算法设计与分析(2025年春季学期)
Advanced Topics

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Outline

1. Randomized Algorithms
   - Freivald's matrix multiplication verification algorithm
   - Randomized Select and Quicksort
   - Randomized Algorithm for Global Min-Cut
   - $\frac{7}{8}$-Approximation Algorithm for Max 3-SAT
2. Extending the Limits of Tractability
3. Approximation Algorithms using Greedy
4. Arbitrarily Good Approximation Using Rounding Data
5. Approximation Using LP Rounding and Primal Dual

Why do we use randomized algorithms?

- simpler algorithms: quick-sort, minimum-cut, and Max 3-SAT.
- faster algorithms: polynomial identity testing, Freivald's matrix multiplication verification algorithm, sampling and fingerprinting.
- mathematical beauty: Nash equilibrium for 0-sum game
- proof of existence of objects: union bound, Lovasz local lemma.

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Price of using randomness

- The algorithm may be incorrect with some probability (Monto Carlo Algorithm)
- The algorithm may take a long time to terminate (Las Vegas Algorithm)

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  - Best known algorithm for matrix multiplication: $O(n^{2.3719})$.
- Freivald's algorithm: randomized algorithm with $O(n^2)$ time.

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Analysis of correctness

- $AB = C$: algorithm outputs true with probability 1.
- $AB \neq C$: algorithm may incorrectly output true.

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Proof.

- $D := C - AB \neq 0$ $\qquad Cr = ABr \iff Dr = 0$
- $\exists i, j \in [n], D_{i,j} \neq 0$

$$D_i r = \sum_{j'=1}^{n} D_{i,j'} r_{j'} = X + Y, \quad X = \sum_{j' \in [n], j' \neq j} D_{i,j'} r_{j'}, Y = D_{i,j} r_j$$

$$\begin{aligned} \Pr[D_i r \neq 0] &= \Pr[Y \neq -X] \\ &= \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \Pr[Y \neq -x | X = x] \\ &= \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \Pr[D_{i,j} r_j \neq -x | X = x] \\ &\geq \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

□

- probabilities:

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Freivald's Algorithm: k experiments

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for t ← 1 to k do
    randomly choose a vector r ∈ {0,1}^n
    if ABr ≠ Cr then return false
return true
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- to achieve δ probability of mistake, need $\log_2 \frac{1}{\delta} = O(\log \frac{1}{\delta})$ experiments.

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Def. A Monta Carlo algorithm is a randomized algorithm whose output may be incorrect with some probability.

- For a Monta Carlo algorithm that outputs true/false, we say the algorithm has one-sided error if it makes error only if the correct output is true (or false).

Outline

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

29	82	75	64	38	45	94	69	25	76	15	92	37	17	85

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Quicksort

quicksort(A, n)

1: if $n \le 1$ then return A
2: $x \leftarrow$ lower median of A
3: $A_L \leftarrow$ elements in A that are less than x \\ Divide
4: $A_R \leftarrow$ elements in A that are greater than x \\ Divide
5: $B_L \leftarrow$ quicksort$(A_L, A_L.\text{size})$ \\ Conquer
6: $B_R \leftarrow$ quicksort$(A_R, A_R.\text{size})$ \\ Conquer
7: $t \leftarrow$ number of times x appear A
8: return the array obtained by concatenating B_L, the array containing t copies of x, and B_R

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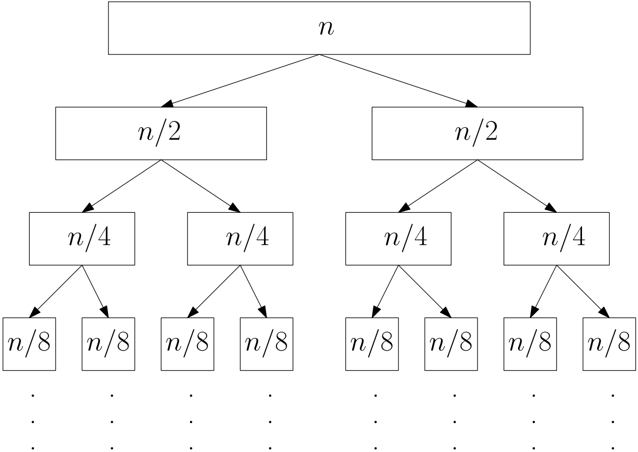
- Recurrence $T(n) \le 2T(n/2) + O(n)$
- Running time $= O(n \log n)$



- Each level has total running time $O(n)$
- Number of levels $= O(\log n)$
- Total running time $= O(n \log n)$

Randomized Quicksort Algorithm

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Variant of Randomized Quicksort Algorithm

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until A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$
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Analysis of Variant

1: $x \leftarrow$ a random element of A
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Q: What is the probability that A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$?

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A: At least $1/2$

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Q: What is the expected number of iterations the above procedure takes?

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Q: What is the expected number of iterations the above procedure takes?

A: At most 2

- Suppose an experiment succeeds with probability $p \in (0, 1]$, independent of all previous experiments.

```
repeat
    run an experiment
until the experiment succeeds
```

Lemma The expected number of experiments we run in the above procedure is $1/p$.

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Proof

$$\begin{aligned}
\text{Expectation} &= p + (1-p)p \times 2 + (1-p)^2 p \times 3 + (1-p)^3 p \times 4 \\
&\quad + \cdots \\
&= p \sum_{i=1}^{\infty} (1-p)^{i-1} i \quad = \quad p \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} (1-p)^{i-1} \\
&= p \sum_{j=1}^{\infty} (1-p)^{j-1} \frac{1}{1-(1-p)} \quad = \quad \sum_{j=1}^{\infty} (1-p)^{j-1} \\
&= (1-p)^0 \frac{1}{1-(1-p)} = 1/p
\end{aligned}$$

Variant Randomized Quicksort Algorithm

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repeat
    x ← a random element of A (x is called a pivot)
    A_L ← elements in A that are less than x          \\ Divide
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until A_L.size ≤ 3n/4 and A_R.size ≤ 3n/4
B_L ← quicksort(A_L, A_L.size)                        \\ Conquer
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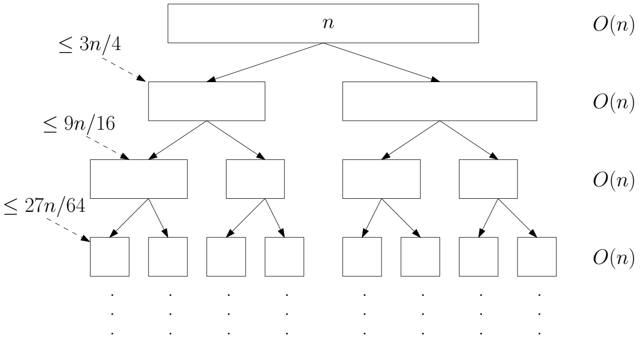
Analysis of Variant

- Divide and Combine: takes $O(n)$ time
- Conquer: break an array of size n into two arrays, each has size at most $3n/4$. Recursively sort the 2 sub-arrays.



- Number of levels $\leq \log_{4/3} n = O(\log n)$

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- Intuition: the quicksort algorithm should be better than the variant.

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- Overall, we have

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- Can prove $T(n) \le c(n\log n)$ for some constant c by reduction

Analysis of Randomized Quicksort Algorithm

The induction step of the proof:

$$
\begin{aligned}
T(n) &\le \frac{2}{n}\sum_{i=0}^{n-1} T(i) + c'n \le \frac{2}{n}\sum_{i=0}^{n-1} ci\log i + c'n \\
&\le \frac{2c}{n}\left(\sum_{i=0}^{\lfloor n/2\rfloor -1} i\log\frac{n}{2} + \sum_{i=\lfloor n/2\rfloor}^{n-1} i\log n\right) + c'n \\
&\le \frac{2c}{n}\left(\frac{n^2}{8}\log\frac{n}{2} + \frac{3n^2}{8}\log n\right) + c'n \\
&= c\left(\frac{n}{4}\log n - \frac{n}{4} + \frac{3n}{4}\log n\right) + c'n \\
&= cn\log n - \frac{cn}{4} + c'n \le cn\log n \qquad \text{if } c \ge 4c'
\end{aligned}
$$

Indirect Analysis Using Number of Comparisons

- Running time $= O(\text{number of comparisons})$
- $\forall 1 \leq i < j \leq n$, $D_{i,j}$ indicates if we compared the i-th smallest element with the j-th smallest element
- number of comparisons $= \sum_{1 \leq i < j \leq n} D_{i,j}$

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Lemma $\mathbb{E}[D_{i,j}] = \frac{2}{j-i+1}$.

Proof.

- A': sorted array for A. Focus on $A'[i..j]$.
- pivot outside $A'[i]$: $A'[i \cdots j]$ will be passed to left or right recursion; go to that recursion
- pivot inside $A'[i]$: $A'[i]$ and $A'[j]$ will be separated; call this critical recursion
- $A[i]$ and $A[j]$ are compared in the critical recursion with probability $\frac{2}{j-i+1}$.

□

$$
\begin{aligned}
\mathbb{E}\left[\text{number of comparisons}\right] &= \mathbb{E}\left[\sum_{1\le i<j\le n} D_{i,j}\right] \\
&= \sum_{1\le i<j\le n} \mathbb{E}\left[D_{i,j}\right] = 2\sum_{1\le i<j\le n}\frac{1}{j-i+1} \\
&\le 2n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) \\
&= \Theta\left(n\log n\right).
\end{aligned}
$$

- The algorithm is a Las-Vegas algorithm:

Def. A Las-Vegas algorithm is a randomized algorithm that always outputs a correct solution but has randomized running time.

Table: Comparisons between Monta Carlo and Las Vegas Algorithms.

	correctness	running time
Monta Carlo	may be wrong	usually has good worst-case running time
Las Vegas	always correct	may take a long time and usually only has good "expected running time"

Lemma Given a Las Vegas algorithm $\mathcal{A}$ with expected running time at most $T(n)$, we can design a Monta Carlo algorithm $\mathcal{A}'$ with worst-case running time $O(T(n))$ and error at most 0.99.

- 0.99 can be changed to any $c < 1$

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Proof.

- run $\mathcal{A}$ for $100T(n)$ time
- if $\mathcal{A}$ terminated, output what $\mathcal{A}$ outputs
- otherwise, declare failure
- Markov Inequality:
  $\Pr[\mathcal{A}$ runs for more than $100T(n)$ time$] \leq 1/100$ □

Randomized Selection Algorithm

selection(A, n, i)

if $n = 1$ **then return** A
$x \leftarrow$ random element of A (called pivot)
$A_L \leftarrow$ elements in A that are less than x ▷ Divide
$A_R \leftarrow$ elements in A that are greater than x ▷ Divide
if $i \leq A_L$.size **then**
    return selection$(A_L, A_L.\text{size}, i)$ ▷ Conquer
else if $i > n - A_R$.size **then**
    return selection$(A_R, A_R.\text{size}, i - (n - A_R.\text{size}))$ ▷ Conquer
else
    return x

- expected running time $= O(n)$

Randomized Selection

- $X_j, j = 0, 1, 2, \cdots$: the size of A in the j-th recursion

$$
\begin{aligned}
\mathbb{E}[X_{j+1}|X_j = n'] &\leq \frac{1}{n'} \sum_{k=1}^{n'} \max\{k-1, n'-k\} \\
&\leq \frac{1}{n'} \left(\int_{k=0}^{n'/2} (n'-k)\mathsf{d}k + \int_{k=n'/2}^{n'} k\mathsf{d}k \right) \\
&= \frac{1}{n'} \left(\left(n'k - \frac{k^2}{2} \right) \Big|_0^{n'/2} + \frac{k^2}{2} \Big|_{n'/2}^{n'} \right) \\
&= \frac{1}{n'} \left(\frac{n'^2}{2} - \frac{n'^2}{8} + \frac{n'^2}{2} - \frac{n'^2}{8} \right) = \frac{3n'}{4}.
\end{aligned}
$$

- $\mathbb{E}[X_{j+1}] \leq \frac{3}{4}\,\mathbb{E}[X_j]$
- $X_0 = n \quad \Longrightarrow \quad \mathbb{E}[X_j] \leq \left(\frac{3}{4}\right)^j n$

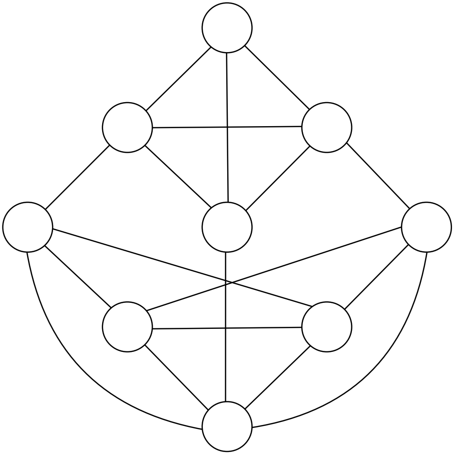
$$
\begin{aligned}
&\mathbb{E}[\text{running time of randomized selection}] \\
&\le \mathbb{E}\left[O(1)\sum_{j=0}^{\infty} X_j\right] \le O(1)\sum_{j=0}^{\infty}\mathbb{E}[X_j] \\
&\le O(1)\sum_{j=0}^{\infty}\left(\frac{3}{4}\right)^j n = O(1)\cdot 4n = O(n).
\end{aligned}
$$

Outline

Global Min-Cut Problem

Input: a connected graph $G = (V, E)$

Output: the minimum number of edges whose removal will disconnect G



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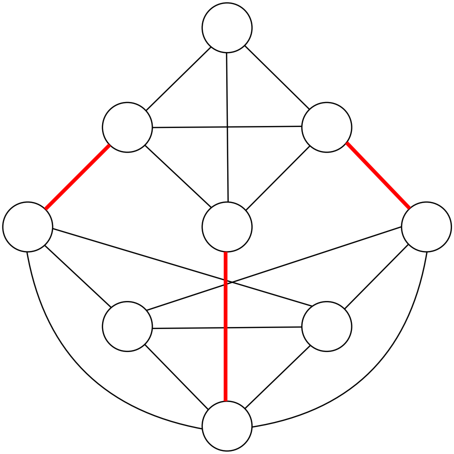

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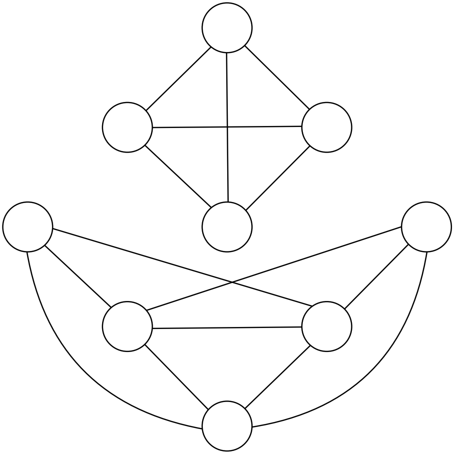

Solving Global Min-Cut Using s-t Min-Cut

let G' be the directed graph obtained from G by replacing every edge with two anti-parallel edges
for a fixed $s \in V$ and every pair $t \in V \setminus \{s\}$ **do**
    obtain the minimum cut separating s and t in G, by solving the maximum flow instance with graph G',source s and sink t
output the smallest minimum cut we found

- Time $= O(n) \times$ (Time for Maximum Flow)

Karger's Randomized Algorithm for Min-Cut

$G' = (V', E') \leftarrow G$
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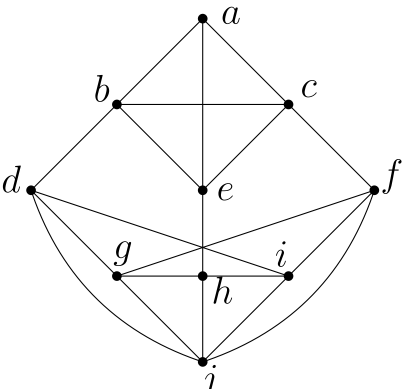

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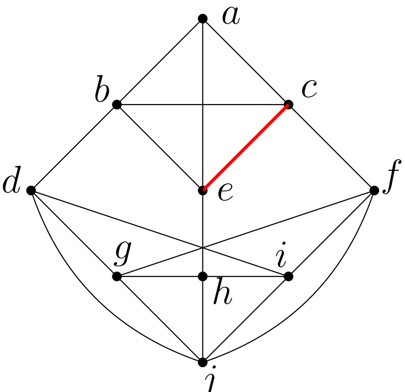

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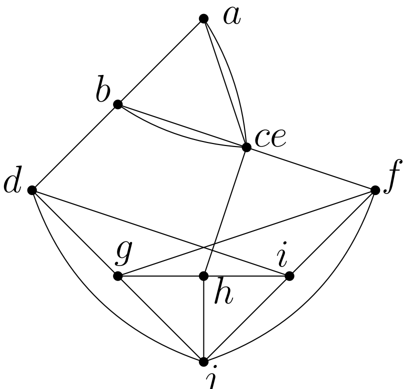

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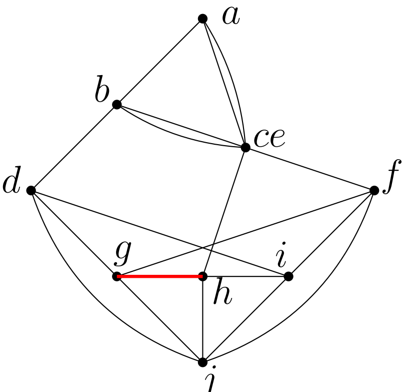

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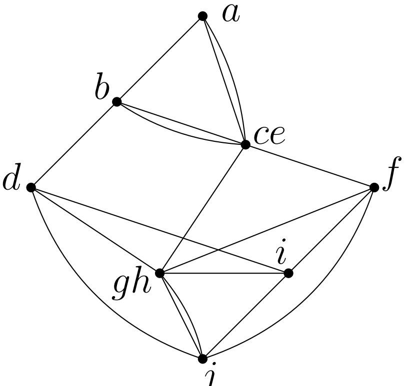

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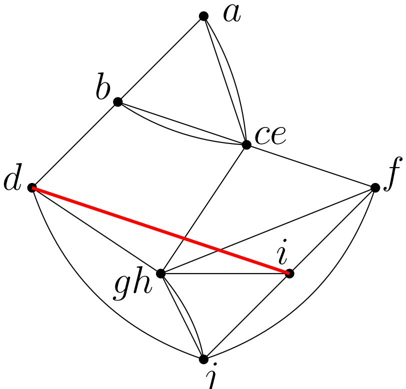

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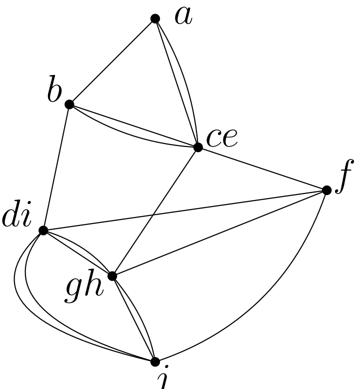

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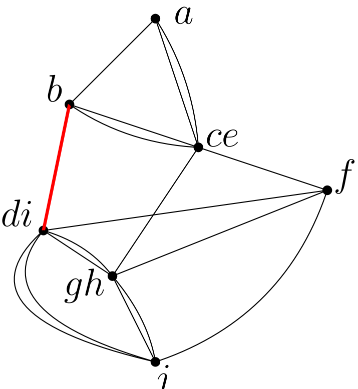

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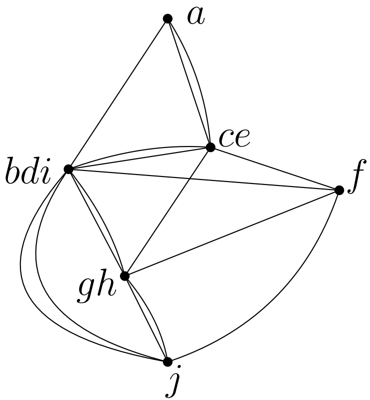

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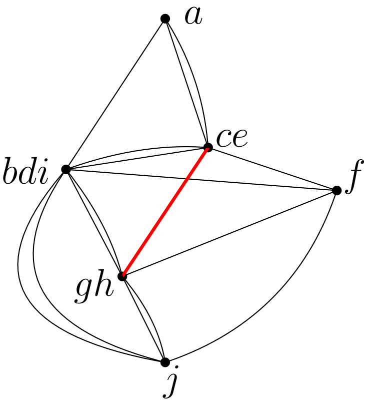

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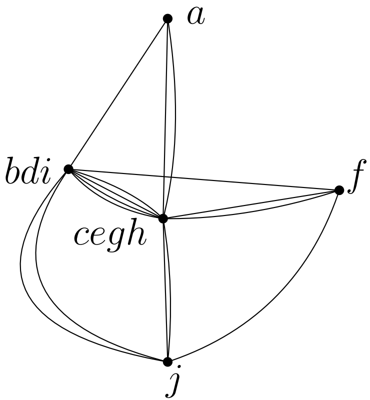

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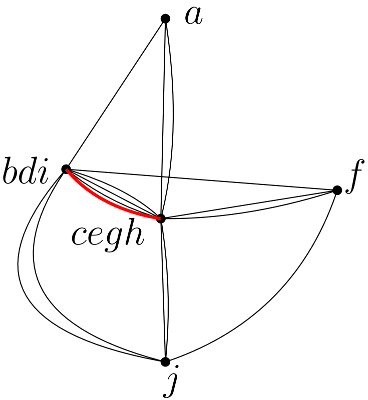

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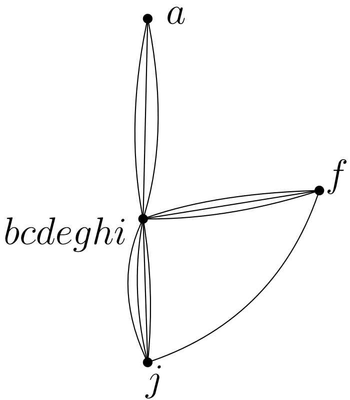

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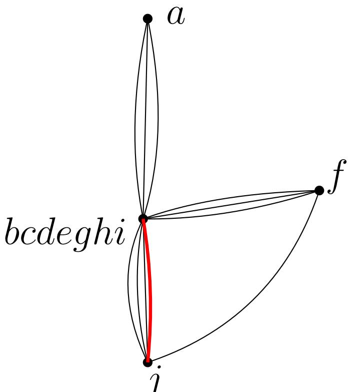

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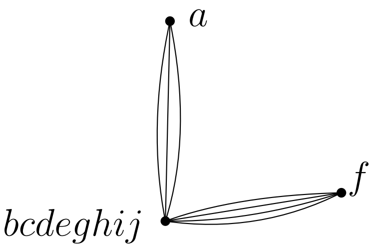

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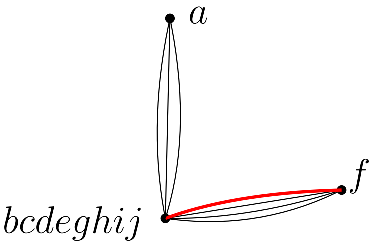

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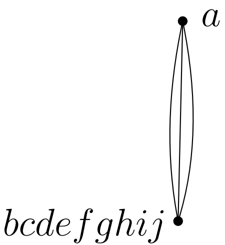

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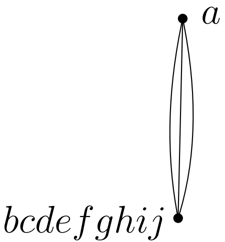

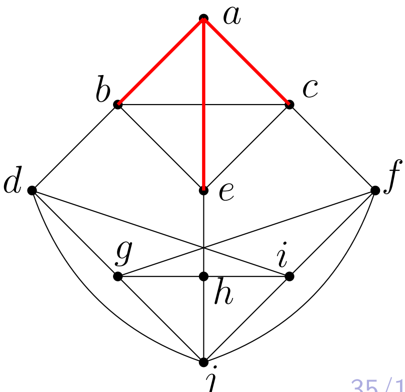

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Coro. Focus on some iteration where we have the graph $G' = (V', E')$ with $n' = |V'|$ at the beginning. Suppose all previous iterations succeed. Then the probability this iteration fails is at most $\frac{c}{n'c/2} = \frac{2}{n'}$.

- The probability that the algorithm succeeds is at least

$$\left(1-\frac{2}{n}\right)\left(1-\frac{2}{n-1}\right)\left(1-\frac{2}{n-2}\right)\cdots\left(1-\frac{2}{3}\right)$$
$$=\frac{n-2}{n}\times\frac{n-3}{n-1}\times\frac{n-4}{n-2}\times\cdots\times\frac{1}{3}=\frac{2}{n(n-1)}$$

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- $A := \frac{n(n-1)}{2}$: algorithm succeeds with probability at least $\frac{1}{A}$
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- To get a success probability of $1-\delta$, run the algorithm for $O(n^2 \log \frac{1}{\delta})$ times.

Equivalent Algorithm

1: give every edge a weight in $[0, 1]$ uniformly at random.
2: solve the MST on the graph G with the weights, using either Kruskal or Prim's algorithm
3: remove the heaviest edge in the MST,
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- run it once: time $= O(m + n \log n)$
- to get success probability $1 - \delta$: time $= O(n^2(m + n \log n) \log \frac{1}{\delta})$

Karger-Stein: A Faster Algorithm

Karger-Stein$(G = (V, E))$

if $|V| \le 6$ **then return** min cut of G directly
repeat twice and return the smaller cut:
    run Karger(G) down to $\lceil n/\sqrt{2} \rceil$ vertices, to obtain G'
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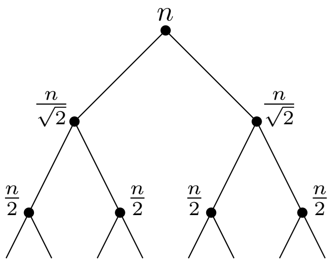

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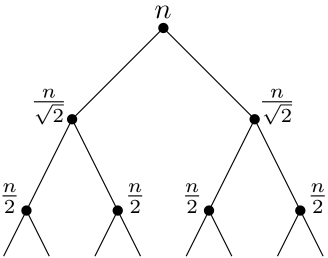

- Running time:
  $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2)$
- $T(n) = O(n^2 \log n)$

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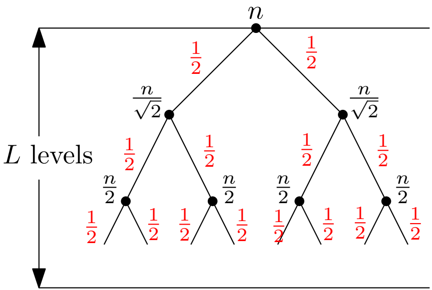
- recursion for Probability: $P(n) \ge 1 - \left(1 - \frac{1}{2} P(\frac{n}{\sqrt{2}})\right)^2$



- every edge is chosen w.p $1/2$
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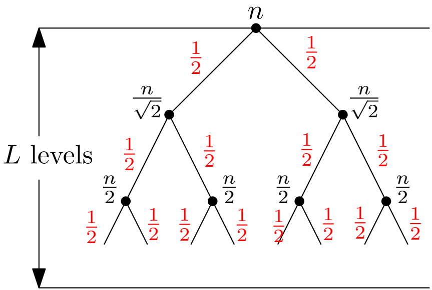

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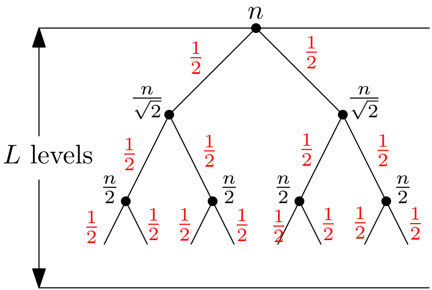

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Proof.

- $L = 0$: a singleton, holds trivially.
- induction:

$$P_L = 1 - \left(1 - \frac{1}{2}P_{L-1}\right)^2 \geq 1 - \left(1 - \frac{1}{2L}\right)^2 = \frac{1}{L} - \frac{1}{4L^2}$$
$$= \frac{4L-1}{4L^2} \geq \frac{1}{L+1}$$

□

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Outline

1. Randomized Algorithms
   - Freivald's matrix multiplication verification algorithm
   - Randomized Select and Quicksort
   - Randomized Algorithm for Global Min-Cut
   - $\frac{7}{8}$-Approximation Algorithm for Max 3-SAT

2. Extending the Limits of Tractability

3. Approximation Algorithms using Greedy

4. Arbitrarily Good Approximation Using Rounding Data

5. Approximation Using LP Rounding and Primal Dual

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Max 3-SAT

Input: n boolean variables $x_1, x_2, \cdots, x_n$

m clauses, each clause is a disjunction of 3 literals from 3 distinct variables

Output: an assignment so as to satisfy as many clauses as possible

Example:

- clauses: $x_2 \vee \neg x_3 \vee \neg x_4, \quad x_2 \vee x_3 \vee \neg x_4,$ $\neg x_1 \vee x_2 \vee x_4, \quad x_1 \vee \neg x_2 \vee x_3, \quad \neg x_1 \vee \neg x_2 \vee \neg x_4$
- We can satisfy all the 5 clauses: $x = (1, 1, 1, 0, 1)$

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Theorem ([Hastad 97]) Unless P $=$ NP, there is no ρ-approximation algorithm for MAX-3-SAT for any $\rho > 7/8$.

Outline

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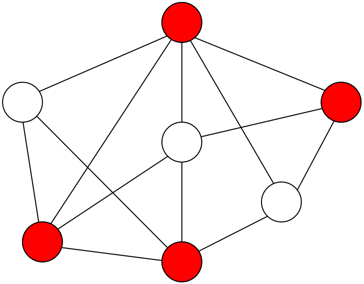
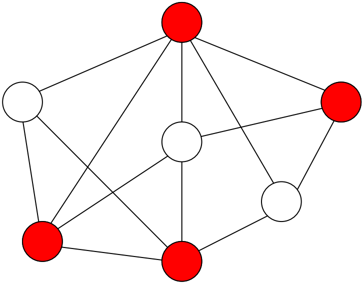
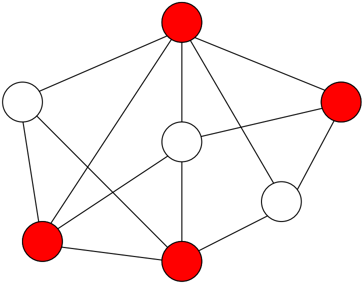




Vertex-Cover Problem

Input: $G = (V, E)$

Output: a vertex cover C with minimum $|C|$



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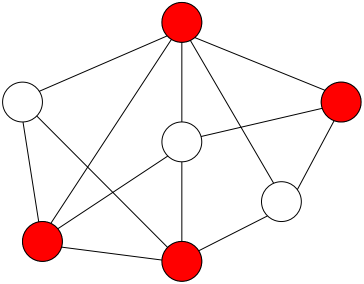

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- Remark: m does not appear in the running time. Indeed, if $m > kn$, then there is no vertex cover of size k.

Vertex-Cover($G' = (V', E'), k$)

if $|E'| = \emptyset$ **then return true**
if $k = 0$ **then return false**
pick any edge $(u, v) \in E'$
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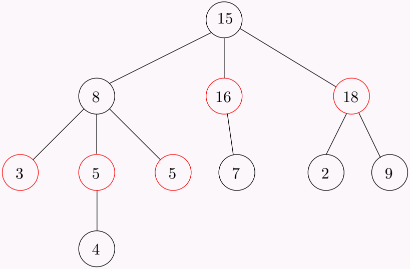
Example: Maximum-Weight Independent Set



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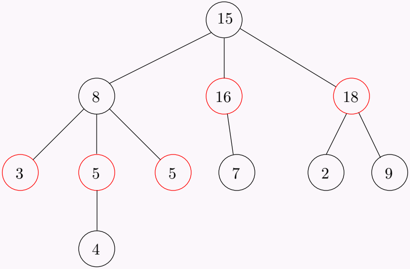

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- Reason why many problems can be solved using DP on trees: the child-trees of a vertex i are only connected through i.

Bounded-Tree-Width Graphs

Def. A tree decomposition of a graph $G = (V, E)$ consists of

- a tree T with node set U, and
- a subset $V_t \subseteq V$ for every $t \in U$, which we call the bag for t,

satisfying the following properties:

- (Vertex Coverage) Every $v \in V$ appears in at least one bag.
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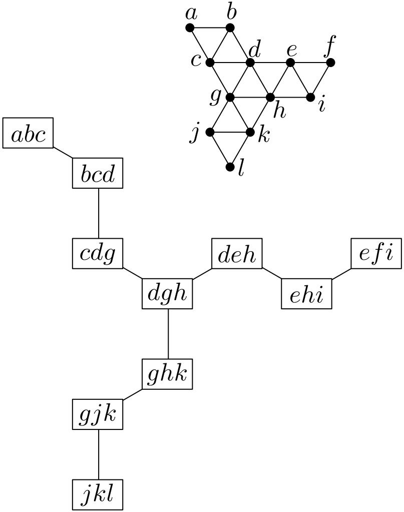

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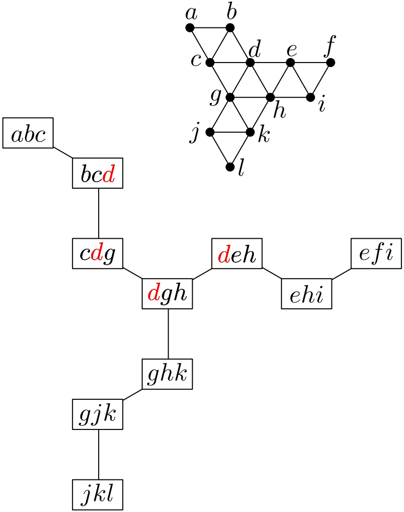

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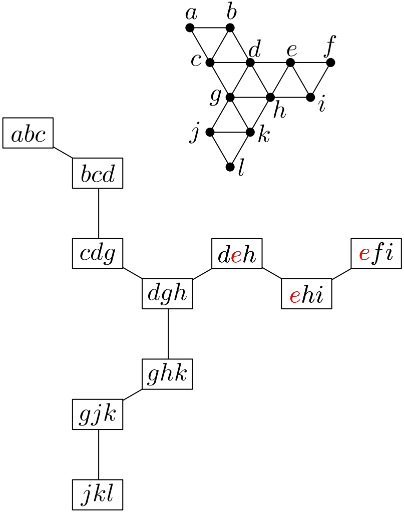

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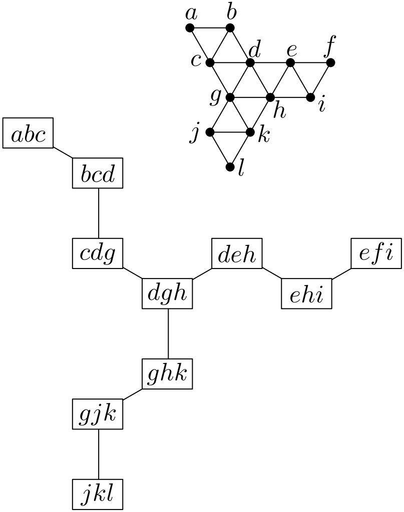

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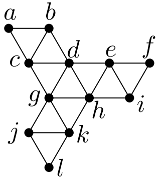

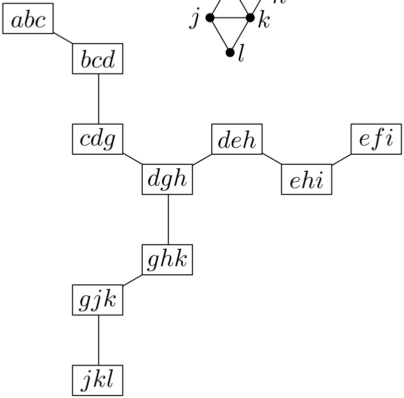

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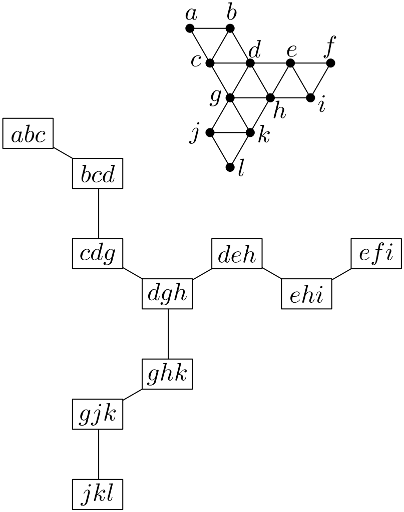
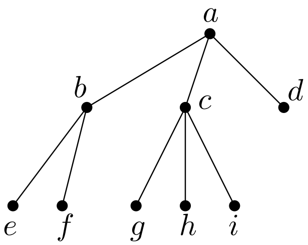


Obs. A (non-empty) tree has tree-width 1.



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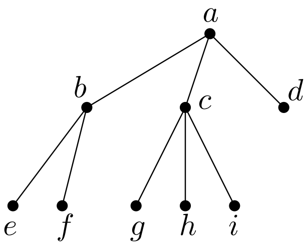

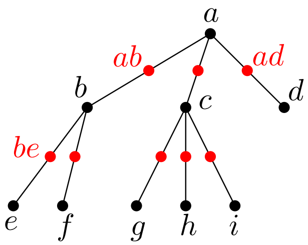

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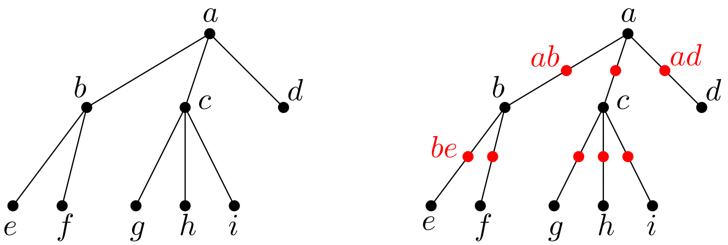

Lemma A graph has tree-width 1 if and only if it is a (non-empty) forest.

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Example: Maximum Weight Independent Set

- given $G = (V, E)$, a tree-decomposition $(T, (V_t)_{t \in U})$ of G with tree-width tw.
- vertex weights $w \in \mathbb{R}^V_{>0}$.
- find an independent set S of G with the maximum total weight.

Assumption: every node in T has at most 2 children. Moreover, every internal nodes in T is one of the following types:

- Splitter: a node t with two children t' and t'', $V_t = V_{t'} = V_{t''}$
- Insertion node: a node t with one child t', $\exists u \notin V_t, V_{t'} = V_t \cup \{u\}$
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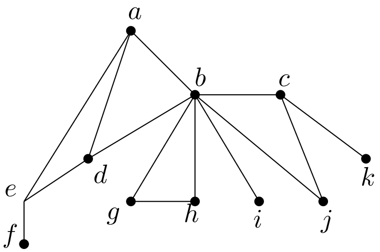

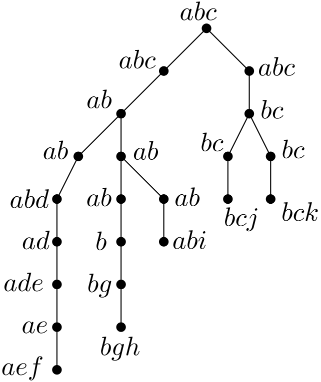

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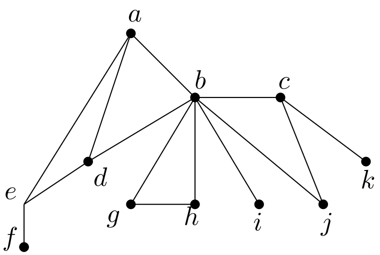

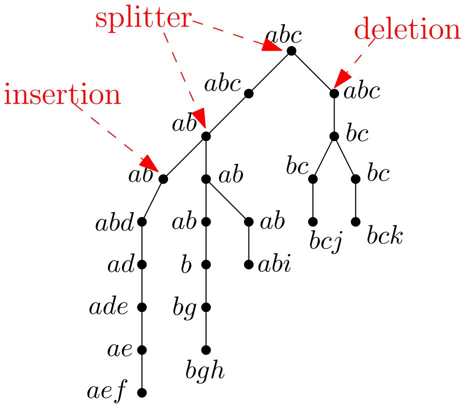

Def. Given a graph $G = (V, E)$, and a tree decomposition $(T, (V_t)_{t \in U})$, a valid labeling of T is a vector $(R_t)_{t \in U}$ of sets, one for every node t, such that the following conditions hold.

- $R_t \subseteq V_t, \forall t \in U$, and R_t is an independent set in G
- $R_t = R_{t'} = R_{t''}$ for a S-node t, and its two children t', t''.
- $R_{t'} \setminus \{u\} = R_t$ for an I-node t and its child t' with $V_{t'} = V_t \cup \{u\}$.
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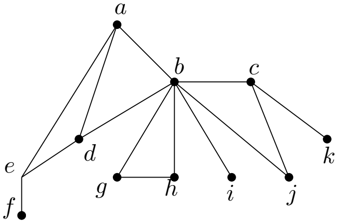

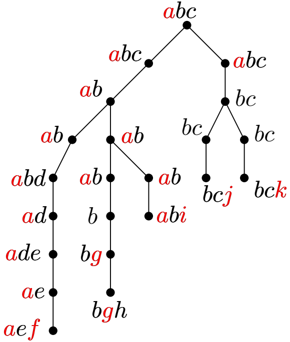

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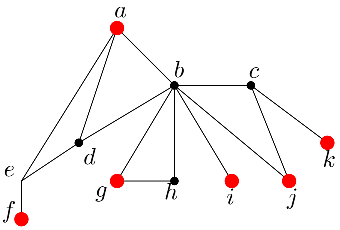

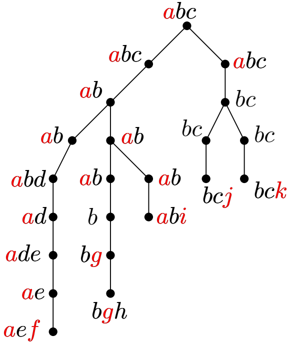

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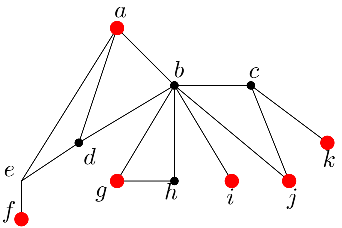
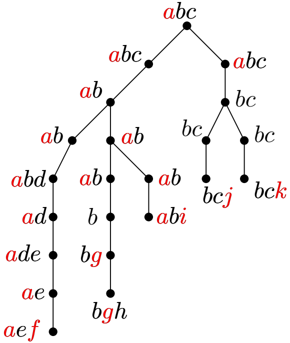
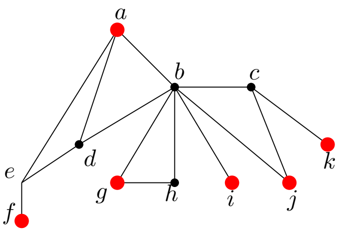
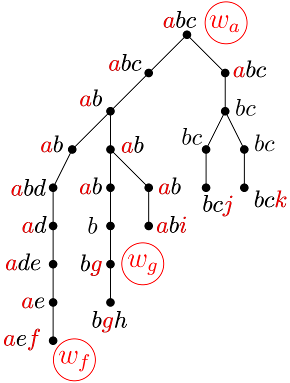




- Therefore, there is an one-to-one mapping between independent sets and valid labelings.

- For every $t \in U$, every $R \subseteq V_t$ that is an IS in G (we call R a label for t), we define a weight $w_t(R)$.
- for the root t and a label R for t, $w_t(R) = \sum_{v \in R} w_r$.
- for an insertion node t with the child t' such that $V_{t'} = V_t \cup \{u\}$, a label R for t', we define $w_{t'}(R) = w_u$ if $u \in R$ and 0 otherwise.
- For all other cases, the weights are defined as 0





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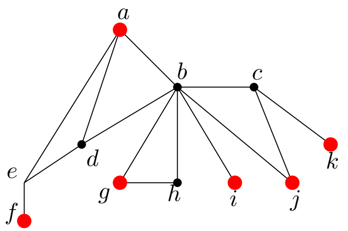

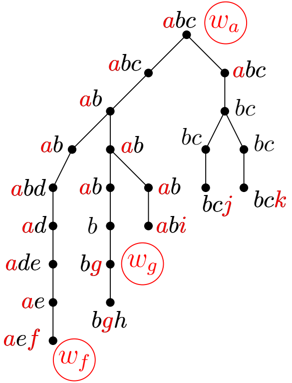

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Dynamic Programming

- $\forall t \in U$, a label R for t: let $f(t,R)$ be the maximum weight of a valid (partial) labeling for the sub-tree of T rooted at t.

$$f(t,R) := \begin{cases} w_t(R) & t \text{ is a leaf} \\ w_t(R) + f(t',R) + f(t'',R) & t \text{ is an S-node with children } t' \text{ and } t'' \\ w_t(R) + \max\{f(t',R), f(t',R\cup\{u\})\} & t \text{ is I-node w. child } t',\ V_{t'} = V_t \cup \{u\} \\ w_t(R) + f(t',R\setminus\{u\}) & t \text{ is D-node w. child } t',\ V_{t'} = V_t \setminus \{u\} \end{cases}$$

- In I-node case, if $R \cup \{u\}$ is an invalid label, then $f(t, R\cup\{u\}) = -\infty$.

- The running time of the dynamic programming: $O\left(2^{\mathsf{tw}} \cdot \mathsf{tw} \cdot n\right)$.
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Q: Suppose we are only given G with tree-width tw, how can we find a tree-decomposition of width tw?

- This is an NP-hard problem.
- We can achieve a weaker goal: find a tree-decomposition of with at most 4tw in time $f(\text{tw}) \cdot \mathrm{poly}(n)$, where $f(\text{tw})$ is a function of tw.
- If tw $= O(1)$, the algorithm runs in polynomial time.
- The constant 4 is acceptable.

Outline

1. Randomized Algorithms

2. Extending the Limits of Tractability

3. Approximation Algorithms using Greedy
   - 2-Approximation Algorithm for Vertex Cover
   - f-Approximation for Set-Cover with Frequency f
   - $(\ln n + 1)$-Approximation for Set-Cover
   - $(1 - \frac{1}{e})$-Approximation for Maximum Coverage

4. Arbitrarily Good Approximation Using Rounding Data

5. Approximation Using LP Rounding and Primal Dual

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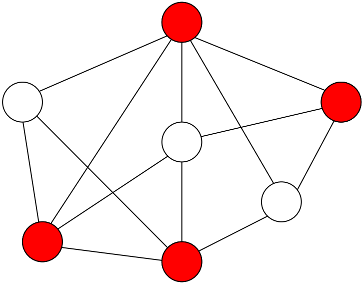
Vertex Cover Problem

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $C \subseteq V$ such that for every $(u, v) \in E$ then $u \in C$ or $v \in C$.



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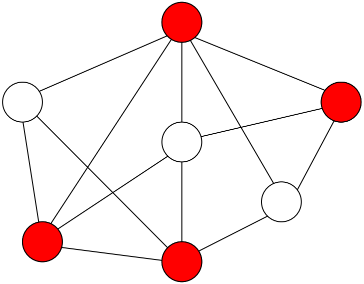

Vertex-Cover Problem

Input: $G = (V, E)$

Output: a vertex cover C with minimum $|C|$

First Try: A "Natural" Greedy Algorithm

Natural Greedy Algorithm for Vertex-Cover

$E' \leftarrow E, C \leftarrow \emptyset$
while $E' \neq \emptyset$ **do**
  let v be the vertex of the maximum degree in (V, E')
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- The logarithmic factor is tight for this algorithm

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- To cover E', the optimum solution needs $|E'|$ vertices □

Outline

Set Cover

Input: $U, |U| = n$: ground set

$S_1, S_2, \cdots, S_m \subseteq U$

Output: minimum size set $C \subseteq [m]$ such that $\bigcup_{i \in C} S_i = U$

Set Cover with Bounded Frequency f

Input: $U, |U| = n$: ground set

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Vertex Cover = Set Cover with Frequency 2

- edges $\Leftrightarrow$ elements
- vertices $\Leftrightarrow$ sets
- every edge (element) can be covered by 2 vertices (sets)

f-Approximation Algorithm for Set Cover with Frequency f

```
C ← ∅
while ⋃_{i∈C} S_i ≠ U do
    let e be any element in U \ ⋃_{i∈C} S_i
    C ← C ∪ {i ∈ [m] : e ∈ S_i}
return C
```

f-Approximation Algorithm for Set Cover with Frequency f

$C \leftarrow \emptyset$
while $\bigcup_{i \in C} S_i \neq U$ **do**
    let e be any element in $U \setminus \bigcup_{i \in C} S_i$
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- Let U' be the set of all elements e considered in Step 3
- Observation: no set S_i contains two elements in U'
- To cover U', the optimum solution needs $|U'|$ sets
- $C \leq f \cdot |U'|$ □

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- g: minimum number of sets needed to cover U

Lemma Let $u_t, t \in \mathbb{Z}_{\geq 0}$ be the number of uncovered elements after t steps. Then for every $t \geq 1$, we have

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- $u_t \leq u_{t-1} - \frac{u_{t-1}}{g} = \left(1 - \frac{1}{g}\right) u_{t-1}$. □

Proof of $(\ln n + 1)$-approximation.

- Let $t = \lceil g \cdot \ln n \rceil$. $u_0 = n$. Then
$$u_t \le \left(1 - \frac{1}{g}\right)^{g \cdot \ln n} \cdot n < e^{-\ln n} \cdot n = n \cdot \frac{1}{n} = 1.$$
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$(1-c)\ln n$-hardness for any $c = \Omega(1)$

Let $c > 0$ be any constant. There is no polynomial-time $(1-c)\ln n$-approximation algorithm for set-cover, unless

- NP $\subseteq$ quasi-poly-time, [Lund, Yannakakis 1994; Feige 1998]
- P = NP. [Dinur, Steuer 2014]

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- $p_k \geq \left(1-\frac{1}{e}\right)\cdot o$ □

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Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \le W.$$

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- Motivation: you have budget W, and want to buy a subset of items of maximum total value

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- Optimum takes item 2 and greedy takes item 1.

DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \cdots, i\}$.

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max\left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W'-w_i] + v_i \end{array} \right\} & i > 0, w_i \leq W' \end{cases}$$

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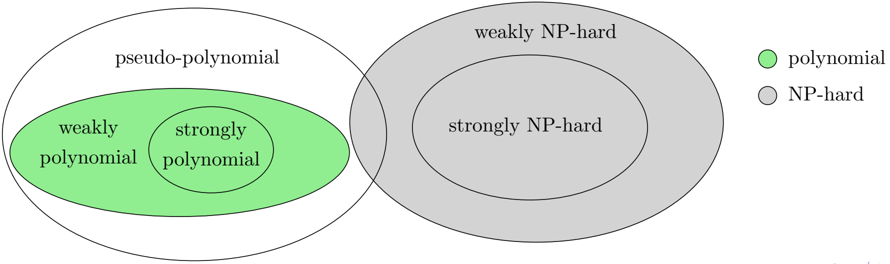
- The input size is polynomial in n and $\log W$; running time is polynomial in n and W.
- The running time is pseudo-polynomial.

- n: number of integers W: maximum value of all integers

- pseudo-polynomial time: $\mathrm{poly}(n, W)$ (e.g., DP for Knapsack)
- weakly polynomial time: $\mathrm{poly}(n, \log W)$ (e.g., Euclidean Algorithm for Greatest Common Divisor)
- strongly polynomial time: $\mathrm{poly}(n)$ time, assuming basic operations on integers taking $O(1)$ time (e.g., Kruskal's)

- weakly NP-hard: NP-hard to solve in time $\mathrm{poly}(n, \log W)$
- strongly NP-hard: NP-hard even if $W = \mathrm{poly}(n)$



Idea for improving the running time to polynomial

- If we make weights upper bounded by $\mathrm{poly}(n)$, then pseudo-polynomial time becomes polynomial time
- Coarsening the weights: $w'_i = \left\lfloor \frac{w_i}{A} \right\rfloor$ for some appropriately defined integer A.

Idea for improving the running time to polynomial

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- Output A times the largest V' such that $f[n, V'] \leq W$.

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Theorem There is a $(1+\epsilon)$-approximation for the knapsack problem in time $O(\frac{n^3}{\epsilon})$.

Def. A polynomial-time approximation scheme (PTAS) is a family of algorithms A_ϵ, where A_ϵ for every $\epsilon > 0$ is a (polynomial-time) $(1 \pm \epsilon)$-approximation algorithm.

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Outline

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Input: n jobs index as $[n]$

each job $j \in [n]$ has a processing time $p_j \in \mathbb{Z}_{>0}$

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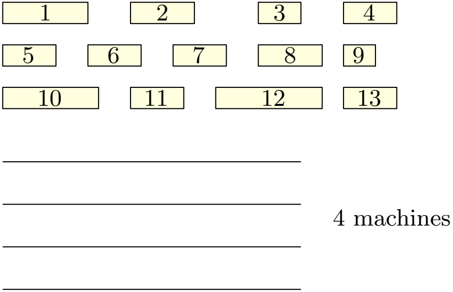

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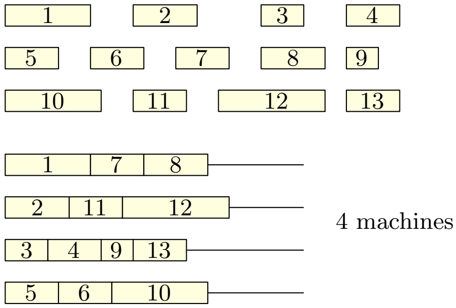

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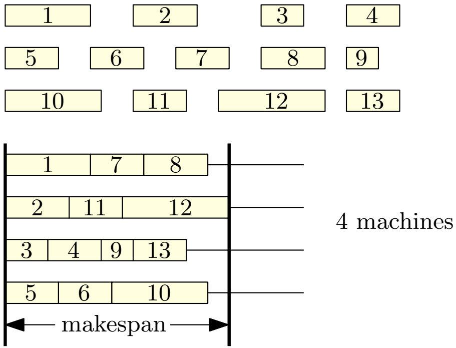

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4: add small jobs to schedule greedily

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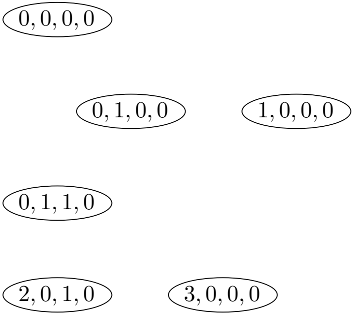
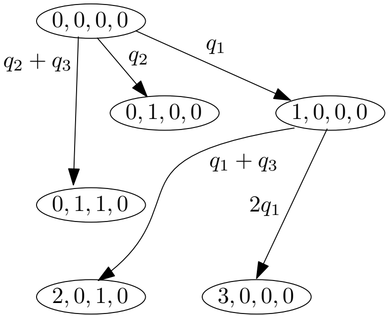
- problem can be solved in $O(m \cdot |E|)$ time using DP
- $O(m \cdot |E|) = O(m \cdot n^{2k}) = n^{O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)}$.


0, 0, 0, 0
0, 1, 0, 0
1, 0, 0, 0
0, 1, 1, 0
2, 0, 1, 0
3, 0, 0, 0



0, 0, 0, 0
$q_2 + q_3$
q_2
q_1
0, 1, 0, 0
1, 0, 0, 0
$q_1 + q_3$
0, 1, 1, 0
$2q_1$
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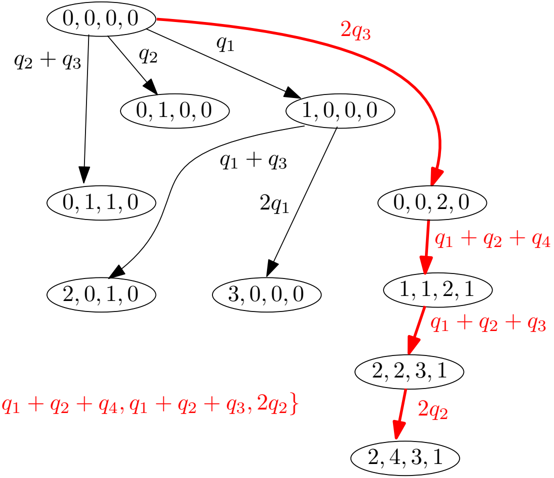

$$\text{cost} = \max\{2q_3, q_1 + q_2 + q_4, q_1 + q_2 + q_3, 2q_2\}$$

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- $\mathcal{I}_B$: instance $(p_j)_{j \in B}$ $\quad \mathrm{opt}_B$: its optimum makespan
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Theorem The dynamic programming algorithm gives a schedule of makespan at most $(1+\epsilon)\mathrm{opt}_B$ in time $n^{O\left(\frac{1}{\epsilon}\log\frac{1}{\epsilon}\right)}$.

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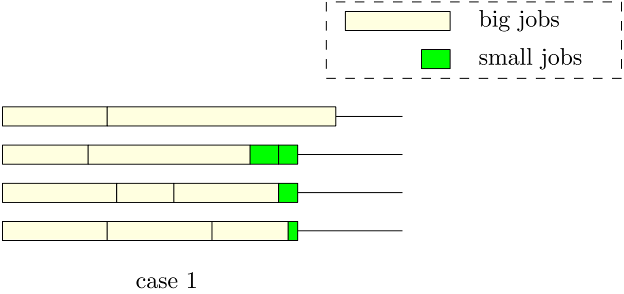
Theorem The dynamic programming algorithm gives a schedule of makespan at most $(1+\epsilon)\mathrm{opt}_B$ in time $n^{O\left(\frac{1}{\epsilon}\log\frac{1}{\epsilon}\right)}$.

Adding small jobs to schedule

```
starting from the schedule for big jobs
for every small job j do
    add j to the machine with the smallest load
```

Analysis of the Final Algorithm



- Case 1: makespan is not increased by small jobs

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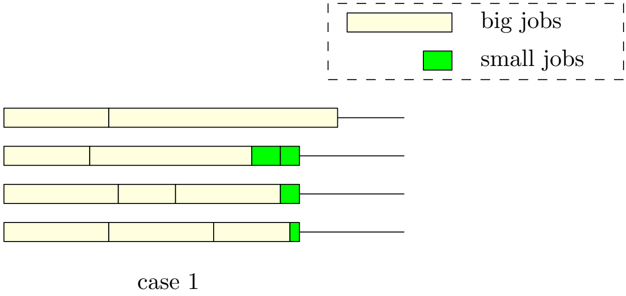

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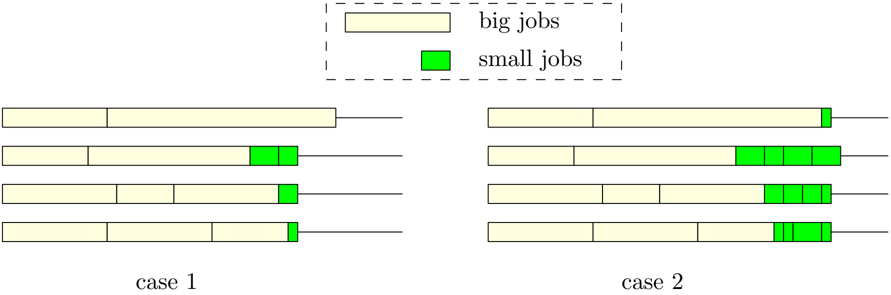

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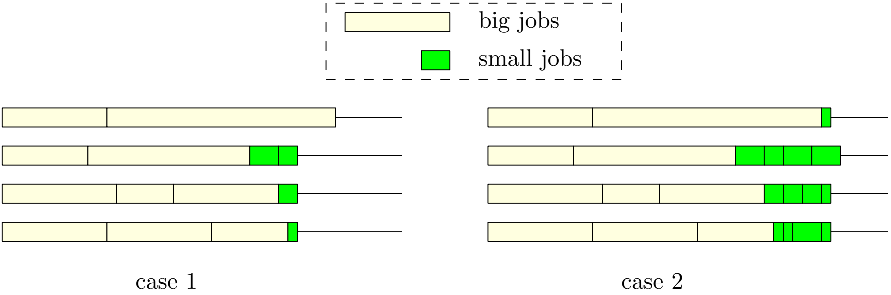

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  - loads between any two machines differ by at most size of a small job, which is at most $\epsilon \cdot p_{\max}$

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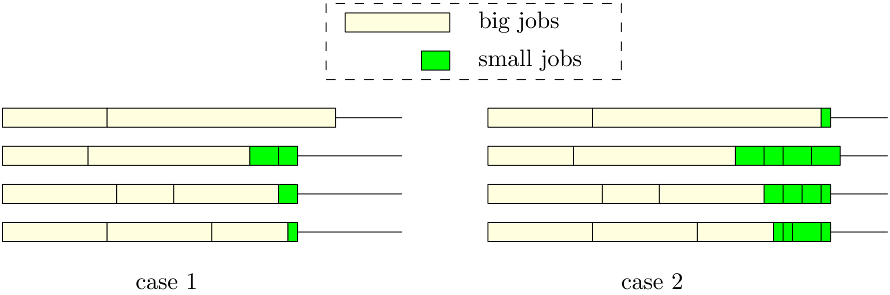

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$$\text{alg} \leq \epsilon \cdot p_{\max} + \frac{1}{m}\sum_{j \in [n]} p_j \leq \epsilon \cdot \text{opt} + \text{opt} = (1+\epsilon) \cdot \text{opt}.$$

Outline

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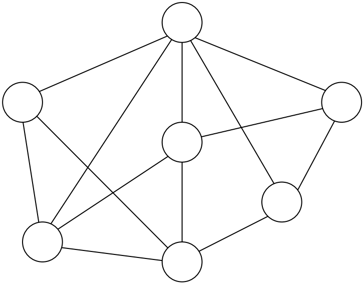
Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$.



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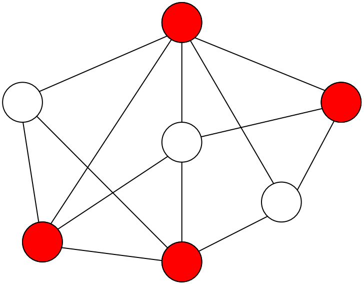

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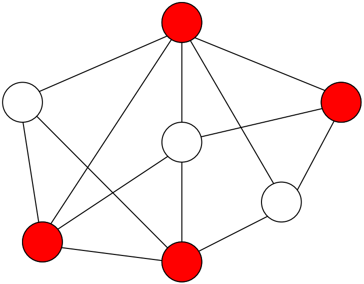

Weighted Vertex-Cover Problem

Input: $G = (V, E)$ with vertex weights $\{w_v\}_{v \in V}$

Output: a vertex cover S with minimum $\sum_{v \in S} w_v$

Integer Programming for Weighted Vertex Cover

- For every $v \in V$, let $x_v \in \{0, 1\}$ indicate whether we select v in the vertex cover S
- The integer programming for weighted vertex cover:

$$(\text{IP}_{\text{WVC}}) \qquad \min \quad \sum_{v \in V} w_v x_v \qquad \text{s.t.}$$

$$x_u + x_v \geq 1 \qquad \forall (u, v) \in E$$

$$x_v \in \{0, 1\} \qquad \forall v \in V$$

- $(\text{IP}_{\text{WVC}}) \Leftrightarrow$ weighted vertex cover
- Thus it is NP-hard to solve integer programmings in general

- Integer programming for WVC:

$$
\begin{aligned}
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- let IP $=$ value of (IP_{WVC}), LP $=$ value of (LP_{WVC})
- Then, LP $\leq$ IP

Algorithm for Weighted Vertex Cover

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1: Solving (LP$_{\text{WVC}}$) to obtain a solution $\{x_u^*\}_{u \in V}$
2:
3:

Algorithm for Weighted Vertex Cover

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1: Solving (LP_{WVC}) to obtain a solution $\{x^*_u\}_{u\in V}$
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- Thus, either $u \in S$ or $v \in S$. □

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$$\begin{aligned} \text{cost}(S) &= \sum_{u \in S} w_u \leq \sum_{u \in S} w_u \cdot 2x_u^* = 2 \sum_{u \in S} w_u \cdot x_u^* \\ &\leq 2 \sum_{u \in V} w_u \cdot x_u^* = 2 \cdot \text{LP}. \end{aligned}$$

□

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Proof.

$\text{cost}(S) \leq 2 \cdot \text{LP} \leq 2 \cdot \text{IP} = 2 \cdot \text{cost(best vertex cover)}$. □

Outline

LP Relaxation

$$\min \quad \sum_{v \in V} w_v x_v$$

$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$

$$x_v \geq 0 \qquad \forall v \in V$$

Dual LP

$$\max \quad \sum_{e \in E} y_e$$

$$\sum_{e \in \delta(v)} y_e \leq w_v \qquad \forall v \in V$$

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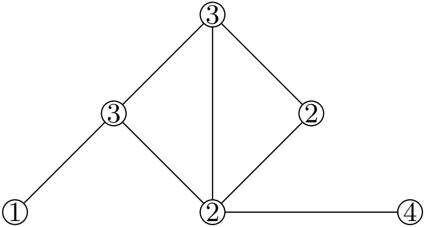
- Algorithm constructs integral primal solution x and dual solution y simultaneously.

Primal-Dual Algorithm for Weighted Vertex Cover Problem

$x \leftarrow 0, y \leftarrow 0$, all edges said to be uncovered
while there exists at least one uncovered edge **do**
    take such an edge e arbitrarily
    increasing y_e until the dual constraint for one end-vertex v of e becomes tight
    $x_v \leftarrow 1$, claim all edges incident to v are covered
return x



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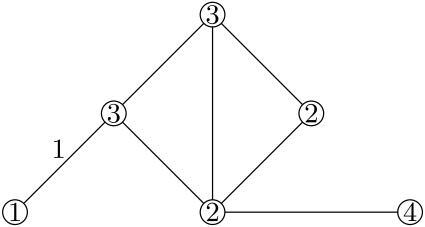

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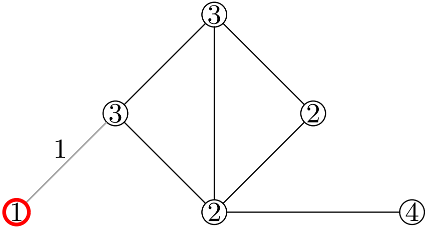

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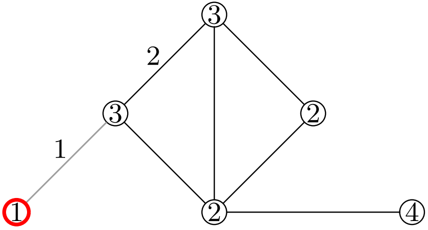

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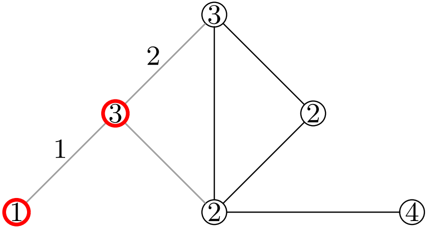

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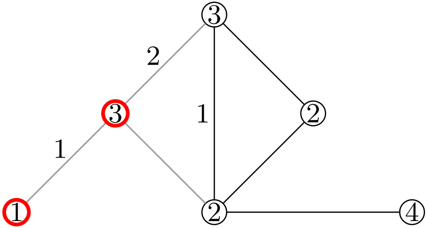

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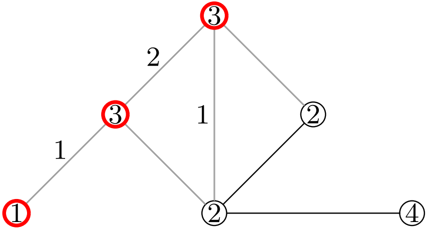

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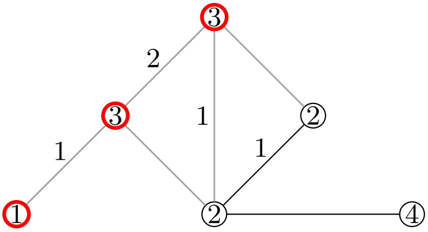

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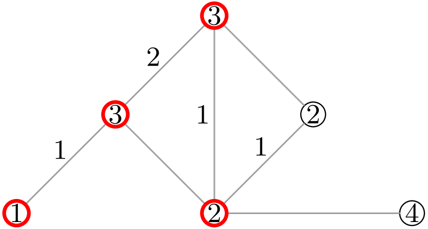

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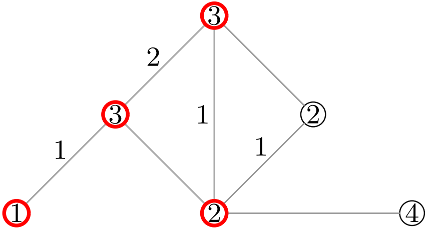

Lemma

1. x satisfies all primal constraints
2. y satisfies all dual constraints
3. $P \leq 2D \leq 2D^* \leq 2 \cdot \text{opt}$

   $P := \sum_{v \in V} x_v$: value of x

   $D := \sum_{e \in E} y_e$: value of y

   D^* : dual LP value

Proof of $P \leq 2D$.

$$P = \sum_{v \in V} w_v x_v \leq \sum_{v \in V} x_v \sum_{e \in \delta(v)} y_e = \sum_{(u,v) \in E} y_{(u,v)} (x_u + x_v)$$

$$\leq 2 \sum_{e \in E} y_e = 2D. \qquad \square$$

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- a more general framework: construct an arbitrary maximal dual solution y; choose the vertices whose dual constraints are tight
- y is maximal: increasing any coordinate y_e makes y infeasible

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- a more general framework: construct an arbitrary maximal dual solution y; choose the vertices whose dual constraints are tight
- y is maximal: increasing any coordinate y_e makes y infeasible

- primal-dual algorithms do not need to solve LPs
- LPs are used in analysis only
- faster than LP-rounding algorithm in general

Outline

Unrelated Machine Scheduling

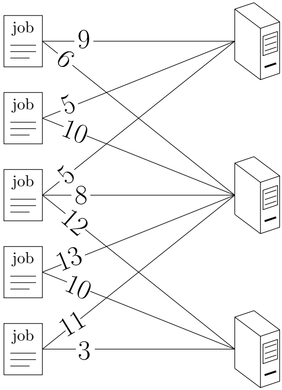
Input: $J, |J| = n$: jobs

$M, |M| = m$: machines

p_{ij}: processing time of job j on machine i

Output: assignment $\sigma : J \mapsto M$:, so as to minimize makespan:

$$\max_{i \in M} \sum_{j \in \sigma^{-1}(i)} p_{ij}$$



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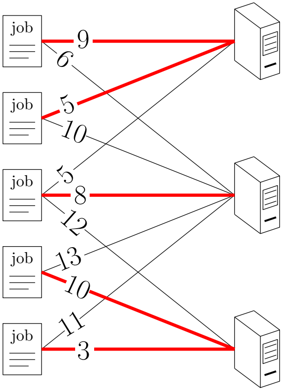

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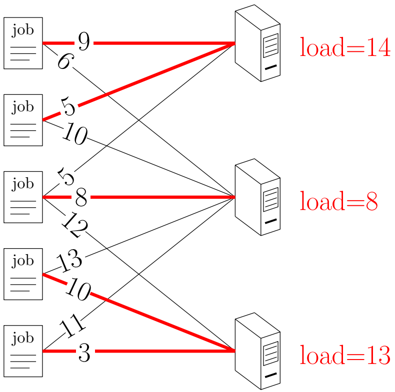

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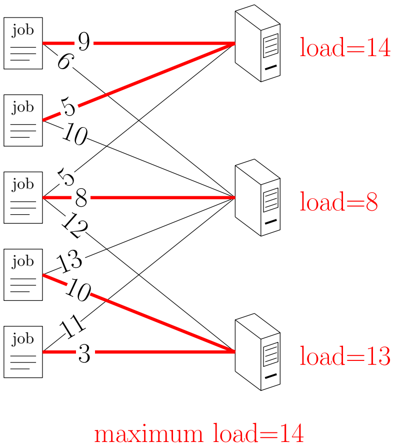
$M, |M| = m$: machines

p_{ij}: processing time of job j on machine i

Output: assignment $\sigma : J \mapsto M$:, so as to minimize makespan:

$$\max_{i \in M} \sum_{j \in \sigma^{-1}(i)} p_{ij}$$



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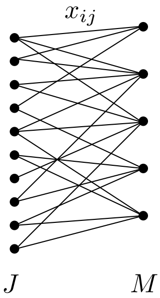
2-Approximate Rounding Algorithm of Shmoys-Tardos



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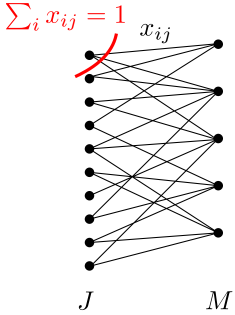

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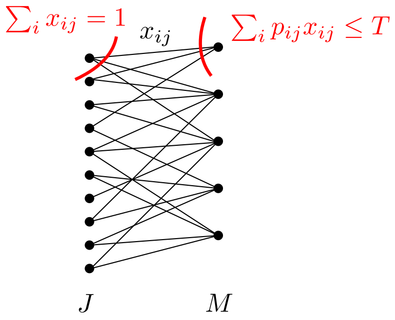

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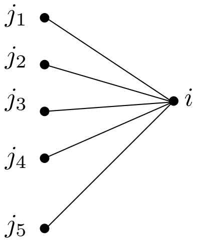

$p_{ij_1} \geq p_{ij_2} \geq \cdots \geq p_{ij_5}$

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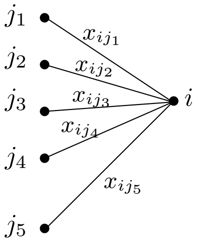

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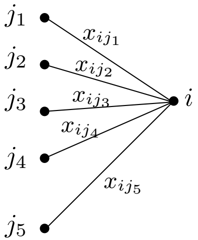

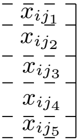

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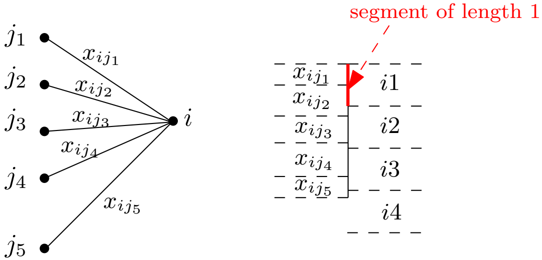

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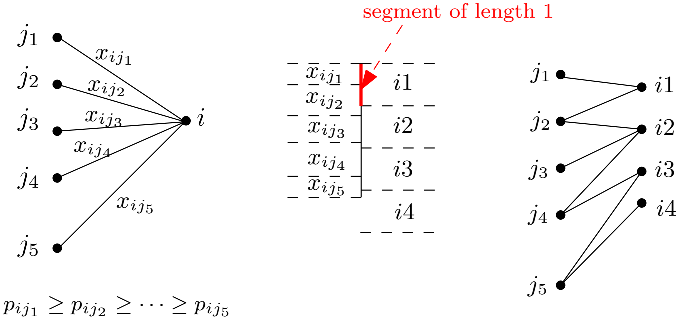

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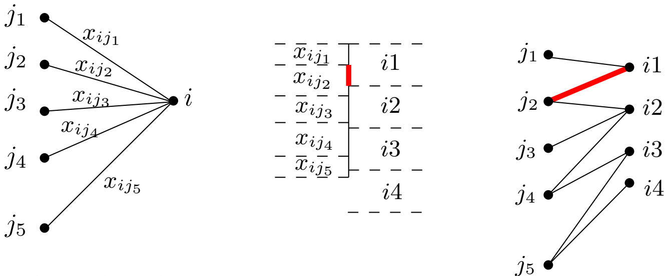

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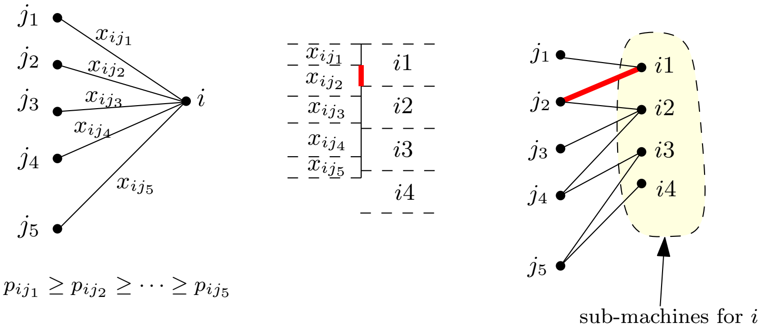

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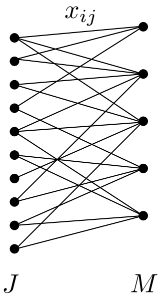

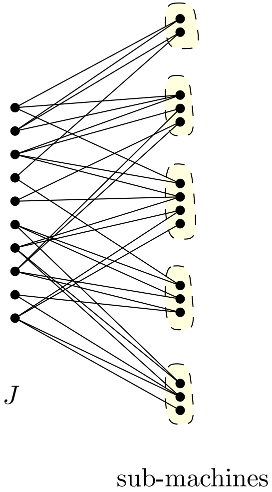

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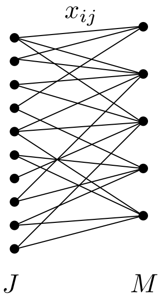

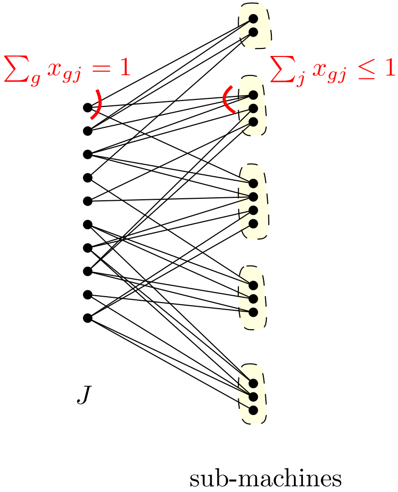

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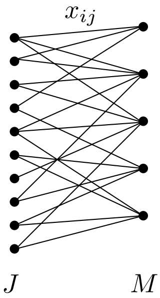

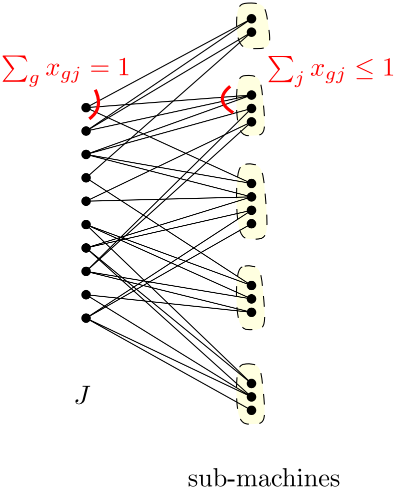

Obs. x between J and sub-machines is a point in the

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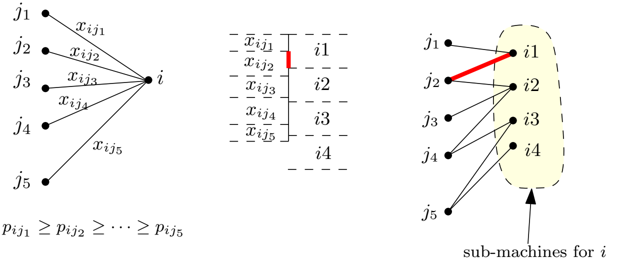

Proof.

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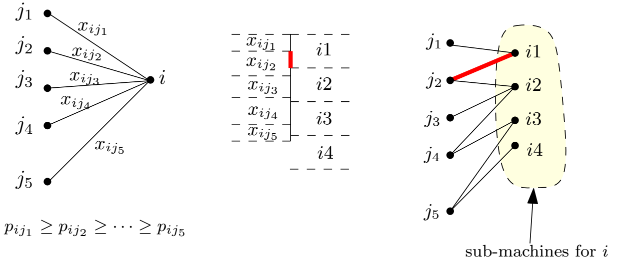

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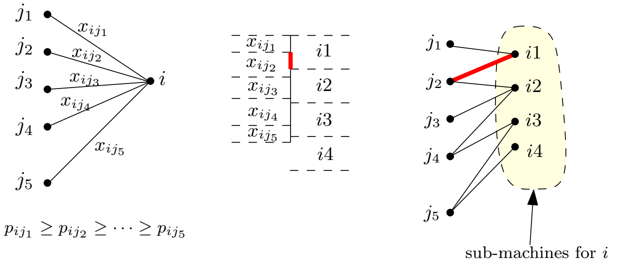

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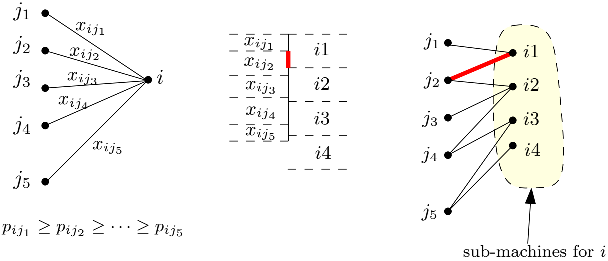

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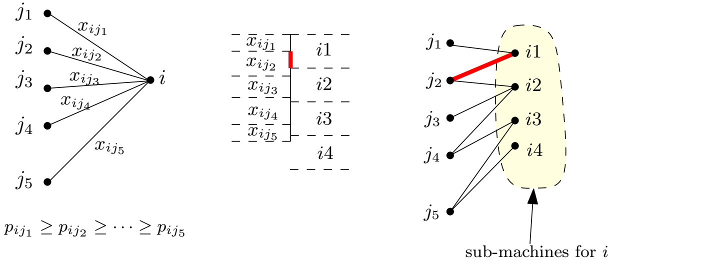

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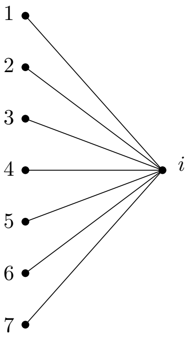
$$\text{(load on } i) = \sum_{t=1}^{a} p_{ik_t} \leq p_{ik_1} + \sum_{t=2}^{a} \sum_{j} x_{i_{t-1}j} \cdot p_{ij}$$

$$\leq p_{ik_1} + \sum_{j} x_{ij} p_{ij} \leq T + T = 2T.$$

□

- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:



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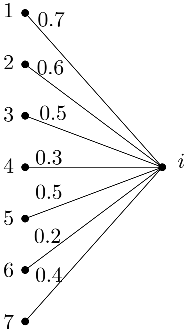

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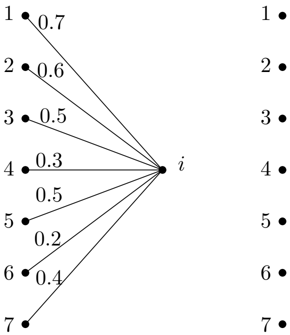

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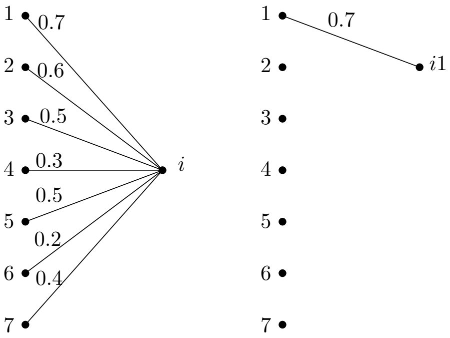

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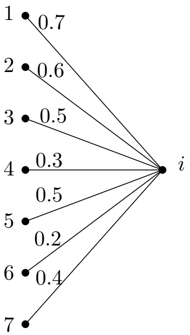

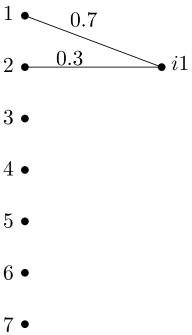

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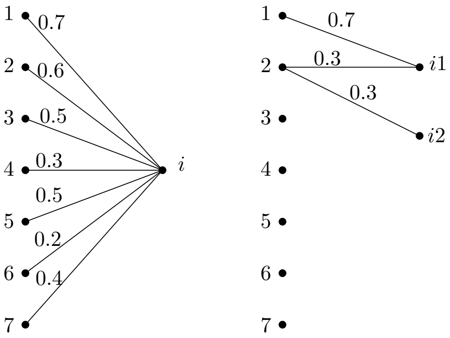

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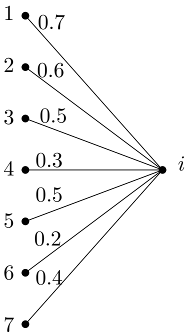

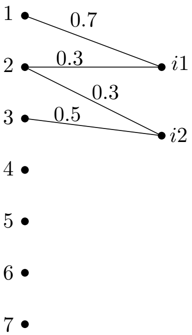

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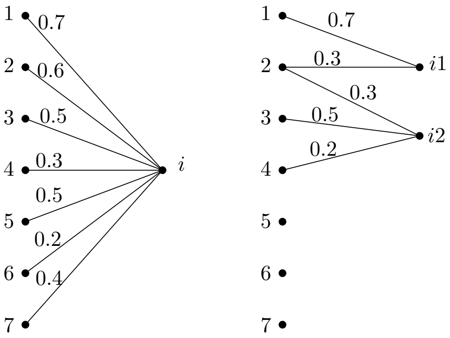

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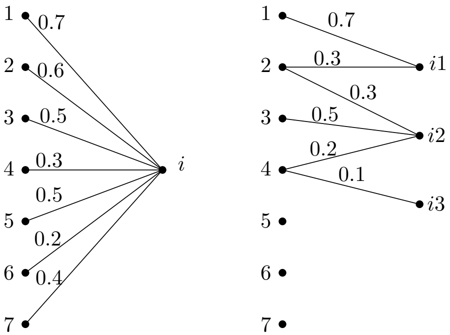

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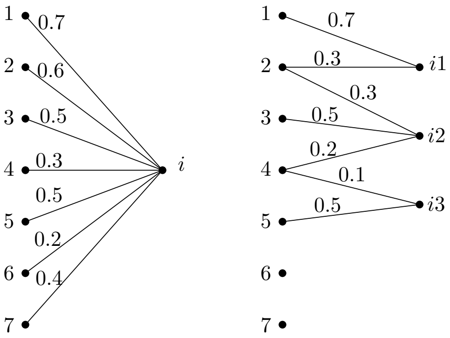

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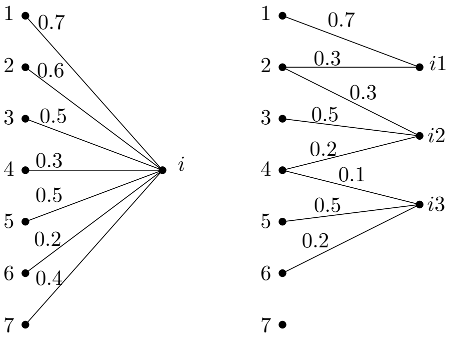

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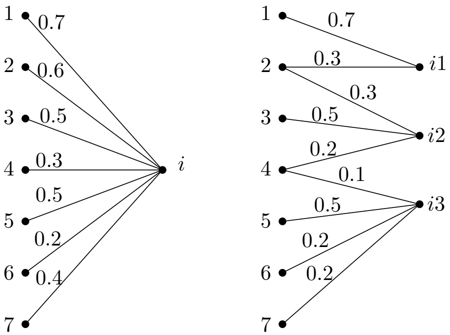

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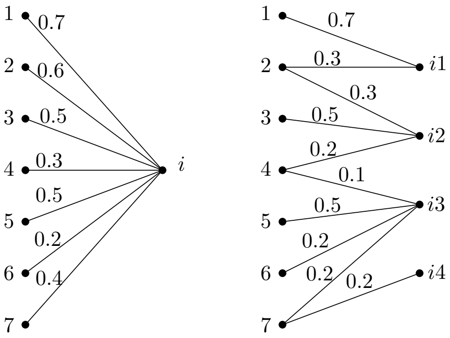

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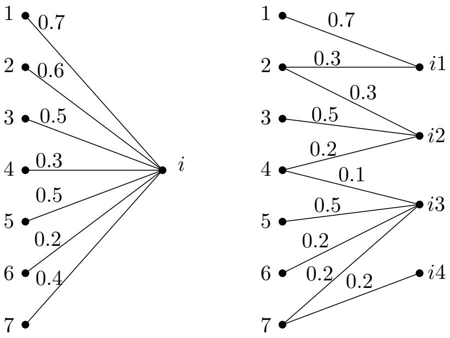
  - $4 \to i3, 7 \to i4$



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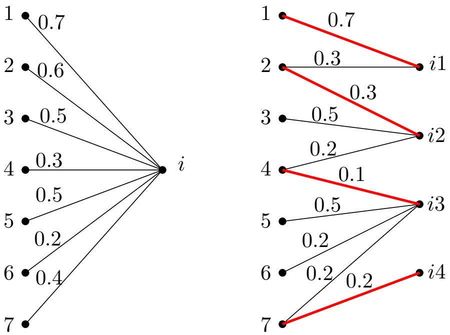

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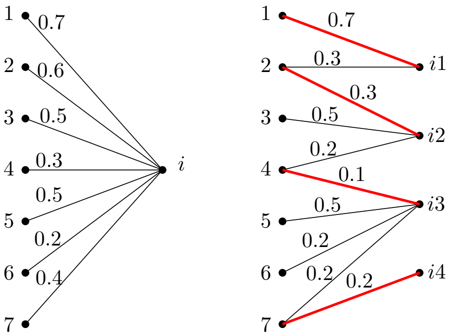

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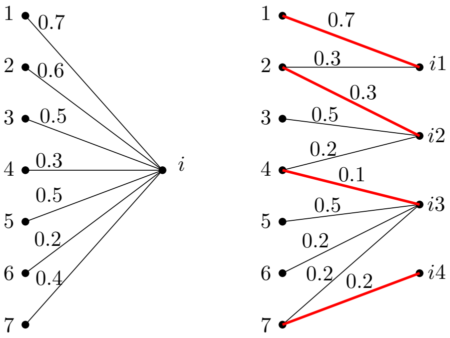

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$$p_1 + p_2 + p_4 + p_7 \leq T + (0.7p_1 + 0.3p_2) + (0.3p_2 + 0.5p_3 + 0.2p_4) + (0.1p_4 + 0.5p_5 + 0.2p_6 + 0.2p_7)$$
$$\leq T + (0.7p_1 + 0.6p_2 + 0.5p_3 + 0.3p_4 + 0.5p_5 + 0.2p_6 + 0.4p_7)$$
$$\leq T + T = 2T$$