people.
- Equalities can be chained: $3n^2 + 2n = O(n^2) = O(n^3)$.

Ω-Notation: Asymptotic Lower Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\}.$$

Ω-**Notation** For a function $g(n)$,

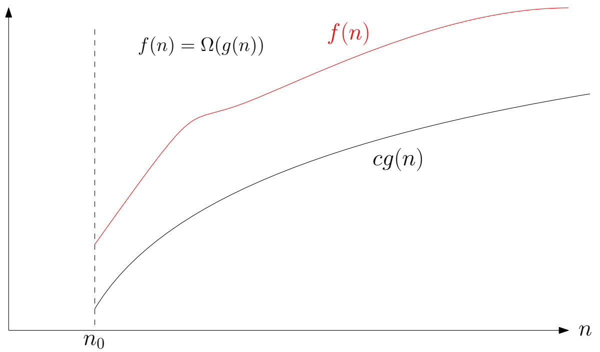
$$\Omega(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \ge cg(n), \forall n \ge n_0\}.$$

- In short, $f(n) \in \Omega(g(n))$ if $f(n) \ge cg(n)$ for some c and large enough n.

Ω-Notation: Asymptotic Lower Bound

Ω-Notation For a function $g(n)$,

$$\Omega(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0\}.$$



Ω-Notation: Asymptotic Lower Bound

- Again, we use "$=$" instead of $\in$.
  - $4n^2 = \Omega(n - 10)$
  - $3n^2 - n + 10 = \Omega(n^2 - 20)$

Asymptotic Notations	O	Ω	Θ
Comparison Relations	$\le$	$\ge$	

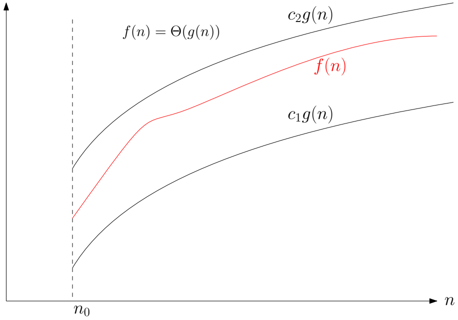
Theorem $f(n) = O(g(n)) \;\Leftrightarrow\; g(n) = \Omega(f(n))$.

Θ-Notation: Asymptotic Tight Bound

Θ-Notation For a function $g(n)$,

$$\Theta(g(n)) = \{\text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0\}.$$

- $f(n) = \Theta(g(n))$, then for large enough n, we have "$f(n) \approx g(n)$".



Θ-Notation: Asymptotic Tight Bound

Θ-Notation For a function $g(n)$,

$$\Theta(g(n)) = \{\text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0\}.$$

- $3n^2 + 2n = \Theta(n^2 - 20n)$
- $2^{n/3+100} = \Theta(2^{n/3})$

Asymptotic Notations	O	Ω	Θ
Comparison Relations	$\leq$	$\geq$	$=$

Theorem $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

o and ω-Notations

o-Notation For a function $g(n)$,

$$o(g(n)) = \{\text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

ω-Notation For a function $g(n)$,

$$\omega(g(n)) = \{\text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0\}.$$

Example:

- $3n^2 + 5n + 10 = o(n^2 \log n)$.
- $3n^2 + 5n + 10 = \omega(n^2 / \log n)$.

Asymptotic Notations	O	Ω	Θ	o	ω
Comparison Relations	$\leq$	$\geq$	$=$	$<$	$>$

Asymptotic Notations	O	Ω	Θ	o	ω
Comparison Relations	$\leq$	$\geq$	$=$	$<$	$>$

Facts on Comparison Relations

- $a \leq b \iff b \geq a$
- $a = b \iff a \leq b$ and $a \geq b$
- $a < b \implies a \leq b$
- $a < b \iff b > a$

Correct Analogies

- $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
- $f(n) = \Theta(g(n)) \iff f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
- $f(n) = o(g(n)) \implies f(n) = O(g(n))$
- $f(n) = o(g(n)) \iff g(n) = \omega(f(n))$

Asymptotic Notations	O	Ω	Θ	o	ω
Comparison Relations	$\leq$	$\geq$	$=$	$<$	$>$

Facts on Comparison Relations

- $a \leq b$ or $a \geq b$
- $a \leq b \iff a = b$ or $a < b$

Incorrect Analogies

- $f(n) = O(g(n))$ or $f(n) = \Omega(g(n))$
- $f(n) = O(g(n)) \iff f(n) = \Theta(g(n))$ or $f(n) = o(g(n))$

Incorrect Analogy

- $f(n) = O(g(n))$ or $f(n) = \Omega(g(n))$

$$f(n) = n^2$$

$$g(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ n^3 & \text{if } n \text{ is even} \end{cases}$$

Recall: Informal way to define O-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \to 3n^2$
- ignoring leading constant: $3n^2 \to n^2$
- $3n^2 - 10n - 5 = O(n^2)$
- In the formal definition of $O(\cdot)$, nothing tells us to ignore lower order terms and leading constant.
- $3n^2 - 10n - 5 = O(5n^2 - 6n + 5)$ is correct, though weird
- $3n^2 - 10n - 5 = O(n^2)$ is the most natural since n^2 is the simplest term we can have inside $O(\cdot)$.

Notice that O denotes asymptotic upper bound

- $n^2 + 2n = O(n^3)$ is correct.
- The following sentence is correct: the running time of insertion sort is $O(n^4)$.
- Usually we say: The running time of insertion sort is $O(n^2)$ and the bound is tight.
- Also correct: the worst-case running time of insertion sort is $\Theta(n^2)$.

Outline

$O(n)$ (Linear) Running Time

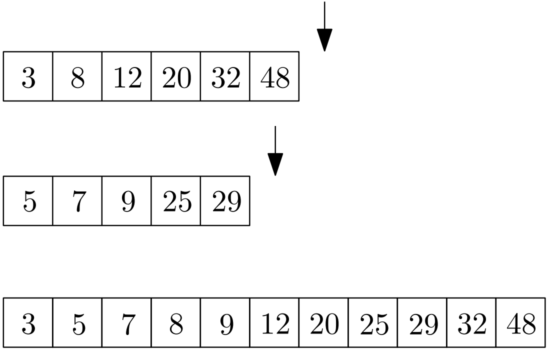
Computing the sum of n numbers

sum(A, n)

```
S ← 0
for i ← 1 to n
        S ← S + A[i]
return S
```

$O(n)$ (Linear) Running Time

- Merge two sorted arrays



$O(n)$ (Linear) Running Time

merge(B, C, n_1, n_2) \\ B and C are sorted, with length n_1 and n_2

$A \leftarrow []$; $i \leftarrow 1$; $j \leftarrow 1$
while $i \leq n_1$ and $j \leq n_2$ **do**
    if $B[i] \leq C[j]$ **then**
        append $B[i]$ to A; $i \leftarrow i + 1$
    else
        append $C[j]$ to A; $j \leftarrow j + 1$
if $i \leq n_1$ then append $B[i..n_1]$ to A
if $j \leq n_2$ then append $C[j..n_2]$ to A
return A

Running time $= O(n)$ where $n = n_1 + n_2$.

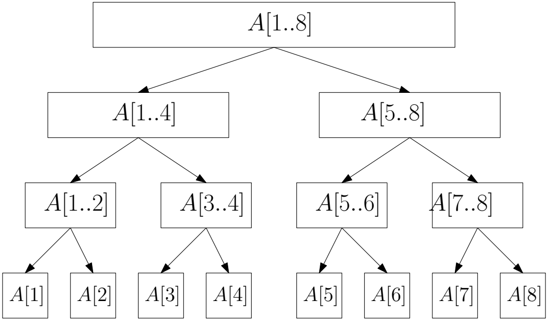
$O(n \log n)$ Running Time

merge-sort(A, n)

```
if n = 1 then
    return A
B ← merge-sort(A[1..⌊n/2⌋], ⌊n/2⌋)
C ← merge-sort(A[⌊n/2⌋ + 1..n], n − ⌊n/2⌋)
return merge(B, C, ⌊n/2⌋, n − ⌊n/2⌋)
```

$O(n \log n)$ Running Time

- Merge-Sort



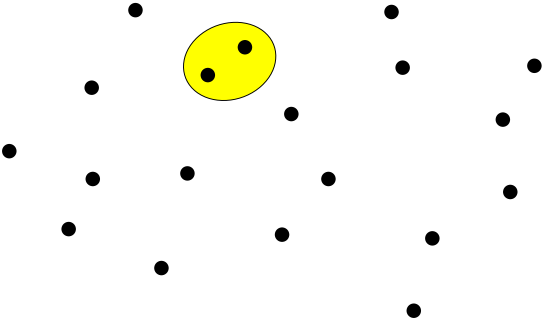
- Each level takes running time $O(n)$
- There are $O(\log n)$ levels
- Running time $= O(n \log n)$

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest



$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest

closest-pair(x, y, n)

$bestd \leftarrow \infty$
for $i \leftarrow 1$ to $n - 1$ **do**
    for $j \leftarrow i + 1$ to n **do**
        $d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$
        if $d < bestd$ **then**
            $besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d$
return $(besti, bestj)$

Closest pair can be solved in $O(n \log n)$ time!

$O(n^3)$ (Cubic) Running Time

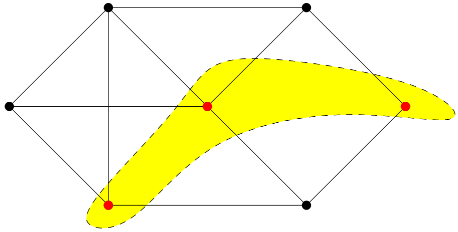
Multiply two matrices of size $n \times n$

matrix-multiplication(A, B, n)

```
C ← matrix of size n × n, with all entries being 0
for i ← 1 to n do
    for j ← 1 to n do
        for k ← 1 to n do
            C[i, k] ← C[i, k] + A[i, j] × B[j, k]
return C
```

Beyond Polynomial Time: 2^n

Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$.



Beyond Polynomial Time: 2^n

Maximum Independent Set Problem

Input: graph $G = (V, E)$

Output: the maximum independent set of G

max-independent-set($G = (V, E)$)

$R \leftarrow \emptyset$
for every set $S \subseteq V$ **do**
    $b \leftarrow$ true
    for every $u, v \in S$ **do**
        if $(u, v) \in E$ then $b \leftarrow$ false
    if b and $|S| > |R|$ then $R \leftarrow S$
return R

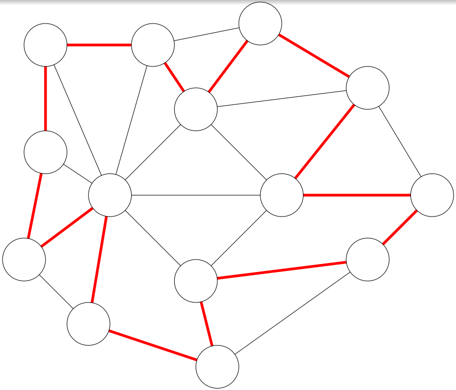
Running time $= O(2^n n^2)$.

Beyond Polynomial Time: $n!$

Hamiltonian Cycle Problem

Input: a graph with n vertices

Output: a cycle that visits each node exactly once, or say no such cycle exists



Beyond Polynomial Time: $n!$

Hamiltonian$(G = (V, E))$

for every permutation $(p_1, p_2, \cdots, p_n)$ of V **do**
    $b \leftarrow$ true
    for $i \leftarrow 1$ to $n-1$ **do**
        if $(p_i, p_{i+1}) \notin E$ then $b \leftarrow$ false
    if $(p_n, p_1) \notin E$ then $b \leftarrow$ false
    if b then return $(p_1, p_2, \cdots, p_n)$
return "No Hamiltonian Cycle"

Running time $= O(n! \times n)$

$O(\log n)$ (Logarithmic) Running Time

- Binary search
  - Input: sorted array A of size n, an integer t;
  - Output: whether t appears in A.
- E.g, search 35 in the following array:

3	8	10	25	29	37	38	42	46	52	59	61	63	75	79

$O(\log n)$ (Logarithmic) Running Time

Binary search

- Input: sorted array A of size n, an integer t;
- Output: whether t appears in A.

binary-search(A, n, t)

$i \leftarrow 1, j \leftarrow n$
while $i \leq j$ **do**
    $k \leftarrow \lfloor (i+j)/2 \rfloor$
    if $A[k] = t$ return true
    if $t < A[k]$ then $j \leftarrow k-1$ else $i \leftarrow k+1$
return false

Running time $= O(\log n)$

Comparing the Orders of Running Times

- Sort the functions from smallest to largest asymptotically
  $\log n, \;\; n, \;\; n^2, \;\; n\log n, \;\; n!, \;\; 2^n, \;\; e^n, \;\; n^n, \;\; \log(n!)$
- $\log n \quad n \quad \{n\log n, \log(n!)\} \quad n^2 \quad 2^n \quad e^n \quad n! \quad n^n$
- $\log n = o(n), \quad n = o(n\log n), \quad n\log n = \Theta(\log(n!))$
- $\log(n!) = o(n^2), \quad n^2 = o(2^n), \quad 2^n = o(e^n)$
- $e^n = o(n!), \quad n! = o(n^n)$

Terminologies

When we talk about upper bounds:

- Logarithmic time: $O(\lg n)$
- Linear time: $O(n)$
- Quadratic time: $O(n^2)$
- Cubic time: $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant k
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
- Sub-quadratic time: $o(n^2)$

When we talk about lower bounds:

- Super-linear time: $\omega(n)$
- Super-quadratic time: $\omega(n^2)$
- Super-polynomial time: $\bigcap_{k>0} \omega(n^k) = n^{\omega(1)}$

Goal of Algorithm Design

- Design algorithms to minimize the order of the running time.

- Using asymptotic analysis allows us to ignore the leading constants and lower order terms
- Makes our life much easier! (E.g., the leading constant depends on the implementation, complier and computer architecture of computer.)

Q: Can constants really be ignored?

- e.g, how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

A:

- Sometimes no
- For most natural and simple algorithms, constants are not so big.
- Algorithm with lower order running time beats algorithm with higher order running time for reasonably large n.