Homework 5

Course: Algorithm Design and Analysis | **Semester**: Spring 2025

Instructor: Shi Li | **Due Date**: **2025/6/15**

Student Name: _________ Student ID: _________

Problems	1	2	3	4	5	Total
Max. Score	20	20	20	20	20	100
Your Score						

- Remark: You can give the answers in either Chinese or English.

Problem 1. Consider the following linear program:

$$\min \quad 2x_2 - x_3$$
$$x_1 + x_2 - x_3 \geq 5$$
$$x_1 + 2x_3 = 6$$
$$x_1, x_2 \geq 0$$

(1a) Give the optimum solution and value of the linear program.

(1b) Write down the dual of the linear program.

(1c) Give the optimum solution and value of the dual linear program.

Problem 2. Consider the following two lemmas.

- **Separation Lemma**: Let P be a closed polytope in $\mathbb{R}^n$, and $x \in \mathbb{R}^n \setminus P$. Then there exists a hyperplane separating P and x. Formally, there exists a vector $w \in \mathbb{R}^n$ and a real $g \in \mathbb{R}$ such that $w^T x' > g$ for every $x' \in P$, and $w^T x < g$.
- **Farkas Lemma**: Consider the linear system $Ax = b$, $x \geq 0$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given, and $x \in \mathbb{R}^n$ is the vector of variables. Then the system is infeasible if and only if there exists a vector $y \in \mathbb{R}^n$ such that $A^T y \geq 0$ and $b^T y < 0$.

Your goal is to use the Separation Lemma to prove the Farkas Lemma.

Problem 3. In class, you learned that the number of global minimum cuts is at most $\binom{n}{2} \leq n^2$ in a connected graph $G = (V, E)$. Let $\alpha \geq 1$ be an integer. We say a cut in G is an α-approximate cut, if its cut value is at most α times the value of the global minimum cut. Prove that the number of α-approximate minimum cuts in a connected graph $G = (V, E)$ is at most $n^{2\alpha}$.

Problem 4. Recall that an independent set in a graph $G = (V, E)$ is a set $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$. In the problem, we are given an $n \times n$ grid graph $G = (V, E)$. Namely, $V = [n] \times [n]$, and $((i, j), (i', j')) \in E$ if and only if $|i - i'| + |j - j'| = 1$. Moreover, we are given a non-negative weight $w_v \geq 0$ for every vertex $v \in V$. Our goal is to find an independent set S in G with the maximum total weight, i.e., $\sum_{v \in S} w_v$.

Design a polynomial time 4-approximation algorithm for the problem.

Problem 5. Recall that in the weighted vertex cover problem, we are given a ground set U of m elements and a collection $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets of U. Each set S_i has a non-negative weight $w_i \geq 0$. Our goal is to find a set $I \subseteq [n]$ satisfying $\bigcup_{i \in I} S_i = U$, so as to minimize $\sum_{i \in I} w_i$.

Now assume additionally we have that each element $e \in U$ appears in at most f sets in $\mathcal{S}$, for some integer $f \geq 1$.

Design a polynomial time f-approximation for the problem, using either LP rounding or primal-dual.