Homework 4

Course: Algorithm Design and Analysis **Semester**: Spring 2025

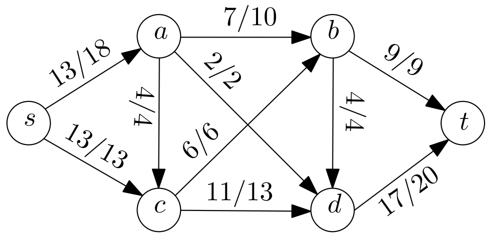
Instructor: Shi Li **Due Date**: **2025/5/25**

Student Name: __________ Student ID: __________

Problems	1	2	3	4	5	6	Total
Max. Score	15	15	15	20	15	20	100
Your Score							

- Remark: You can use give the answers in either Chinese or English.

Problem 1. Consider the following network flow instance with graph $G = (V, E)$, source s, sink t and capacity vector $c \in \mathbb{Z}_{>0}^E$. We already have a valid flow $f : E \to \mathbb{Z}_{\geq 0}$ on the graph. (The first number on each edge e is $f(e)$ and the second number is c_e.)



(1a) Draw the residual graph G_f, with residual capacities on edges.

(1b) Give an augmenting path in the residual graph.

(1c) Show the flow obtained after the augmenting f along the path you found for question (1b).

Problem 2. Given an undirected graph $G = (V, E)$ and an integer $d \geq 1$, we say G is d-orientable if we can make the edges in E directed so that each vertex has at most d incoming edges. For example, a tree is 1-orientable. The graph $G = (V, E)$ on the left side of Figure 1 is 2-orientable since we can make the edges directed (right side of the figure) so that each vertex has at most 2 incoming edges. Given $G = (V, E)$ and a positive integer $d \leq |V|$, design a polynomial time algorithm that decides if G is d-orientable or not.

Problem 3. You are given n jobs which needs to be processed on one machine. Each job j has an arrive time r_j, a deadline d_j and a processing time p_j, where $r_j \in \mathbb{Z}_{\geq 0}$ and $p_j, d_j \in \mathbb{Z}_{>0}$. A job j can only be processed during the time interval $(r_j, d_j]$. The processing of jobs can be interrupted and resumed later, and a job j is completed if it is processed for p_j units of time during the interval $(r_j, d_j]$. Your goal is to check if all the jobs can be completed.

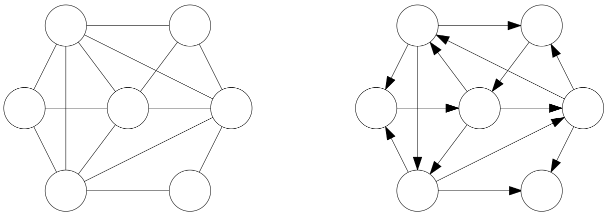


Figure 1: A 2-orientable graph

More formally, the time is slotted into unit time slots $1, 2, 3, \cdots$. During any time slot t, you can choose to process a job j on the time slot, if $r_j < t \le d_j$. You can process at most one job in a time slot. The jobs can be completed if every job j can be processed during p_j time slots.

job indices	a	b	c
r_j	0	3	7
d_j	5	10	9
p_j	4	5	1

Table 1: Instance for Problem 3.

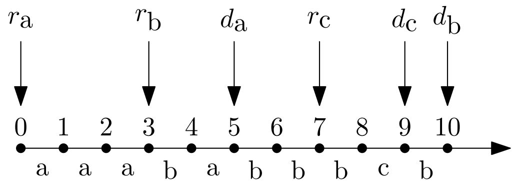


Figure 2: A feasible schedule of the instance.

For example, consider the instance with 3 jobs given by Table 1. The 3 jobs can all be completed: A feasible solution is given in Figure 2. So, for the instance you should output "yes". However, if p_c is 2 instead of 1, then you should output "no".

(You can solve the problem using a greedy algorithm; you can try to solve it yourself, but there is no need to write down the solution.)

Use the maximum-flow-minimum-cut theorem or Hall's theorem for bipartite matching to prove the following statement:

- The instance is feasible (i.e, there is a schedule that completes all jobs) if and only if for every two integers $0 \le r < d$, we have

$$\sum_{j \in [n]: r \le r_j < d_j \le d} p_j \le d - r.$$

Problem 4. For problems (4a)-(4e), state

(i) whether the problem is known to be in NP, and

(ii) whether the problem is known to be in Co-NP.

For problems (4c), (4d) and (4e), if your answer for (i) and/or (ii) is yes, you need to give the certificate and the certifier that proves the problem (or its complement) is in NP.

(4a) Given a graph $G = (V, E)$ and $s \le |V|$, the problem asks whether G contains an independent set of size s.

(4b) Given two circuits C_1 and C_2, each with n input variables $x_1, x_2, \cdots, x_n$, decide if the two circuits compute the same function. That is, whether C_1 and C_2 give the same output for every boolean assignment of x-variables.

(4c) Given a graph $G = (V, E)$, decide if G is 3-colorable.

(4d) Given a graph $G = (V, E)$, decide if G is 2-colorable.

(4e) An undirected graph $G = (V, E)$ is called a 1-expander if for every $U \subseteq V$, the number of edges between U and $V \setminus U$ in G is at least $\min\{|U|, |V \setminus U|\}$. Given a graph G, the problem asks if G is a 1-expander.

Problem 5. Let NPC be the set of NP-Complete problems. Prove the following statements:

(5a) If P $\neq$ NP, then P $\cap$ NPC $= \emptyset$.

(5b) If P $=$ NP, then P $=$ NPC.

(5c) If P $=$ NP, then P $=$ Co-NP.

Problem 6. Given a graph $G = (V, E)$, the degree-3 spanning tree (D3ST) problem asks whether G contains a spanning tree T of degree at most 3. (The degree of a vertex v in a spanning tree T is the number of edges incident to v in T; the degree of T is the maximum degree of v, over all vertices $v \in V$.)

Prove that Hamiltonian-Path $\leq_P$ D3ST.