算法设计与分析(2025年春季学期)
Advanced Topics

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Outline

1. Randomized Algorithms
   - Freivald's matrix multiplication verification algorithm
   - Randomized Select and Quicksort
   - Randomized Algorithm for Global Min-Cut
   - $\frac{7}{8}$-Approximation Algorithm for Max 3-SAT
2. Extending the Limits of Tractability
3. Approximation Algorithms using Greedy
4. Arbitrarily Good Approximation Using Rounding Data
5. Approximation Using LP Rounding and Primal Dual

Why do we use randomized algorithms?

- simpler algorithms: quick-sort, minimum-cut, and Max 3-SAT.
- faster algorithms: polynomial identity testing, Freivald's matrix multiplication verification algorithm, sampling and fingerprinting.
- mathematical beauty: Nash equilibrium for 0-sum game
- proof of existence of objects: union bound, Lovasz local lemma.

Price of using randomness

- The algorithm may be incorrect with some probability (Monto Carlo Algorithm)
- The algorithm may take a long time to terminate (Las Vegas Algorithm)

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Matrix Multiplication Verification

Input: 3 matrices $A, B, C \in \mathbb{Z}^{n\times n}$

Output: whether if $C = AB$

- trivial: compute $C' = AB$ and check if $C' = C$.
- time $=$ matrix multiplication time
  - naive algorithm: $O(n^3)$
  - Strassen's algorithm: $O(n^{2.81})$
  - Best known algorithm for matrix multiplication: $O(n^{2.3719})$.
- Freivald's algorithm: randomized algorithm with $O(n^2)$ time.

Freivald's Algorithm: one experiment

1: randomly choose a vector $r \in \{0, 1\}^n$
2: **return** $ABr = Cr$

Q: What is the running time of the algorithm?

- $(AB)r$: matrix-multiplication time
- $A(Br)$: $O(n^2)$ time

Analysis of correctness

- $AB = C$: algorithm outputs true with probability 1.
- $AB \neq C$: algorithm may incorrectly output true.

Lemma If $AB \neq C$, then the algorithm outputs false with probability at least $1/2$.

Lemma If $AB \neq C$, then the algorithm outputs false with probability at least $1/2$.

Proof.

- $D := C - AB \neq 0$ $\qquad Cr = ABr \iff Dr = 0$
- $\exists i, j \in [n], D_{i,j} \neq 0$

$$D_i r = \sum_{j'=1}^{n} D_{i,j'} r_{j'} = X + Y, \quad X = \sum_{j' \in [n], j' \neq j} D_{i,j'} r_{j'}, Y = D_{i,j} r_j$$

$$\begin{aligned}
\Pr[D_i r \neq 0] &= \Pr[Y \neq -X] \\
&= \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \Pr[Y \neq -x | X = x] \\
&= \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \Pr[D_{i,j} r_j \neq -x | X = x] \\
&\geq \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

□

- probabilities:

	true	false
$AB = C$	1	0
$AB \neq C$	$\leq 1/2$	$\geq 1/2$

Freivald's Algorithm: k experiments

1: **for** $t \leftarrow 1$ to k **do**
2: randomly choose a vector $r \in \{0, 1\}^n$
3: **if** $ABr \neq Cr$ **then return false**
4: **return true**

- probabilities with k experiments:

	true	false
$AB = C$	1	0
$AB \neq C$	$\leq 1/2^k$	$\geq 1 - 1/2^k$

- to achieve δ probability of mistake, need $\log_2 \frac{1}{\delta} = O(\log \frac{1}{\delta})$ experiments.

- Frievald's algorithm is a Monta Carlo algorithm.

Def. A Monta Carlo algorithm is a randomized algorithm whose output may be incorrect with some probability.

- For a Monta Carlo algorithm that outputs true/false, we say the algorithm has one-sided error if it makes error only if the correct output is true (or false).

Outline

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

29	82	75	64	38	45	94	69	25	76	15	92	37	17	85

29	38	45	25	15	37	17	64	82	75	94	92	69	76	85

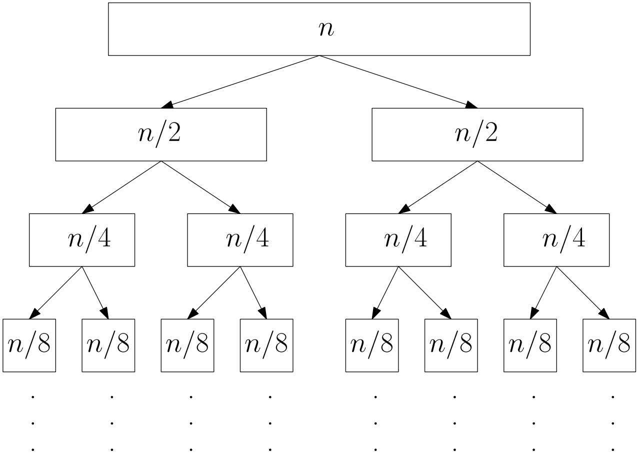
25	15	17	29	38	45	37	64	82	75	94	92	69	76	85

Quicksort

quicksort(A, n)

if $n \leq 1$ then return A
$x \leftarrow$ lower median of A
$A_L \leftarrow$ elements in A that are less than x \\ Divide
$A_R \leftarrow$ elements in A that are greater than x \\ Divide
$B_L \leftarrow$ quicksort$(A_L, A_L.\text{size})$ \\ Conquer
$B_R \leftarrow$ quicksort$(A_R, A_R.\text{size})$ \\ Conquer
$t \leftarrow$ number of times x appear A
return the array obtained by concatenating B_L, the array containing t copies of x, and B_R

- Recurrence $T(n) \leq 2T(n/2) + O(n)$
- Running time $= O(n \log n)$



- Each level has total running time $O(n)$
- Number of levels $= O(\log n)$
- Total running time $= O(n \log n)$

Randomized Quicksort Algorithm

quicksort(A, n)

1. if $n \leq 1$ then return A
2. $x \leftarrow$ a random element of A (x is called a pivot)
3. $A_L \leftarrow$ elements in A that are less than x \\ Divide
4. $A_R \leftarrow$ elements in A that are greater than x \\ Divide
5. $B_L \leftarrow$ quicksort$(A_L, A_L.\text{size})$ \\ Conquer
6. $B_R \leftarrow$ quicksort$(A_R, A_R.\text{size})$ \\ Conquer
7. $t \leftarrow$ number of times x appear A
8. return the array obtained by concatenating B_L, the array containing t copies of x, and B_R

Variant of Randomized Quicksort Algorithm

quicksort(A, n)

if $n \leq 1$ then return A
repeat
    $x \leftarrow$ a random element of A (x is called a pivot)
    $A_L \leftarrow$ elements in A that are less than x \\\\ Divide
    $A_R \leftarrow$ elements in A that are greater than x \\\\ Divide
until A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$
$B_L \leftarrow$ quicksort$(A_L, A_L$.size$)$ \\\\ Conquer
$B_R \leftarrow$ quicksort$(A_R, A_R$.size$)$ \\\\ Conquer
$t \leftarrow$ number of times x appear A
return the array obtained by concatenating B_L, the array containing t copies of x, and B_R

Analysis of Variant

1: $x \leftarrow$ a random element of A
2: $A_L \leftarrow$ elements in A that are less than x
3: $A_R \leftarrow$ elements in A that are greater than x

Q: What is the probability that A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$?

A: At least $1/2$

Analysis of Variant

repeat
    $x \leftarrow$ a random element of A
    $A_L \leftarrow$ elements in A that are less than x
    $A_R \leftarrow$ elements in A that are greater than x
until A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$

Q: What is the expected number of iterations the above procedure takes?

A: At most 2

- Suppose an experiment succeeds with probability $p \in (0, 1]$, independent of all previous experiments.

```
repeat
    run an experiment
until the experiment succeeds
```

Lemma The expected number of experiments we run in the above procedure is $1/p$.

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Proof

$$
\begin{aligned}
\text{Expectation} &= p + (1-p)p \times 2 + (1-p)^2 p \times 3 + (1-p)^3 p \times 4 \\
&\quad + \cdots \\
&= p \sum_{i=1}^{\infty} (1-p)^{i-1} i \quad = \quad p \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} (1-p)^{i-1} \\
&= p \sum_{j=1}^{\infty} (1-p)^{j-1} \frac{1}{1-(1-p)} \quad = \quad \sum_{j=1}^{\infty} (1-p)^{j-1} \\
&= (1-p)^0 \frac{1}{1-(1-p)} = 1/p
\end{aligned}
$$

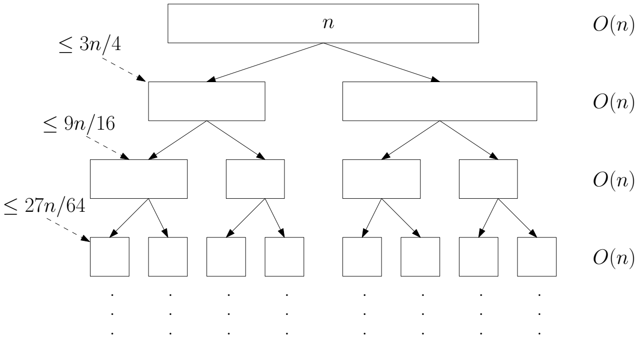
Variant Randomized Quicksort Algorithm

quicksort(A, n)

if $n \leq 1$ then return A
repeat
    $x \leftarrow$ a random element of A (x is called a pivot)
    $A_L \leftarrow$ elements in A that are less than x \\ Divide
    $A_R \leftarrow$ elements in A that are greater than x \\ Divide
until A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$
$B_L \leftarrow$ quicksort$(A_L, A_L$.size$)$ \\ Conquer
$B_R \leftarrow$ quicksort$(A_R, A_R$.size$)$ \\ Conquer
$t \leftarrow$ number of times x appear A
return the array obtained by concatenating B_L, the array containing t copies of x, and B_R

Analysis of Variant

- Divide and Combine: takes $O(n)$ time
- Conquer: break an array of size n into two arrays, each has size at most $3n/4$. Recursively sort the 2 sub-arrays.



- Number of levels $\leq \log_{4/3} n = O(\log n)$

Randomized Quicksort Algorithm

quicksort(A, n)

if $n \leq 1$ then return A
$x \leftarrow$ a random element of A (x is called a pivot)
$A_L \leftarrow$ elements in A that are less than x \\ Divide
$A_R \leftarrow$ elements in A that are greater than x \\ Divide
$B_L \leftarrow$ quicksort$(A_L, A_L.\text{size})$ \\ Conquer
$B_R \leftarrow$ quicksort$(A_R, A_R.\text{size})$ \\ Conquer
$t \leftarrow$ number of times x appear A
return the array obtained by concatenating B_L, the array containing t copies of x, and B_R

- Intuition: the quicksort algorithm should be better than the variant.

Analysis of Randomized Quicksort Algorithm

- $T(n)$: an upper bound on the expected running time of the randomized quicksort algorithm on n elements
- Assuming we choose the element of rank i as the pivot.
- The left sub-instance has size at most $i-1$
- The right sub-instance has size at most $n-i$
- Thus, the expected running time in this case is $\big(T(i-1)+T(n-i)\big)+O(n)$
- Overall, we have

$$\begin{aligned} T(n) &= \frac{1}{n}\sum_{i=1}^{n}\big(T(i-1)+T(n-i)\big)+O(n) \\ &= \frac{2}{n}\sum_{i=0}^{n-1}T(i)+O(n) \end{aligned}$$

- Can prove $T(n) \leq c(n\log n)$ for some constant c by reduction

Analysis of Randomized Quicksort Algorithm

The induction step of the proof:

$$
\begin{aligned}
T(n) &\le \frac{2}{n}\sum_{i=0}^{n-1} T(i) + c'n \le \frac{2}{n}\sum_{i=0}^{n-1} ci\log i + c'n \\
&\le \frac{2c}{n}\left(\sum_{i=0}^{\lfloor n/2\rfloor - 1} i\log\frac{n}{2} + \sum_{i=\lfloor n/2\rfloor}^{n-1} i\log n\right) + c'n \\
&\le \frac{2c}{n}\left(\frac{n^2}{8}\log\frac{n}{2} + \frac{3n^2}{8}\log n\right) + c'n \\
&= c\left(\frac{n}{4}\log n - \frac{n}{4} + \frac{3n}{4}\log n\right) + c'n \\
&= cn\log n - \frac{cn}{4} + c'n \le cn\log n \qquad \text{if } c \ge 4c'
\end{aligned}
$$

Indirect Analysis Using Number of Comparisons

- Running time $= O$(number of comparisons)
- $\forall 1 \leq i < j \leq n$, $D_{i,j}$ indicates if we compared the i-th smallest element with the j-th smallest element
- number of comparisons $= \sum_{1 \leq i < j \leq n} D_{i,j}$

Lemma $\mathbb{E}[D_{i,j}] = \frac{2}{j-i+1}$.

Proof.

- A': sorted array for A. Focus on $A'[i..j]$.
- pivot outside $A'[i]$: $A'[i \cdots j]$ will be passed to left or right recursion; go to that recursion
- pivot inside $A'[i]$: $A'[i]$ and $A'[j]$ will be separated; call this critical recursion
- $A[i]$ and $A[j]$ are compared in the critical recursion with probability $\frac{2}{j-i+1}$.

□

$$\mathbb{E}\left[\text{number of comparisons}\right] = \mathbb{E}\left[\sum_{1 \le i < j \le n} D_{i,j}\right]$$

$$= \sum_{1 \le i < j \le n} \mathbb{E}\left[D_{i,j}\right] = 2 \sum_{1 \le i < j \le n} \frac{1}{j - i + 1}$$

$$\le 2n \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$= \Theta\left(n \log n\right).$$

- The algorithm is a Las-Vegas algorithm:

Def. A Las-Vegas algorithm is a randomized algorithm that always outputs a correct solution but has randomized running time.

Table: Comparisons between Monta Carlo and Las Vegas Algorithms.

	correctness	running time
Monta Carlo	may be wrong	usually has good worst-case running time
Las Vegas	always correct	may take a long time and usually only has good "expected running time"

Lemma Given a Las Vegas algorithm $\mathcal{A}$ with expected running time at most $T(n)$, we can design a Monta Carlo algorithm $\mathcal{A}'$ with worst-case running time $O(T(n))$ and error at most 0.99.

- 0.99 can be changed to any $c < 1$

Proof.

- run $\mathcal{A}$ for $100T(n)$ time
- if $\mathcal{A}$ terminated, output what $\mathcal{A}$ outputs
- otherwise, declare failure
- Markov Inequality:
  $\Pr[\mathcal{A}$ runs for more than $100T(n)$ time$] \leq 1/100$ □

Randomized Selection Algorithm

selection(A, n, i)

```
if n = 1 then return A
x ← random element of A (called pivot)
A_L ← elements in A that are less than x                          ▷ Divide
A_R ← elements in A that are greater than x                       ▷ Divide
if i ≤ A_L.size then
    return selection(A_L, A_L.size, i)                            ▷ Conquer
else if i > n − A_R.size then
    return selection(A_R, A_R.size, i − (n − A_R.size))           ▷ Conquer
else
    return x
```

- expected running time $= O(n)$

Randomized Selection

- $X_j, j = 0, 1, 2, \cdots$: the size of A in the j-th recursion

$$
\begin{aligned}
\mathbb{E}[X_{j+1} | X_j = n'] &\leq \frac{1}{n'} \sum_{k=1}^{n'} \max\{k-1, n'-k\} \\
&\leq \frac{1}{n'} \left(\int_{k=0}^{n'/2} (n'-k) \mathrm{d}k + \int_{k=n'/2}^{n'} k \mathrm{d}k \right) \\
&= \frac{1}{n'} \left(\left(n'k - \frac{k^2}{2} \right) \Big|_0^{n'/2} + \frac{k^2}{2} \Big|_{n'/2}^{n'} \right) \\
&= \frac{1}{n'} \left(\frac{n'^2}{2} - \frac{n'^2}{8} + \frac{n'^2}{2} - \frac{n'^2}{8} \right) = \frac{3n'}{4}.
\end{aligned}
$$

- $\mathbb{E}[X_{j+1}] \leq \frac{3}{4} \mathbb{E}[X_j]$
- $X_0 = n \quad \Longrightarrow \quad \mathbb{E}[X_j] \leq \left(\frac{3}{4}\right)^j n$

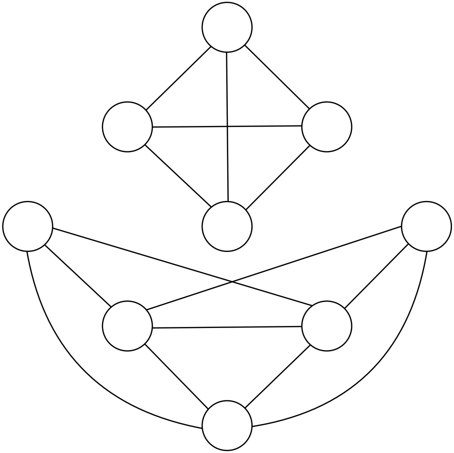
$$
\begin{aligned}
&\mathbb{E}[\text{running time of randomized selection}] \\
&\leq \mathbb{E}\left[O(1)\sum_{j=0}^{\infty} X_j\right] \leq O(1)\sum_{j=0}^{\infty}\mathbb{E}[X_j] \\
&\leq O(1)\sum_{j=0}^{\infty}\left(\frac{3}{4}\right)^j n = O(1)\cdot 4n = O(n).
\end{aligned}
$$

Outline

Global Min-Cut Problem

Input: a connected graph $G = (V, E)$

Output: the minimum number of edges whose removal will disconnect G



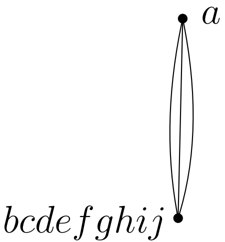
Solving Global Min-Cut Using s-t Min-Cut

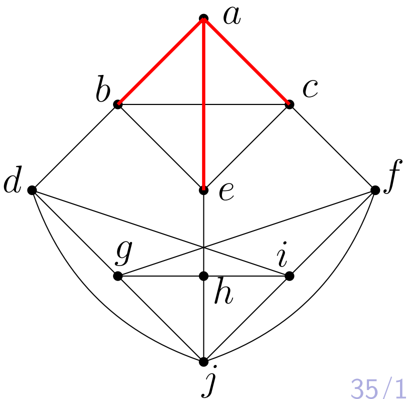
1: let G' be the directed graph obtained from G by replacing every edge with two anti-parallel edges
2: **for** a fixed $s \in V$ and every pair $t \in V \setminus \{s\}$ **do**
3: obtain the minimum cut separating s and t in G, by solving the maximum flow instance with graph G',source s and sink t
4: output the smallest minimum cut we found

- Time $= O(n) \times$ (Time for Maximum Flow)

Karger's Randomized Algorithm for Min-Cut

1: $G' = (V', E') \leftarrow G$
2: **while** $|V'| > 2$ **do**
3: pick $uv \in E'$ uniformly at random
4: contract uv in G', keeping parallel edges, but not self-loops
5: **return** the cut in G correspondent to E'





Obs. Contraction does not decrease size of min-cut.

Lemma If $G' = (V', E')$ has size of min-cut being c, then $|E'| \geq |V'|c/2$

Proof.

Every vertex will have degree at least c, and thus $2|E'| \geq |V'|c$. □

- let $C \subseteq E$ be a fixed min-cut of G
- an iteration fails if we chose some edge $e \in C$ to contract.

Coro. Focus on some iteration where we have the graph $G' = (V', E')$ with $n' = |V'|$ at the beginning. Suppose all previous iterations succeed. Then the probability this iteration fails is at most $\frac{c}{n'c/2} = \frac{2}{n'}$.

- The probability that the algorithm succeeds is at least

$$\left(1-\frac{2}{n}\right)\left(1-\frac{2}{n-1}\right)\left(1-\frac{2}{n-2}\right)\cdots\left(1-\frac{2}{3}\right)$$
$$=\frac{n-2}{n}\times\frac{n-3}{n-1}\times\frac{n-4}{n-2}\times\cdots\times\frac{1}{3}=\frac{2}{n(n-1)}$$

Coro. Any graph G has at most $\frac{n(n-1)}{2}$ distinct minimum cuts.

- $A := \frac{n(n-1)}{2}$: algorithm succeeds with probability at least $\frac{1}{A}$
- Running the algorithm for Ak times will increase the probability to

$$1-(1-\frac{1}{A})^{Ak} \geq 1-e^{-k}.$$

- To get a success probability of $1-\delta$, run the algorithm for $O(n^2 \log \frac{1}{\delta})$ times.

Equivalent Algorithm

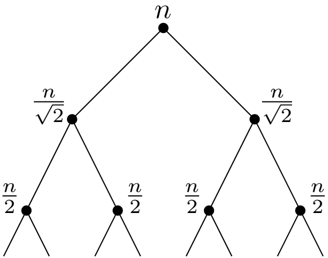
1: give every edge a weight in $[0,1]$ uniformly at random.
2: solve the MST on the graph G with the weights, using either Kruskal or Prim's algorithm
3: remove the heaviest edge in the MST,
4: let U and $V \setminus U$ be the vertex sets of two components
5: **return** the cut in G between U and $V \setminus U$

- run it once: time $= O(m + n\log n)$
- to get success probability $1-\delta$: time $= O(n^2(m + n\log n)\log\frac{1}{\delta})$

Karger-Stein: A Faster Algorithm

Karger-Stein($G = (V, E)$)

if $|V| \leq 6$ **then return** min cut of G directly
repeat twice and return the smaller cut:
    run Karger(G) down to $\lceil n/\sqrt{2} \rceil$ vertices, to obtain G'
    consider the candidate cut returned by Karger-Stein(G')



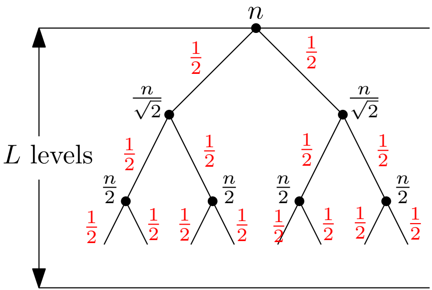
- Running time: $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2)$
- $T(n) = O(n^2 \log n)$

Karger-Stein$(G = (V, E))$

if $|V| \leq 6$ **then return** min cut of G directly
repeat twice and return the smaller cut:
    run Karger(G) down to $\lceil n/\sqrt{2} \rceil + 1$ vertices, to obtain G'
    consider the candidate cut returned by Karger-Stein(G')

Analysis of Probability of Success

- running Karger(G) down to $\lceil n/\sqrt{2} \rceil + 1$ vertices, success probability is at least
$$\frac{n-2}{n} \times \frac{n-3}{n-1} \times \cdots \times \frac{\lceil n/\sqrt{2} \rceil}{\lceil n/\sqrt{2} \rceil + 2} = \frac{(\lceil n/\sqrt{2} \rceil + 1) \lceil n/\sqrt{2} \rceil}{n(n-1)}$$
$$\geq \frac{n^2/2 + n/\sqrt{2}}{n^2 - n} \geq \frac{1}{2}$$
- recursion for Probability: $P(n) \geq 1 - \left(1 - \frac{1}{2} P(\frac{n}{\sqrt{2}})\right)^2$



- every edge is chosen w.p $1/2$
- success if we choose some root-to-leaf path
- what is the success probability in terms of L?

Lemma $P_L \geq \frac{1}{L+1}$.

Proof.

- $L = 0$: a singleton, holds trivially.
- induction:

$$P_L = 1 - \left(1 - \frac{1}{2}P_{L-1}\right)^2 \geq 1 - \left(1 - \frac{1}{2L}\right)^2 = \frac{1}{L} - \frac{1}{4L^2}$$

$$= \frac{4L-1}{4L^2} \geq \frac{1}{L+1}$$

□

Karger-Stein$(G = (V, E))$

if $|V| \leq 6$ **then return** min cut of G directly
repeat twice and return the smaller cut:
    run Karger(G) down to $\lceil n/\sqrt{2} \rceil + 1$ vertices, to obtain G'
    consider the candidate cut returned by Karger-Stein(G')

- Running time: $O(n^2 \log n)$
- Success probability: $\Omega\left(\frac{1}{\log n}\right)$
- Repeat $O(\log n)$ times can increase the probability to a constant

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Approximation Algorithms

An algorithm for an optimization problem is an α-approximation algorithm, if it runs in polynomial time, and for any instance to the problem, it outputs a solution whose cost (or value) is within an α-factor of the cost (or value) of the optimum solution.

- opt: cost (or value) of the optimum solution
- sol: cost (or value) of the solution produced by the algorithm
- α: approximation ratio
- For minimization problems:
  - $\alpha \geq 1$ and we require $\text{sol} \leq \alpha \cdot \text{opt}$
- For maximization problems, there are two conventions:
  - $\alpha \leq 1$ and we require $\text{sol} \geq \alpha \cdot \text{opt}$
  - $\alpha \geq 1$ and we require $\text{sol} \geq \text{opt}/\alpha$

Max 3-SAT

Input: n boolean variables $x_1, x_2, \cdots, x_n$

m clauses, each clause is a disjunction of 3 literals from 3 distinct variables

Output: an assignment so as to satisfy as many clauses as possible

Example:

- clauses: $x_2 \vee \neg x_3 \vee \neg x_4, \quad x_2 \vee x_3 \vee \neg x_4,$
  $\neg x_1 \vee x_2 \vee x_4, \quad x_1 \vee \neg x_2 \vee x_3, \quad \neg x_1 \vee \neg x_2 \vee \neg x_4$
- We can satisfy all the 5 clauses: $x = (1, 1, 1, 0, 1)$

Randomized Algorithm for Max 3-SAT

- Simple idea: randomly set each variable $x_u = 1$ with probability $1/2$, independent of other variables

Lemma Let m be the number of clauses. Then, in expectation, $\frac{7}{8}m$ number of clauses will be satisfied.

Proof.

- for each clause C_j, let $Z_j = 1$ if C_j is satisfied and 0 otherwise
- $Z = \sum_{j=1}^{m} Z_j$ is the total number of satisfied clauses
- $\mathbb{E}[Z_j] = 7/8$: out of 8 possible assignments to the 3 variables in C_j, 7 of them will make C_j satisfied
- $\mathbb{E}[Z] = \mathbb{E}\left[\sum_{j=1}^{m} Z_j\right] = \sum_{j=1}^{m} \mathbb{E}[Z_j] = \sum_{j=1}^{m} \frac{7}{8} = \frac{7}{8}m$, by linearity of expectation. □

Randomized Algorithm for Max 3-SAT

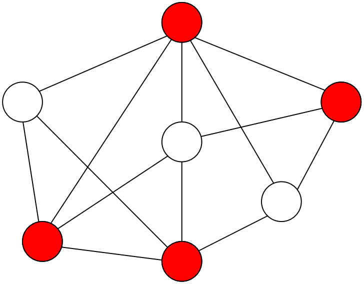
Lemma Let m be the number of clauses. Then, in expectation, $\frac{7}{8}m$ number of clauses will be satisfied.

- Since the optimum solution can satisfy at most m clauses, lemma gives a randomized $7/8$-approximation for Max-3-SAT.

Theorem ([Hastad 97]) Unless P $=$ NP, there is no ρ-approximation algorithm for MAX-3-SAT for any $\rho > 7/8$.

Outline

Outline



Vertex-Cover Problem

Input: $G = (V, E)$

Output: a vertex cover C with minimum $|C|$

- (The decision version of) vertex-cover is NP-complete.

Q: What if we are only interested in a vertex cover of size at most k, for some small number k?

Q: What if we are only interested in a vertex cover of size at most k, for some constant k?

- Motivation: if the minimum vertex cover is too big, then the solution becomes meaningless.
- Enumeration gives a $O(kn^{k+1})$-time algorithm.
- For moderately large k (e.g., $n = 1000, k = 10$), algorithm is impractical.

Lemma There is an algorithm with running time $O(2^k \cdot kn)$ to check if G contains a vertex cover of size at most k or not.

- Remark: m does not appear in the running time. Indeed, if $m > kn$, then there is no vertex cover of size k.

Vertex-Cover$(G' = (V', E'), k)$

```
if |E'| = ∅ then return true
if k = 0 then return false
pick any edge (u, v) ∈ E'
return Vertex-Cover(G' \ u, k − 1) or Vertex-Cover(G' \ v, k − 1)
```

- $G' \setminus u$: the graph obtained from G' by removing u and its incident edges
- Correctness: if $(u, v) \in E'$, we must choose u or choose v to cover (u, v).
- Running time: 2^k recursions and each recursion has running time $O(kn)$.

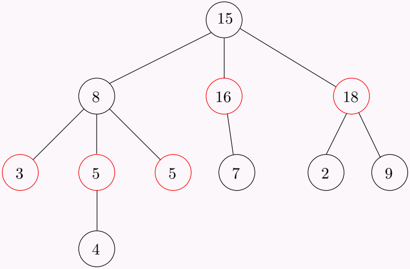
Def. An problem is fixed parameterized tractable (FPT) with respect to a parameter k, if it can be solved in $f(k) \cdot \text{poly}(n)$ time, where n is the size of its input and $\text{poly}(n) = \bigcup_{t=0}^{\infty} O(n^t)$.

- Vertex cover is fixed parameterized tractable with respect to the size k of the optimum solution.

Outline

- Many NP-hard problems on general graphs are easy on trees.
- Greedy algorithms: independent set, vertex cover, dominating set,
- Dynamic programming: weighted versions of above problems

Example: Maximum-Weight Independent Set



- dynamic programming:
- $f[i, 0]$: optimum value in tree i when i is not chosen
- $f[i, 1]$: optimum value in tree i

- Reason why many problems can be solved using DP on trees: the child-trees of a vertex i are only connected through i.

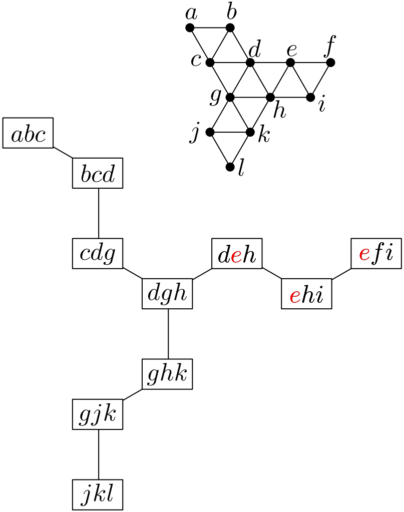
Bounded-Tree-Width Graphs

Def. A tree decomposition of a graph $G = (V, E)$ consists of

- a tree T with node set U, and
- a subset $V_t \subseteq V$ for every $t \in U$, which we call the bag for t,

satisfying the following properties:

- (Vertex Coverage) Every $v \in V$ appears in at least one bag.
- (Edge Coverage) For every $(u, v) \in E$, some bag contains both u and v.
- (Coherence) For every $u \in V$, the nodes $t \in U : u \in V_t$ induce a connected sub-graph of T.

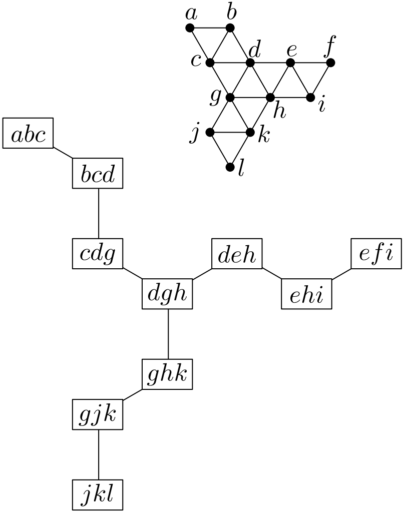


Bounded-Tree-Width Graphs

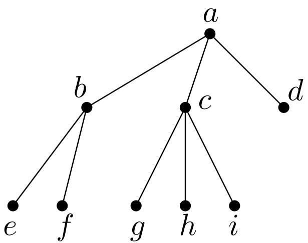
Def. The tree-width of the tree-decomposition $(T, (V_t)_{t \in U})$ is defined as $\max_{t \in U} |V_t| - 1$.

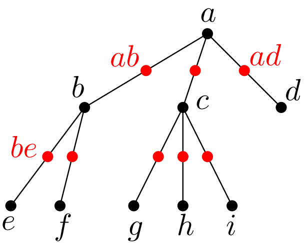
Def. The tree-width of a graph $G = (V, E)$, denoted as $\mathrm{tw}(G)$, is the minimum tree-width of a tree decomposition $(T, (V_t)_{t \in U})$ of G.

- The graph on the top right has tree-width 2.



Obs. A (non-empty) tree has tree-width 1.





Lemma A graph has tree-width 1 if and only if it is a (non-empty) forest.

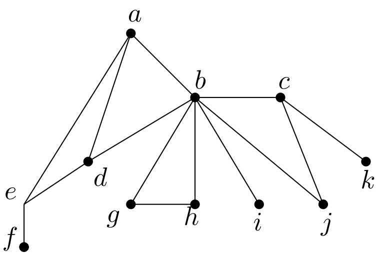
- Many problems on graphs with small tree-width can be solved using dynamic programming.
- Typically, the running time will be exponential in $\mathrm{tw}(G)$.

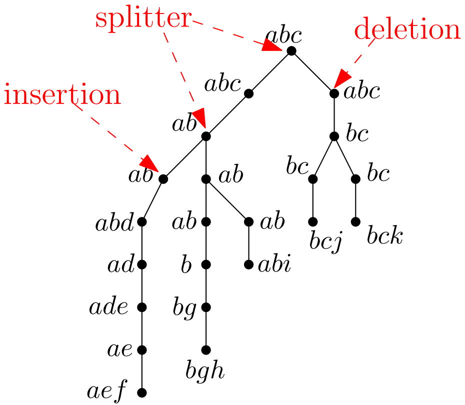
Example: Maximum Weight Independent Set

- given $G = (V, E)$, a tree-decomposition $(T, (V_t)_{t \in U})$ of G with tree-width tw.
- vertex weights $w \in \mathbb{R}^V_{>0}$.
- find an independent set S of G with the maximum total weight.

Assumption: every node in T has at most 2 children. Moreover, every internal nodes in T is one of the following types:

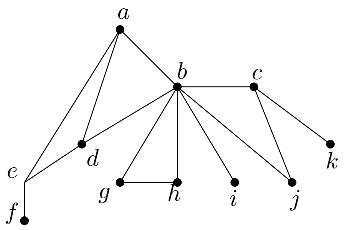
- Splitter: a node t with two children t' and t'', $V_t = V_{t'} = V_{t''}$
- Insertion node: a node t with one child t', $\exists u \notin V_t, V_{t'} = V_t \cup \{u\}$
- Deletion node: a node t with one child t', $\exists u \in V_t, V_{t'} = V_t \setminus \{u\}$

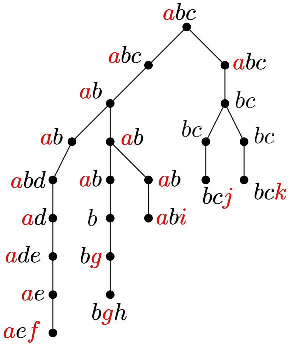




Def. Given a graph $G = (V, E)$, and a tree decomposition $(T, (V_t)_{t \in U})$, a valid labeling of T is a vector $(R_t)_{t \in U}$ of sets, one for every node t, such that the following conditions hold.

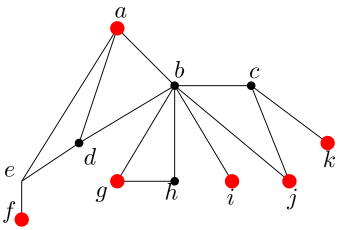
- $R_t \subseteq V_t, \forall t \in U$, and R_t is an independent set in G
- $R_t = R_{t'} = R_{t''}$ for a S-node t, and its two children t', t''.
- $R_{t'} \setminus \{u\} = R_t$ for an I-node t and its child t' with $V_{t'} = V_t \cup \{u\}$.
- $R_{t'} = R_t \setminus \{u\}$ for a D-node t and its child t' with $V_{t'} = V_t \setminus \{u\}$.

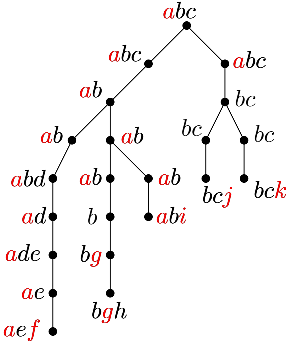




Lemma If S is an IS of G, then $(R_t := S \cap V_t)_{t \in U}$ is a valid labeling.

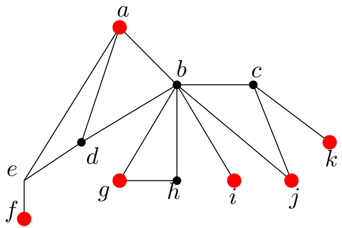
Lemma If $(R_t)_{t \in U}$ is a valid labeling, then $\bigcup_t R_t$ is an IS.

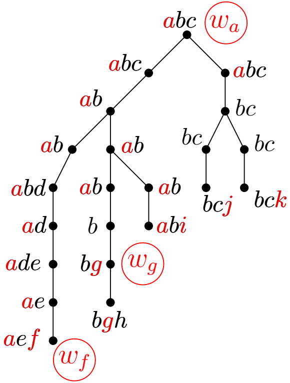




- Therefore, there is an one-to-one mapping between independent sets and valid labelings.

- For every $t \in U$, every $R \subseteq V_t$ that is an IS in G (we call R a label for t), we define a weight $w_t(R)$.
- for the root t and a label R for t, $w_t(R) = \sum_{v \in R} w_r$.
- for an insertion node t with the child t' such that $V_{t'} = V_t \cup \{u\}$, a label R for t', we define $w_{t'}(R) = w_u$ if $u \in R$ and 0 otherwise.
- For all other cases, the weights are defined as 0





- Problem: find a valid labeling for T with maximum weight

Dynamic Programming

- $\forall t \in U$, a label R for t: let $f(t, R)$ be the maximum weight of a valid (partial) labeling for the sub-tree of T rooted at t.

$$f(t,R) := \begin{cases} w_t(R) & t \text{ is a leaf} \\ w_t(R) + f(t',R) + f(t'',R) & \\ \quad t \text{ is an S-node with children } t' \text{ and } t'' & \\ w_t(R) + \max\{f(t',R), f(t', R \cup \{u\})\} & \\ \quad t \text{ is I-node w. child } t',\ V_{t'} = V_t \cup \{u\} & \\ w_t(R) + f(t', R \setminus \{u\}) & \\ \quad t \text{ is D-node w. child } t',\ V_{t'} = V_t \setminus \{u\} & \end{cases}$$

- In I-node case, if $R \cup \{u\}$ is an invalid label, then $f(t, R \cup \{u\}) = -\infty$.

- The running time of the dynamic programming: $O\big(2^{\text{tw}} \cdot \text{tw} \cdot n\big)$.
- It is efficient when tw is $O(\log n)$.

Q: Suppose we are only given G with tree-width tw, how can we find a tree-decomposition of width tw?

- This is an NP-hard problem.
- We can achieve a weaker goal: find a tree-decomposition of with at most 4tw in time $f(\text{tw}) \cdot \text{poly}(n)$, where $f(\text{tw})$ is a function of tw.
- If $\text{tw} = O(1)$, the algorithm runs in polynomial time.
- The constant 4 is acceptable.

Outline

1. Randomized Algorithms

2. Extending the Limits of Tractability

3. Approximation Algorithms using Greedy
   - 2-Approximation Algorithm for Vertex Cover
   - f-Approximation for Set-Cover with Frequency f
   - $(\ln n + 1)$-Approximation for Set-Cover
   - $(1 - \frac{1}{e})$-Approximation for Maximum Coverage

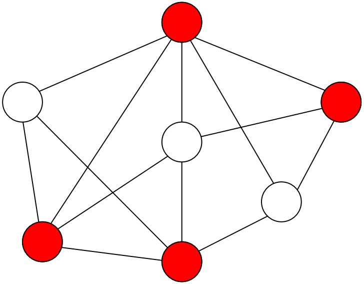
4. Arbitrarily Good Approximation Using Rounding Data

5. Approximation Using LP Rounding and Primal Dual

Outline

Vertex Cover Problem

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $C \subseteq V$ such that for every $(u, v) \in E$ then $u \in C$ or $v \in C$.



Vertex-Cover Problem

Input: $G = (V, E)$

Output: a vertex cover C with minimum $|C|$

First Try: A "Natural" Greedy Algorithm

Natural Greedy Algorithm for Vertex-Cover

$E' \leftarrow E, C \leftarrow \emptyset$
while $E' \neq \emptyset$ **do**
    let v be the vertex of the maximum degree in (V, E')
    $C \leftarrow C \cup \{v\}$,
    remove all edges incident to v from E'
return C

Theorem Greedy algorithm is an $(\ln n + 1)$-approximation for vertex-cover.

- We prove it for the more general set cover problem
- The logarithmic factor is tight for this algorithm

2-Approximation Algorithm for Vertex Cover

$E' \leftarrow E, C \leftarrow \emptyset$
while $E' \neq \emptyset$ **do**
    let (u, v) be any edge in E'
    $C \leftarrow C \cup \{u, v\}$
    remove all edges incident to u and v from E'
return C

- counter-intuitive: adding both u and v to C seems wasteful
- intuition for the 2-approximation ratio:
  - optimum solution C^* must cover edge (u, v), using either u or v
  - we select both, so we are always ahead of the optimum solution
  - we use at most 2 times more vertices than C^* does

2-Approximation Algorithm for Vertex Cover

```
1: E' ← E, C ← ∅
2: while E' ≠ ∅ do
3:     let (u, v) be any edge in E'
4:     C ← C ∪ {u, v}
5:     remove all edges incident to u and v from E'
6: return C
```

Theorem The algorithm is a 2-approximation algorithm for vertex-cover.

Proof.

- Let E' be the set of edges (u, v) considered in Step 3
- Observation: E' is a matching and $|C| = 2|E'|$
- To cover E', the optimum solution needs $|E'|$ vertices □

Outline

Set Cover with Bounded Frequency f

Input: $U, |U| = n$: ground set

$S_1, S_2, \cdots, S_m \subseteq U$

every $j \in U$ appears in at most f subsets in $\{S_1, S_2, \cdots, S_m\}$

Output: minimum size set $C \subseteq [m]$ such that $\bigcup_{i \in C} S_i = U$

Vertex Cover = Set Cover with Frequency 2

- edges $\Leftrightarrow$ elements
- vertices $\Leftrightarrow$ sets
- every edge (element) can be covered by 2 vertices (sets)

f-Approximation Algorithm for Set Cover with Frequency f

1: $C \leftarrow \emptyset$
2: **while** $\bigcup_{i \in C} S_i \neq U$ **do**
3: let e be any element in $U \setminus \bigcup_{i \in C} S_i$
4: $C \leftarrow C \cup \{i \in [m] : e \in S_i\}$
5: **return** C

Theorem The algorithm is a f-approximation algorithm.

Proof.

- Let U' be the set of all elements e considered in Step 3
- Observation: no set S_i contains two elements in U'
- To cover U', the optimum solution needs $|U'|$ sets
- $C \leq f \cdot |U'|$ □

Outline

Set Cover

Input: $U, |U| = n$: ground set

$S_1, S_2, \cdots, S_m \subseteq U$

Output: minimum size set $C \subseteq [m]$ such that $\bigcup_{i \in C} S_i = U$

Greedy Algorithm for Set Cover

```
C ← ∅, U' ← U
while U' ≠ ∅ do
    choose the i that maximizes |U' ∩ S_i|
    C ← C ∪ {i}, U' ← U' \ S_i
return C
```

$C \leftarrow \emptyset, U' \leftarrow U$

while $U' \neq \emptyset$ **do**

  choose the i that maximizes $|U' \cap S_i|$

  $C \leftarrow C \cup \{i\}, U' \leftarrow U' \setminus S_i$

return C

- g: minimum number of sets needed to cover U

Lemma Let $u_t, t \in \mathbb{Z}_{\geq 0}$ be the number of uncovered elements after t steps. Then for every $t \geq 1$, we have

$$u_t \leq \left(1 - \frac{1}{g}\right) \cdot u_{t-1}.$$

Proof.

- Consider the g sets $S_1^*, S_2^*, \cdots, S_g^*$ in optimum solution
- $S_1^* \cup S_2^* \cup \cdots \cup S_g^* = U$
- at beginning of step t, some set in $S_1^*, S_2^*, \cdots, S_g^*$ must contain $\geq \frac{u_{t-1}}{g}$ uncovered elements
- $u_t \leq u_{t-1} - \frac{u_{t-1}}{g} = \left(1 - \frac{1}{g}\right) u_{t-1}$. □

Proof of $(\ln n + 1)$-approximation.

- Let $t = \lceil g \cdot \ln n \rceil$. $u_0 = n$. Then

$$u_t \le \left(1 - \frac{1}{g}\right)^{g \cdot \ln n} \cdot n < e^{-\ln n} \cdot n = n \cdot \frac{1}{n} = 1.$$

- So $u_t = 0$, approximation ratio $\le \frac{\lceil g \cdot \ln n \rceil}{g} \le \ln n + 1$. □

- A more careful analysis gives a H_n-approximation, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the n-th harmonic number.
- $\ln(n+1) < H_n < \ln n + 1$.

$(1-c)\ln n$-hardness for any $c = \Omega(1)$

Let $c > 0$ be any constant. There is no polynomial-time $(1-c)\ln n$-approximation algorithm for set-cover, unless

- NP $\subseteq$ quasi-poly-time, [Lund, Yannakakis 1994; Feige 1998]
- P $=$ NP. [Dinur, Steuer 2014]

Outline

- set cover: use smallest number of sets to cover all elements.
- maximum coverage: use k sets to cover maximum number of elements

Maximum Coverage

Input: $U, |U| = n$: ground set,

$S_1, S_2, \cdots, S_m \subseteq U$, $\quad k \in [m]$

Output: $C \subseteq [m], |C| = k$ with the maximum $\bigcup_{i \in C} S_i$

Greedy Algorithm for Maximum Coverage

1: $C \leftarrow \emptyset, U' \leftarrow U$
2: **for** $t \leftarrow 1$ to k **do**
3: choose the i that maximizes $|U' \cap S_i|$
4: $C \leftarrow C \cup \{i\}, U' \leftarrow U' \setminus S_i$
5: **return** C

Theorem Greedy algorithm gives $(1 - \frac{1}{e})$-approximation for maximum coverage.

Proof.

- o: max. number of elements that can be covered by k sets.
- p_t: #(covered elements) by greedy algorithm after step t
- $p_t \geq p_{t-1} + \dfrac{o - p_{t-1}}{k}$
- $o - p_t \leq o - p_{t-1} - \frac{o - p_{t-1}}{k} = \left(1 - \frac{1}{k}\right)(o - p_{t-1})$
- $o - p_k \leq \left(1 - \frac{1}{k}\right)^k (o - p_0) \leq \frac{1}{e} \cdot o$
- $p_k \geq \left(1 - \frac{1}{e}\right) \cdot o$ □

Outline

Outline

Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

- Motivation: you have budget W, and want to buy a subset of items of maximum total value

Greedy Algorithm

1: sort items according to non-increasing order of v_i/w_i
2: **for** each item in the ordering **do**
3: take the item if we have enough budget

- Bad example: $W = 100, n = 2, w = (1, 100), v = (1.1, 100)$.
- Optimum takes item 2 and greedy takes item 1.

DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \cdots, i\}$.

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max \left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W' - w_i] + v_i \end{array} \right\} & i > 0, w_i \le W' \end{cases}$$

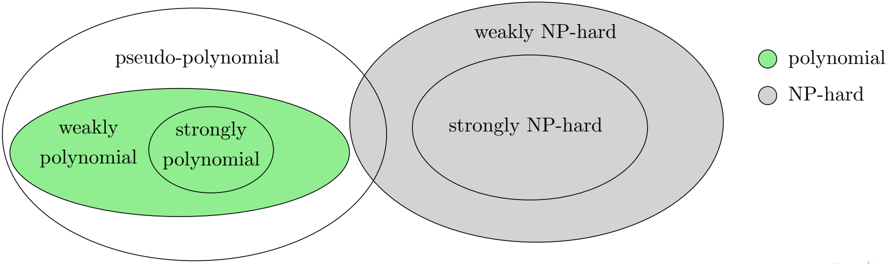
- Running time of the algorithm is $O(nW)$.

Q: Is this a polynomial time?

A: No.

- The input size is polynomial in n and $\log W$; running time is polynomial in n and W.
- The running time is pseudo-polynomial.

- n: number of integers W: maximum value of all integers
- pseudo-polynomial time: $\mathrm{poly}(n, W)$ (e.g., DP for Knapsack)
- weakly polynomial time: $\mathrm{poly}(n, \log W)$ (e.g., Euclidean Algorithm for Greatest Common Divisor)
- strongly polynomial time: $\mathrm{poly}(n)$ time, assuming basic operations on integers taking $O(1)$ time (e.g., Kruskal's)
- weakly NP-hard: NP-hard to solve in time $\mathrm{poly}(n, \log W)$
- strongly NP-hard: NP-hard even if $W = \mathrm{poly}(n)$



Idea for improving the running time to polynomial

- If we make weights upper bounded by $\text{poly}(n)$, then pseudo-polynomial time becomes polynomial time
- Coarsening the weights: $w'_i = \left\lfloor \frac{w_i}{A} \right\rfloor$ for some appropriately defined integer A.
- However, coarsening weights will change the problem.
- $$\frac{\text{weight budget constraint} \quad : \quad \text{hard}}{\text{maximum value requirement} \quad : \quad \text{soft}}$$
- We coarsen the values instead
- In the DP, we use values as parameters

- Let A be some integer to be defined later
- $v'_i := \left\lfloor \frac{v_i}{A} \right\rfloor$ be the scaled value of item i
- Definition of DP cells: $f[i, V'] = \min_{S \subseteq [i] : v'(S) \geq V'} w(S)$

$$f[i, V'] = \begin{cases} 0 & V' \leq 0 \\ \infty & i = 0, V' > 0 \\ \min \left\{ \begin{array}{c} f[i-1, V'] \\ f[i-1, V' - v'_i] + w_i \end{array} \right\} & i > 0, V' > 0 \end{cases}$$

- Output A times the largest V' such that $f[n, V'] \leq W$.

- Instance $\mathcal{I}$: $(v_1, v_2, \cdots, v_n)$ $\quad$ opt: optimum value of $\mathcal{I}$
- Instance $\mathcal{I}'$: $(Av'_1, \cdots, AV'_n)$ $\quad$ opt$'$: optimum value of $\mathcal{I}'$

$$\begin{aligned} & v_i - A < Av'_i \le v_i, \qquad \forall i \in [n] \\ \implies \quad & \mathrm{opt} - nA < \mathrm{opt}' \le \mathrm{opt} \end{aligned}$$

- $\mathrm{opt} \ge v_{\max} := \max_{i \in [n]} v_i$ (assuming $w_i \le W, \forall i$)
- setting $A := \left\lfloor \frac{\epsilon \cdot v_{\max}}{n} \right\rfloor$: $(1-\epsilon)\mathrm{opt} \le \mathrm{opt}' \le \mathrm{opt}$

- $\forall i, v'_i = O(\frac{n}{\epsilon}) \quad \implies \quad$ running time $= O(\frac{n^3}{\epsilon})$

Theorem There is a $(1+\epsilon)$-approximation for the knapsack problem in time $O(\frac{n^3}{\epsilon})$.

Def. A polynomial-time approximation scheme (PTAS) is a family of algorithms A_ϵ, where A_ϵ for every $\epsilon > 0$ is a (polynomial-time) $(1 \pm \epsilon)$-approximation algorithm.

- Remark: the approximation ratio is $1 + \epsilon$ or $1 - \epsilon$, depending on whether the problem is a minimization/maximization problem

Def. A fully polynomial-time approximation scheme (FPTAS) is an approximation scheme A_ϵ such that the running time of A_ϵ is $\mathrm{poly}(n, \frac{1}{\epsilon})$ for input instances of n.

- So, Knapsack admits an FPTAS.

Q: Assume P $\neq$ NP. What is a neccesary condition for a NP-hard problem to admit an FPTAS?

- Vertex cover? Maximum independent set?

Outline

Makespan Minimization on Identical Machines

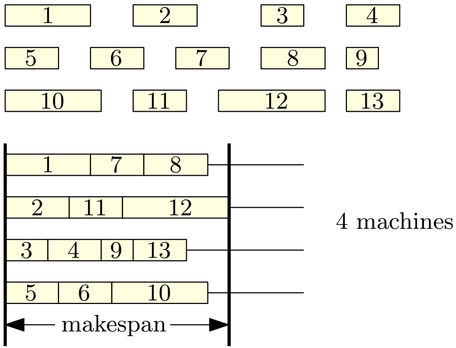
Input: n jobs index as $[n]$

each job $j \in [n]$ has a processing time $p_j \in \mathbb{Z}_{>0}$

m machines

Output: schedule of jobs on machines with minimum makespan

$\sigma : [n] \to [m]$ with minimum $\max_{i \in [m]} \sum_{j \in \sigma^{-1}(i)} p_j$



Greedy Algorithm

```
start from an empty schedule
for j = 1 to n do
    put job j on the machine with the smallest load
```

Analysis of $(2 - \frac{1}{m})$-Approximation for Greedy Algorithm

$$p_{\max} := \max_{j \in [n]} p_j$$

$$\mathrm{alg} \leq p_{\max} + \frac{1}{m} \cdot \Big(\sum_{j \in [n]} p_j - p_{\max}\Big) = \Big(1 - \frac{1}{m}\Big) p_{\max} + \frac{1}{m} \sum_{j \in [n]} p_j$$

$$\left.\begin{array}{lcl} \mathrm{opt} & \geq & p_{\max} \\ \mathrm{opt} & \geq & \frac{1}{m} \sum_{j \in [n]} p_j \end{array}\right\} \implies \quad \mathrm{alg} \leq \Big(2 - \frac{1}{m}\Big) \mathrm{opt}$$

Q: What happens if all items have size at most $\epsilon \cdot \mathrm{opt}$?

A: $\mathrm{alg} \leq \frac{1}{m}\sum_{j\in[n]} p_j + p_{\max} \leq \mathrm{opt} + \epsilon \cdot \mathrm{opt} = (1+\epsilon)\mathrm{opt}$.

Q: What can we do if all items have size at least $\epsilon \cdot \mathrm{opt}$?

A: We can round the sizes, so that #(distinct sizes) is small

Overview of Algorithm

1: declare j small if $p_j < \epsilon \cdot p_{\max}$ and big otherwise
2: use trunction + DP to solve the instance defined by big jobs
3: use DP for instance $(p'_j)_{j \text{ big}}$ to schedule big jobs
4: add small jobs to schedule greedily

Dynamic Programming for Big Jobs

- $B := \{j \in [n] : p_j \geq \epsilon p_{\max}\}$: set of big jobs
- $p'_j := \max\{\epsilon p_{\max}(1+\epsilon)^t \leq p_j : t \in \mathbb{Z}\}, \forall j \in B$

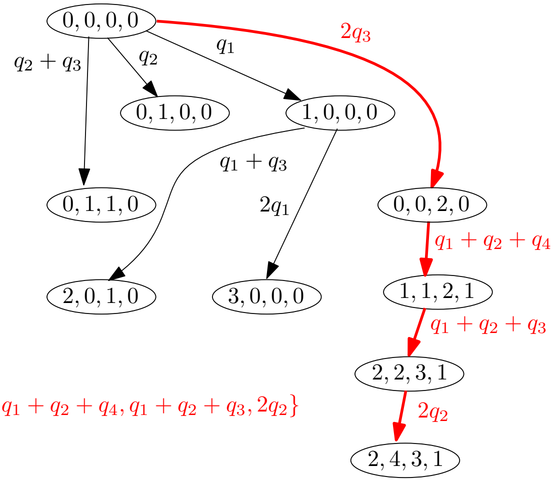
  p'_j is the rounded size of j
- $k := |\{p'_j : j \in B\}|$: #(distinct rounded sizes)

  $$k \leq 1 + \log_{1+\epsilon} \frac{p_{\max}}{\epsilon p_{\max}} = O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)$$
- $\{q_1, q_2, \cdots, q_k\} := \{p'_j : j \in B\}$: the k distinct rounded sizes
- $n_1, \cdots, n_k$: #(big jobs) with rounded sizes being $q_1, \cdots, q_k$

Constructing a Directed Acyclic Graph $G = (V, E)$

- a vertex $(a_1, \cdots, a_k)$, $a_i \in [0, n_i], \forall i \in [k]$
  - denotes the instance with a_1 jobs of size q_1, a_2 jobs of size q_2, $\cdots$, a_k jobs of size q_k
- an arc $(a_1, \cdots, a_k) \to (b_1, \cdots b_k)$ of weight $\sum_{i=1}^{k}(b_i - a_i)q_i$, if $a_i \le b_i, \forall i \in [k]$, and $a_i < b_i$ for some $i \in [k]$
  - reducing instance $(b_1, \cdots b_k)$ to $(a_1, \cdots, a_k)$ requires 1 machine of load $\sum_{i=1}^{k}(b_i - a_i)q_i$

- Goal: find a path from $(0, \cdots, 0)$ to $(n_1, \cdots, n_k)$ of at most m edges, so as to minimize the maximum weight on the path.
- problem can be solved in $O(m \cdot |E|)$ time using DP
- $O(m \cdot |E|) = O(m \cdot n^{2k}) = n^{O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)}$.



$\text{cost} = \max\{2q_3, q_1 + q_2 + q_4, q_1 + q_2 + q_3, 2q_2\}$

Analysis of Algorithm for Big Jobs

- $\mathcal{I}_B$: instance $(p_j)_{j \in B}$ $\quad \mathrm{opt}_B$: its optimum makespan
- $\mathcal{I}'_B$: instance $(p'_j)_{j \in B}$ $\quad \mathrm{opt}'_B$: its optimum makespan
- $\mathrm{opt}'_B \leq \mathrm{opt}_B$
- schedule for $\mathcal{I}'_B \Rightarrow$ schedule for $\mathcal{I}_B$:

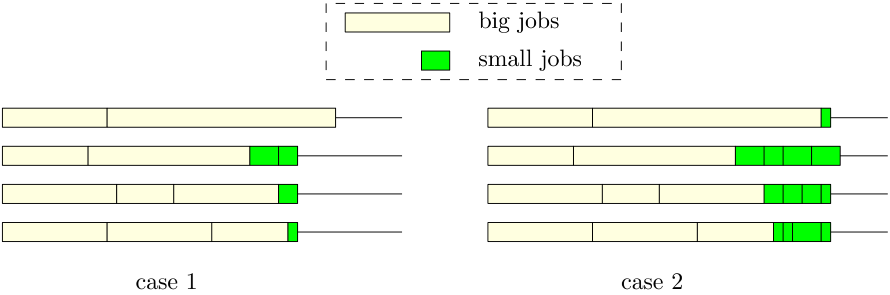
  $(1+\epsilon)$-blowup in makespan

Theorem The dynamic programming algorithm gives a schedule of makespan at most $(1+\epsilon)\mathrm{opt}_B$ in time $n^{O\left(\frac{1}{\epsilon}\log\frac{1}{\epsilon}\right)}$.

Adding small jobs to schedule

1: starting from the schedule for big jobs
2: **for** every small job j **do**
3: add j to the machine with the smallest load

Analysis of the Final Algorithm



- Case 1: makespan is not increased by small jobs

$$\text{alg} \leq (1+\epsilon)\text{opt}_B \leq (1+\epsilon)\text{opt}.$$

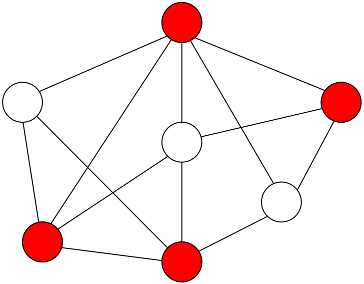
- Case 2: makespan is increased by small jobs
  - loads between any two machines differ by at most size of a small job, which is at most $\epsilon \cdot p_{\max}$

$$\text{alg} \leq \epsilon \cdot p_{\max} + \frac{1}{m}\sum_{j\in[n]} p_j \leq \epsilon \cdot \text{opt} + \text{opt} = (1+\epsilon)\cdot \text{opt}.$$

Outline

Outline

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$.



Weighted Vertex-Cover Problem

Input: $G = (V, E)$ with vertex weights $\{w_v\}_{v \in V}$

Output: a vertex cover S with minimum $\sum_{v \in S} w_v$

Integer Programming for Weighted Vertex Cover

- For every $v \in V$, let $x_v \in \{0, 1\}$ indicate whether we select v in the vertex cover S
- The integer programming for weighted vertex cover:

$$
\begin{aligned}
(\text{IP}_{\text{WVC}}) \qquad \min \quad & \sum_{v \in V} w_v x_v \qquad \text{s.t.} \\
x_u + x_v \geq 1 & \qquad \forall (u, v) \in E \\
x_v \in \{0, 1\} & \qquad \forall v \in V
\end{aligned}
$$

- $(\text{IP}_{\text{WVC}}) \Leftrightarrow$ weighted vertex cover
- Thus it is NP-hard to solve integer programmings in general

- Integer programming for WVC:

$$(\text{IP}_{\text{WVC}}) \qquad \min \quad \sum_{v \in V} w_v x_v \qquad \text{s.t.}$$
$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$
$$x_v \in \{0,1\} \qquad \forall v \in V$$

- Linear programming relaxation for WVC:

$$(\text{LP}_{\text{WVC}}) \qquad \min \quad \sum_{v \in V} w_v x_v \qquad \text{s.t.}$$
$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$
$$x_v \in [0,1] \qquad \forall v \in V$$

- let IP $=$ value of (IP_{WVC}), LP $=$ value of (LP_{WVC})
- Then, LP $\leq$ IP

Algorithm for Weighted Vertex Cover

Algorithm for Weighted Vertex Cover

1: Solving (LP_{WVC}) to obtain a solution $\{x^*_u\}_{u \in V}$
2: Thus, $\text{LP} = \sum_{u \in V} w_u x^*_u \leq \text{IP}$
3: Let $S = \{u \in V : x_u \geq 1/2\}$ and output S

Lemma S is a vertex cover of G.

Proof.

- Consider any edge $(u, v) \in E$: we have $x^*_u + x^*_v \geq 1$
- Thus, either $x^*_u \geq 1/2$ or $x^*_v \geq 1/2$
- Thus, either $u \in S$ or $v \in S$. □

Algorithm for Weighted Vertex Cover

Algorithm for Weighted Vertex Cover

1: Solving (LP_{WVC}) to obtain a solution $\{x^*_u\}_{u \in V}$
2: Thus, $\text{LP} = \sum_{u \in V} w_u x^*_u \leq \text{IP}$
3: Let $S = \{u \in V : x_u \geq 1/2\}$ and output S

Lemma S is a vertex cover of G.

Lemma $\text{cost}(S) := \sum_{u \in S} w_u \leq 2 \cdot \text{LP}$.

Proof.

$$\text{cost}(S) = \sum_{u \in S} w_u \leq \sum_{u \in S} w_u \cdot 2x^*_u = 2 \sum_{u \in S} w_u \cdot x^*_u$$
$$\leq 2 \sum_{u \in V} w_u \cdot x^*_u = 2 \cdot \text{LP}.$$

□

Algorithm for Weighted Vertex Cover

Algorithm for Weighted Vertex Cover

1: Solving (LP_{WVC}) to obtain a solution $\{x_u^*\}_{u \in V}$
2: Thus, $\text{LP} = \sum_{u \in V} w_u x_u^* \leq \text{IP}$
3: Let $S = \{u \in V : x_u^* \geq 1/2\}$ and output S

Lemma S is a vertex cover of G.

Lemma $\text{cost}(S) := \sum_{u \in S} w_u \leq 2 \cdot \text{LP}$.

Theorem Algorithm is a 2-approximation algorithm for WVC.

Proof.

$\text{cost}(S) \leq 2 \cdot \text{LP} \leq 2 \cdot \text{IP} = 2 \cdot \text{cost(best vertex cover)}$. □

Outline

LP Relaxation

$$\min \sum_{v \in V} w_v x_v$$

$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$

$$x_v \geq 0 \qquad \forall v \in V$$

Dual LP

$$\max \sum_{e \in E} y_e$$

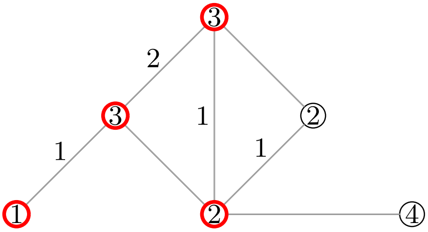
$$\sum_{e \in \delta(v)} y_e \leq w_v \qquad \forall v \in V$$

$$y_e \geq 0 \qquad \forall e \in E$$

- Algorithm constructs integral primal solution x and dual solution y simultaneously.

Primal-Dual Algorithm for Weighted Vertex Cover Problem

$x \leftarrow 0, y \leftarrow 0$, all edges said to be uncovered
while there exists at least one uncovered edge **do**
    take such an edge e arbitrarily
    increasing y_e until the dual constraint for one end-vertex v of e becomes tight
    $x_v \leftarrow 1$, claim all edges incident to v are covered
return x



Lemma

1. x satisfies all primal constraints
2. y satisfies all dual constraints
3. $P \le 2D \le 2D^* \le 2 \cdot \text{opt}$

   $P := \sum_{v \in V} x_v$: value of x

   $D := \sum_{e \in E} y_e$: value of y

   D^*: dual LP value

Proof of $P \leq 2D$.

$$P = \sum_{v\in V} w_v x_v \leq \sum_{v\in V} x_v \sum_{e\in\delta(v)} y_e = \sum_{(u,v)\in E} y_{(u,v)}(x_u + x_v)$$

$$\leq 2\sum_{e\in E} y_e = 2D.$$

□

- a more general framework: construct an arbitrary maximal dual solution y; choose the vertices whose dual constraints are tight
- y is maximal: increasing any coordinate y_e makes y infeasible

- primal-dual algorithms do not need to solve LPs
- LPs are used in analysis only
- faster than LP-rounding algorithm in general

Outline

1. Randomized Algorithms
2. Extending the Limits of Tractability
3. Approximation Algorithms using Greedy
4. Arbitrarily Good Approximation Using Rounding Data
5. Approximation Using LP Rounding and Primal Dual
   - 2-Approximation Algorithm for (Weighted) Vertex Cover Via Linear Programming
   - 2-Approximation Algorithm for Weighted Vertex Cover Using Primal-Dual
   - 2-Approximation Algorithm for Unrelated Machine Scheduling

Unrelated Machine Scheduling

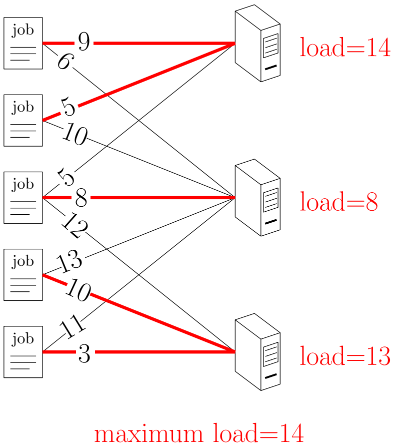
Input: $J, |J| = n$: jobs

$M, |M| = m$: machines

p_{ij}: processing time of job j on machine i

Output: assignment $\sigma : J \mapsto M$:, so as to minimize makespan:

$$\max_{i \in M} \sum_{j \in \sigma^{-1}(i)} p_{ij}$$

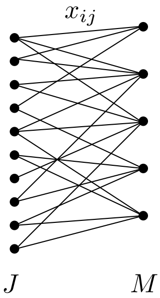


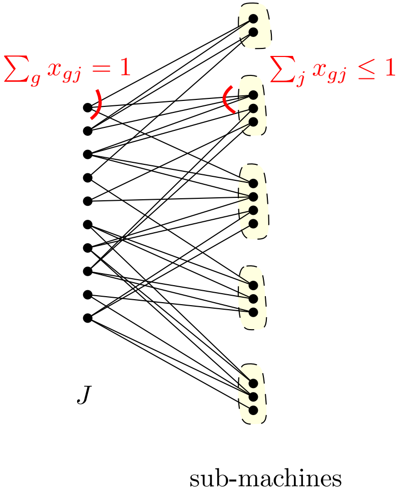
- Assumption: we are given a target makespan T, and $p_{ij} \in [0, T] \cup \{\infty\}$

- x_{ij}: fraction of j assigned to i

$$
\begin{aligned}
\sum_i x_{ij} &= 1 && \forall j \in J \\
\sum_j p_{ij} x_{ij} &\le T && \forall i \in M \\
x_{ij} &\ge 0 && \forall ij
\end{aligned}
$$

2-Approximate Rounding Algorithm of Shmoys-Tardos



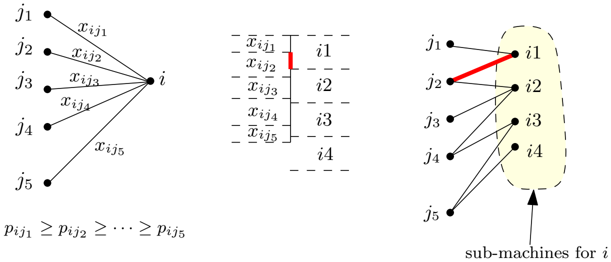


Obs. x between J and sub-machines is a point in the

- Recall bipartite matching polytope is integral.
- x is a convex combination of matchings.
- Any matching in the combination covers all jobs J.

Lemma Any matching in the combination gives an schedule of makespan $\leq 2T$.

Lemma Any matching in the combination gives an schedule of makespan $\leq 2T$.



Proof.

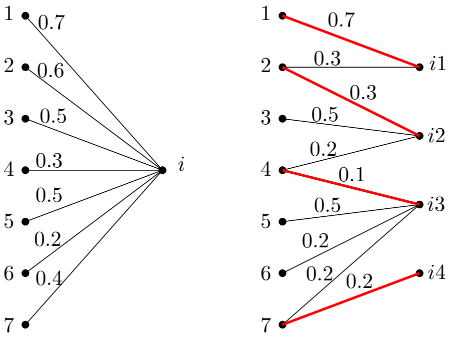
- focus on machine i, let $i_1, i_2, \cdots, i_a$ be the sub-machines for i
- assume job k_t is assigned to sub-machine i_t.

$$\text{(load on } i\text{)} = \sum_{t=1}^{a} p_{ik_t} \leq p_{ik_1} + \sum_{t=2}^{a} \sum_{j} x_{i_{t-1}j} \cdot p_{ij}$$

$$\leq p_{ik_1} + \sum_{j} x_{ij} p_{ij} \leq T + T = 2T.$$

□

- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:
  - $1 \to i1, 2 \to i2$
  - $4 \to i3, 7 \to i4$



$p_1 \leq T$

$p_2 \leq 0.7p_1 + 0.3p_2$

$p_4 \leq 0.3p_2 + 0.5p_3 + 0.2p_4$

$p_7 \leq 0.1p_4 + 0.5p_5 + 0.2p_6 + 0.2p_7$

$$p_1 + p_2 + p_4 + p_7 \leq T + (0.7p_1 + 0.3p_2) + (0.3p_2 + 0.5p_3 + 0.2p_4) + (0.1p_4 + 0.5p_5 + 0.2p_6 + 0.2p_7)$$

$$\leq T + (0.7p_1 + 0.6p_2 + 0.5p_3 + 0.3p_4 + 0.5p_5 + 0.2p_6 + 0.4p_7)$$

$$\leq T + T = 2T$$