算法设计与分析(2025年春季学期)
Divide-and-Conquer

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Outline

Greedy Algorithm

- mainly for combinatorial optimization problems
- trivial algorithm runs in exponential time
- greedy algorithm gives an efficient algorithm
- main focus of analysis: correctness of algorithm

Divide-and-Conquer

- not necessarily for combinatorial optimization problems
- trivial algorithm already runs in polynomial time
- divide-and-conquer gives a more efficient algorithm
- main focus of analysis: running time

Divide-and-Conquer

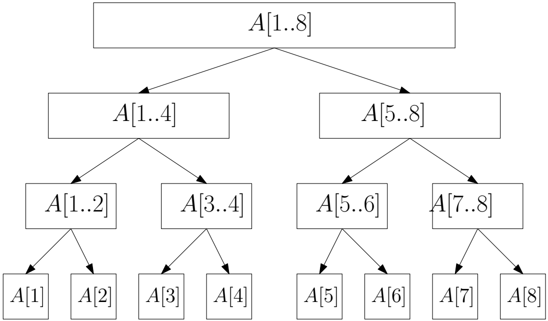
- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance

merge-sort(A, n)

```
1: if n = 1 then
2:     return A
3: else
4:     B ← merge-sort(A[1..⌊n/2⌋], ⌊n/2⌋)
5:     C ← merge-sort(A[⌊n/2⌋ + 1..n], ⌈n/2⌉)
6:     return merge(B, C, ⌊n/2⌋, ⌈n/2⌉)
```

- Divide: trivial
- Conquer: 4, 5
- Combine: 6

Running Time for Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\log n)$ levels
- Running time $= O(n \log n)$
- Better than insertion sort

Running Time for Merge-Sort Using Recurrence

- $T(n) =$ running time for sorting n numbers,then

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) & \text{if } n \geq 2 \end{cases}$$

- With some tolerance of informality:

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + O(n) & \text{if } n \geq 2 \end{cases}$$

- Even simpler: $T(n) = 2T(n/2) + O(n)$. (Implicit assumption: $T(n) = O(1)$ if n is at most some constant.)
- Solving this recurrence, we have $T(n) = O(n \log n)$ (we shall show how later)

Outline

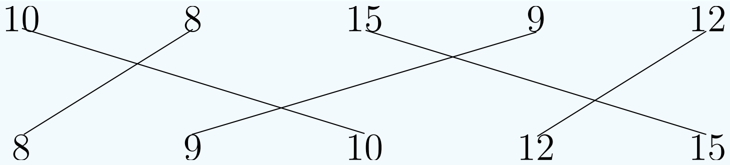
Def. Given an array A of n integers, an inversion in A is a pair (i, j) of indices such that $i < j$ and $A[i] > A[j]$.

Counting Inversions

Input: an sequence A of n numbers

Output: number of inversions in A

Example:



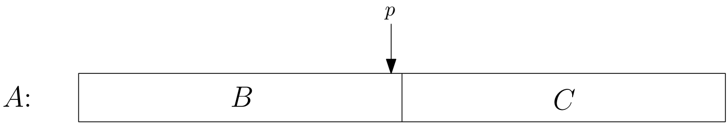
- 4 inversions (for convenience, using numbers, not indices): $(10, 8), (10, 9), (15, 9), (15, 12)$

Naive Algorithm for Counting Inversions

count-inversions(A, n)

```
c ← 0
for every i ← 1 to n − 1 do
    for every j ← i + 1 to n do
        if A[i] > A[j] then c ← c + 1
return c
```

Divide-and-Conquer



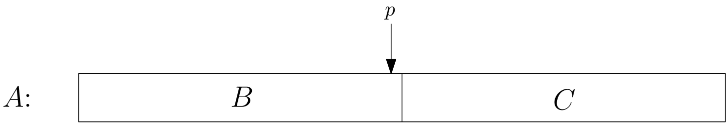
- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p+1..n]$
- $$\#\text{invs}(A) = \#\text{invs}(B) + \#\text{invs}(C) + m$$
  $$m = \left|\left\{(i,j) : B[i] > C[j]\right\}\right|$$

Q: How fast can we compute m, via trivial algorithm?

A: $O(n^2)$

- Can not improve the $O(n^2)$ time for counting inversions.

Divide-and-Conquer

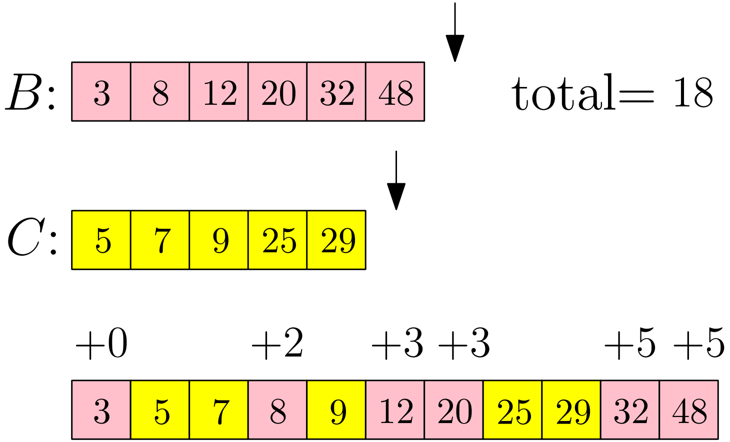


- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p+1..n]$
- $$\#\text{invs}(A) = \#\text{invs}(B) + \#\text{invs}(C) + m$$ $$m = \left|\left\{(i,j) : B[i] > C[j]\right\}\right|$$

Lemma If both B and C are sorted, then we can compute m in $O(n)$ time!

Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:



Count Inversions between B and C

- Procedure that merges B and C and counts inversions between B and C at the same time

merge-and-count(B, C, n_1, n_2)

```
count ← 0;
A ← array of size n1 + n2; i ← 1; j ← 1
while i ≤ n1 or j ≤ n2 do
    if j > n2 or (i ≤ n1 and B[i] ≤ C[j]) then
        A[i + j − 1] ← B[i]; i ← i + 1
        count ← count + (j − 1)
    else
        A[i + j − 1] ← C[j]; j ← j + 1
return (A, count)
```

Sort and Count Inversions in A

- A procedure that returns the sorted array of A and counts the number of inversions in A:

sort-and-count(A, n)

```
1: if n = 1 then
2:     return (A, 0)
3: else
4:     (B, m_1) ← sort-and-count(A[1..⌊n/2⌋], ⌊n/2⌋)
5:     (C, m_2) ← sort-and-count(A[⌊n/2⌋+1..n], ⌈n/2⌉)
6:     (A, m_3) ← merge-and-count(B, C, ⌊n/2⌋, ⌈n/2⌉)
7:     return (A, m_1 + m_2 + m_3)
```

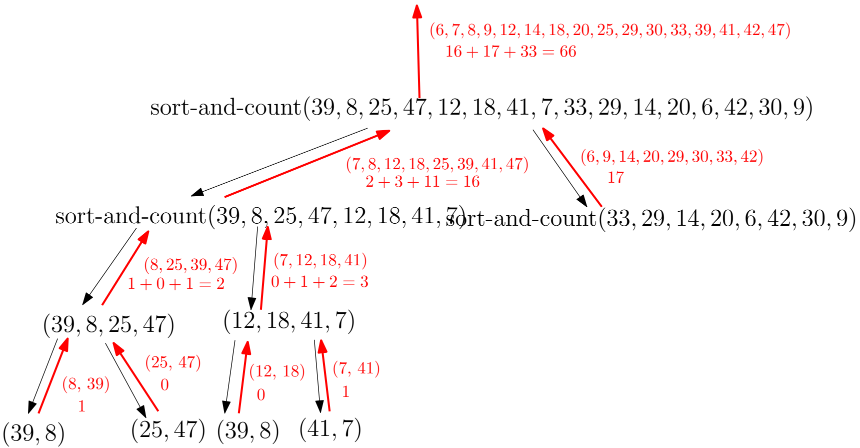
- Divide: trivial
- Conquer: 4, 5
- Combine: 6, 7

sort-and-count(A, n)

if $n = 1$ **then**
    return $(A, 0)$
else
    $(B, m_1) \leftarrow$ sort-and-count$\left(A\left[1..\lfloor n/2 \rfloor\right], \lfloor n/2 \rfloor\right)$
    $(C, m_2) \leftarrow$ sort-and-count$\left(A\left[\lfloor n/2 \rfloor + 1..n\right], \lceil n/2 \rceil\right)$
    $(A, m_3) \leftarrow$ merge-and-count$(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$
    return $(A, m_1 + m_2 + m_3)$

- Recurrence for the running time: $T(n) = 2T(n/2) + O(n)$
- Running time $= O(n \log n)$

Example


(6, 7, 8, 9, 12, 14, 18, 20, 25, 29, 30, 33, 39, 41, 42, 47)
16 + 17 + 33 = 66
sort-and-count(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)
(7, 8, 12, 18, 25, 39, 41, 47)
2 + 3 + 11 = 16
(6, 9, 14, 20, 29, 30, 33, 42)
17
sort-and-count(39, 8, 25, 47, 12, 18, 41, 7)
sort-and-count(33, 29, 14, 20, 6, 42, 30, 9)
(8, 25, 39, 47)
1 + 0 + 1 = 2
(7, 12, 18, 41)
0 + 1 + 2 = 3
(39, 8, 25, 47)
(12, 18, 41, 7)
(8, 39)
1
(25, 47)
0
(12, 18)
0
(7, 41)
1
(39, 8)
(25, 47)
(39, 8)
(41, 7)

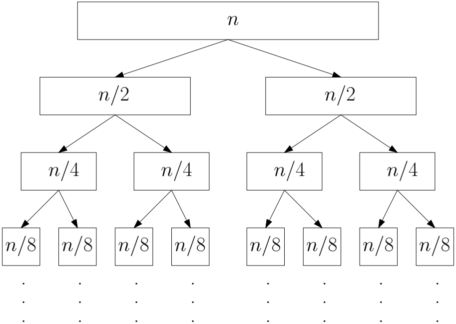
Outline

Methods for Solving Recurrences

- The recursion-tree method
- The master theorem

Recursion-Tree Method

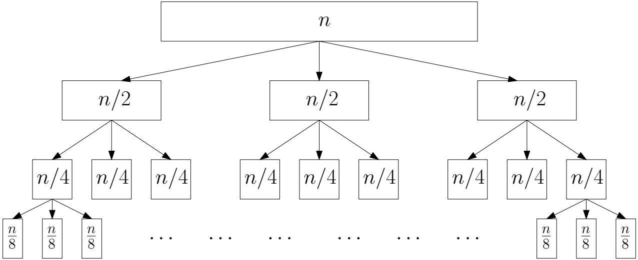
- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$
- There are $O(\log n)$ levels
- Running time $= O(n \log n)$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$

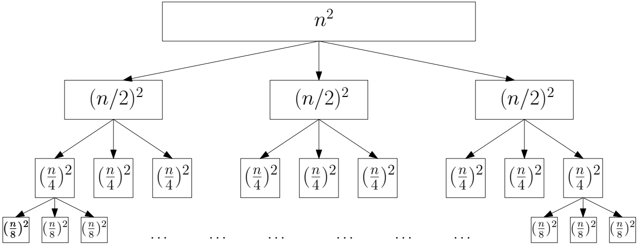


- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
- Index of last level? $\log_2 n$
- Total running time?

$$\sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i n = O\left(n\left(\frac{3}{2}\right)^{\log_2 n}\right) = O(3^{\log_2 n}) = O(n^{\log_2 3}).$$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$



- Total running time at level i? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$
- Index of last level? $\log_2 n$
- Total running time?

$$\sum_{i=0}^{\log_2 n} \left(\frac{3}{4}\right)^i n^2 = O(n^2).$$

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \log n)$
$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\log_2 3})$
$T(n) = 3T(n/2) + O(n^2)$	3	2	2	$O(n^2)$

Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

$$T(n) = \begin{cases} O(n^{\log_b a}) & \text{if } c < \log_b a \\ O(n^c \log n) & \text{if } c = \log_b a \\ O(n^c) & \text{if } c > \log_b a \end{cases}$$

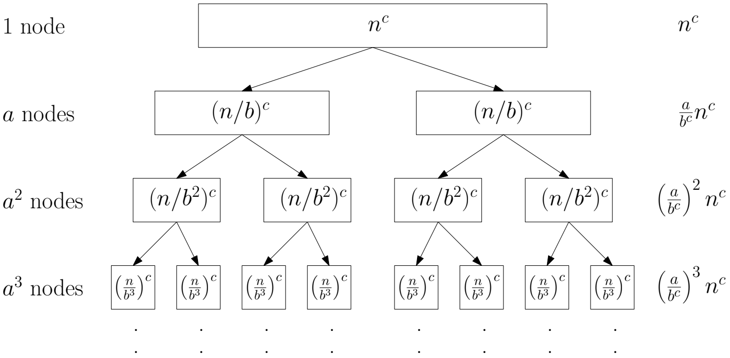
Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

$$T(n) = \begin{cases} O(n^{\log_b a}) & \text{if } c < \log_b a \\ O(n^c \log n) & \text{if } c = \log_b a \\ O(n^c) & \text{if } c > \log_b a \end{cases}$$

- Ex: $T(n) = 4T(n/2) + O(n^2)$. Case 2. $T(n) = O(n^2 \log n)$
- Ex: $T(n) = 3T(n/2) + O(n)$. Case 1. $T(n) = O(n^{\log_2 3})$
- Ex: $T(n) = T(n/2) + O(1)$. Case 2. $T(n) = O(\log n)$
- Ex: $T(n) = 2T(n/2) + O(n^2)$. Case 3. $T(n) = O(n^2)$

Proof of Master Theorem Using Recursion Tree

$$T(n) = aT(n/b) + O(n^c)$$



- $c < \log_b a$: bottom-level dominates: $\left(\frac{a}{b^c}\right)^{\log_b n} n^c = n^{\log_b a}$
- $c = \log_b a$: all levels have same time: $n^c \log_b n = O(n^c \log n)$
- $c > \log_b a$: top-level dominates: $O(n^c)$

Outline

Outline

Quicksort vs Merge-Sort

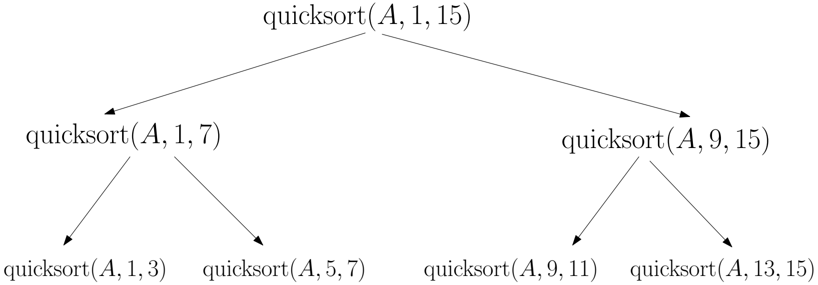
	Merge Sort	**Quicksort**
Divide	Trivial	Separate small and big numbers
Conquer	Recurse	Recurse
Combine	Merge 2 sorted arrays	Trivial

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

A:

15	17	25	29	37	38	45	64	69	75	76	82	85	92	94



Quicksort

quicksort(A, n)

if $n \leq 1$ **then return** A
$x \leftarrow$ lower median of A
$A_L \leftarrow$ array of elements in A that are less than x \\ Divide
$A_R \leftarrow$ array of elements in A that are greater than x \\ Divide
$B_L \leftarrow$ quicksort$(A_L$, length of $A_L)$ \\ Conquer
$B_R \leftarrow$ quicksort$(A_R$, length of $A_R)$ \\ Conquer
$t \leftarrow$ number of times x appear A
return concatenation of B_L, t copies of x, and B_R

- Recurrence $T(n) \leq 2T(n/2) + O(n)$
- Running time $= O(n \log n)$

Assumption We can choose median of an array of size n in $O(n)$ time.

Q: How to remove this assumption?

A:

1. There is an algorithm to find median in $O(n)$ time, using divide-and-conquer (we shall not talk about it; it is complicated and not practical)
2. Choose a pivot randomly and pretend it is the median (it is practical)

Quicksort Using A Random Pivot

quicksort(A, n)

if $n \leq 1$ **then return** A
$x \leftarrow$ a random element of A (x is called a pivot)
$A_L \leftarrow$ array of elements in A that are less than x \\ Divide
$A_R \leftarrow$ array of elements in A that are greater than x \\ Divide
$B_L \leftarrow$ quicksort$(A_L,$ length of $A_L)$ \\ Conquer
$B_R \leftarrow$ quicksort$(A_R,$ length of $A_R)$ \\ Conquer
$t \leftarrow$ number of times x appear A
return concatenation of B_L, t copies of x, and B_R

Randomized Algorithm Model

Assumption There is a procedure to produce a random real number in $[0, 1]$.

Q: Can computers really produce random numbers?

A: No! The execution of a computer programs is deterministic!

- In practice: use pseudo-random-generator, a deterministic algorithm returning numbers that "look like" random
- In theory: assume they can.

Quicksort Using A Random Pivot

quicksort(A, n)

if $n \leq 1$ **then return** A
$x \leftarrow$ a random element of A (x is called a pivot)
$A_L \leftarrow$ array of elements in A that are less than x \\ Divide
$A_R \leftarrow$ array of elements in A that are greater than x \\ Divide
$B_L \leftarrow$ quicksort$(A_L,$ length of $A_L)$ \\ Conquer
$B_R \leftarrow$ quicksort$(A_R,$ length of $A_R)$ \\ Conquer
$t \leftarrow$ number of times x appear A
return concatenation of B_L, t copies of x, and B_R

Lemma The expected running time of the algorithm is $O(n \log n)$.

Quicksort Can Be Implemented as an "In-Place" Sorting Algorithm

- In-Place Sorting Algorithm: an algorithm that only uses "small" extra space.

$i\ j$

↓

17	37	15	29	38	45	25	64	69	76	94	92	75	82	85

- To partition the array into two parts, we only need $O(1)$ extra space.

partition(A, ℓ, r)

$p \leftarrow$ random integer between ℓ and r, swap $A[p]$ and $A[\ell]$
$i \leftarrow \ell, j \leftarrow r$
while true **do**
    while $i < j$ and $A[i] < A[j]$ do $j \leftarrow j - 1$
    if $i = j$ **then** break
    swap $A[i]$ and $A[j]$; $i \leftarrow i + 1$
    while $i < j$ and $A[i] < A[j]$ **do** $i \leftarrow i + 1$
    if $i = j$ **then** break
    swap $A[i]$ and $A[j]$; $j \leftarrow j - 1$
return i

In-Place Implementation of Quick-Sort

quicksort(A, ℓ, r)

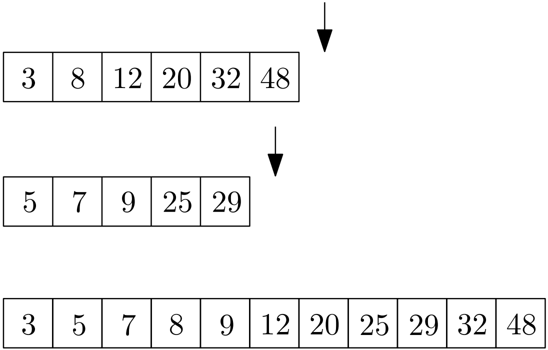
```
if ℓ ≥ r then return
m ← patition(A, ℓ, r)
quicksort(A, ℓ, m − 1)
quicksort(A, m + 1, r)
```

- To sort an array A of size n, call quicksort$(A, 1, n)$.

Note: We pass the array A by reference, instead of by copying.

Merge-Sort is Not In-Place

- To merge two arrays, we need a third array with size equaling the total size of two arrays



Outline

Comparison-Based Sorting Algorithms

Q: Can we do better than $O(n \log n)$ for sorting?

A: No, for comparison-based sorting algorithms.

Comparison-Based Sorting Algorithms

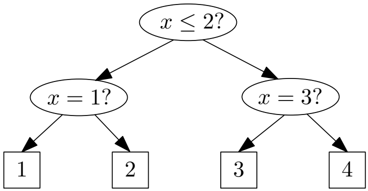
- To sort, we are only allowed to compare two elements
- We can not use "internal structures" of the elements

Lemma The (worst-case) running time of any comparison-based sorting algorithm is $\Omega(n \log n)$.

- Bob has one number x in his hand, $x \in \{1, 2, 3, \cdots, N\}$.
- You can ask Bob "yes/no" questions about x.

Q: How many questions do you need to ask Bob in order to know x?

A: $\lceil \log_2 N \rceil$.



Comparison-Based Sorting Algorithms

Q: Can we do better than $O(n \log n)$ for sorting?

A: No, for comparison-based sorting algorithms.

- Bob has a permutation π over $\{1, 2, 3, \cdots, n\}$ in his hand.
- You can ask Bob "yes/no" questions about π.

Q: How many questions do you need to ask in order to get the permutation π?

A: $\log_2 n! = \Theta(n \log n)$

Comparison-Based Sorting Algorithms

Q: Can we do better than $O(n \log n)$ for sorting?

A: No, for comparison-based sorting algorithms.

- Bob has a permutation π over $\{1, 2, 3, \cdots, n\}$ in his hand.
- You can ask Bob questions of the form "does i appear before j in π?"

Q: How many questions do you need to ask in order to get the permutation π?

A: At least $\log_2 n! = \Theta(n \log n)$

Outline

Selection Problem

Input: a set A of n numbers, and $1 \leq i \leq n$

Output: the i-th smallest number in A

- Sorting solves the problem in time $O(n \log n)$.
- Our goal: $O(n)$ running time

Recall: Quicksort with Median Finder

quicksort(A, n)

```
if n ≤ 1 then return A
x ← lower median of A
A_L ← elements in A that are less than x                ▷ Divide
A_R ← elements in A that are greater than x             ▷ Divide
B_L ← quicksort(A_L, A_L.size)                          ▷ Conquer
B_R ← quicksort(A_R, A_R.size)                          ▷ Conquer
t ← number of times x appear A
return the array obtained by concatenating B_L, the array
    containing t copies of x, and B_R
```

Selection Algorithm with Median Finder

selection(A, n, i)

if $n = 1$ **then return** A
$x \leftarrow$ lower median of A
$A_L \leftarrow$ elements in A that are less than x ▷ Divide
$A_R \leftarrow$ elements in A that are greater than x ▷ Divide
if $i \leq A_L$.size **then**
    return selection$(A_L, A_L.\text{size}, i)$ ▷ Conquer
else if $i > n - A_R$.size **then**
    return selection$(A_R, A_R.\text{size}, i - (n - A_R.\text{size}))$ ▷ Conquer
else
    return x

- Recurrence for selection: $T(n) = T(n/2) + O(n)$
- Solving recurrence: $T(n) = O(n)$

Randomized Selection Algorithm

selection(A, n, i)

if $n = 1$ **thenreturn** A
$x \leftarrow$ random element of A (called pivot)
$A_L \leftarrow$ elements in A that are less than x ▷ Divide
$A_R \leftarrow$ elements in A that are greater than x ▷ Divide
if $i \leq A_L$.size **then**
    return selection$(A_L, A_L.\text{size}, i)$ ▷ Conquer
else if $i > n - A_R$.size **then**
    return selection$(A_R, A_R.\text{size}, i - (n - A_R.\text{size}))$ ▷ Conquer
else
    return x

- expected running time $= O(n)$

Outline

Polynomial Multiplication

Input: two polynomials of degree $n-1$

Output: product of two polynomials

Example:

$$
\begin{aligned}
&(3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5) \\
= {}& 6x^6 - 9x^5 + 18x^4 - 15x^3 \\
&+ 4x^5 - 6x^4 + 12x^3 - 10x^2 \\
&- 10x^4 + 15x^3 - 30x^2 + 25x \\
&+ 8x^3 - 12x^2 + 24x - 20 \\
= {}& 6x^6 - 5x^5 + 2x^4 + 20x^3 - 52x^2 + 49x - 20
\end{aligned}
$$

- **Input**: $(4, -5, 2, 3), (-5, 6, -3, 2)$
- **Output**: $(-20, 49, -52, 20, 2, -5, 6)$

Discrete Convolution on Finite Domain

- $f : \{0, 1, \cdots, n-1\} \to \mathbb{R}, g : \{0, 1, \cdots, m-1\} \to \mathbb{R}$
- the convolution of f and g, denoted as $h := f \times g$, is defined as

$$h(k) := \sum_{i,j:i+j=k} f(i)g(j) \qquad \forall k \in \{0, 1, 2, \cdots, m+n-2\}$$

	0	1	2	3	4	5	6
f	4	-5	2	3			
g	-5	6	-3	2			
$f \times g$	-20	49	-52	20	2	-5	6

Applications of Convolutions

- Polynomial and integer multiplication
- Signal and Image Processing
- Probability theory: Sum of two distributions
- Convolutional neural network

- Polynomial multiplication $\Leftrightarrow$ Convolution
- We shall focus on multiplication.

Big Integer Multiplication Using Polynomial Multiplication

- $16103416169 \times 424317167$
- $(16x^3 + 103x^2 + 416x + 169) \times (424x^2 + 317x + 167)$
- $6784x^5 + 48744x^4 + 211707x^3 + 220729x^2 + 123045x + 28223$
- $6784, 48744, 211707, 220729, 123045, 282236784, 48744, 211707, 220$
- 6832955927852073223

Naïve Algorithm

polynomial-multiplication(A, B, n)

let $C[k] \leftarrow 0$ for every $k = 0, 1, 2, \cdots, 2n-2$
for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
        $C[i+j] \leftarrow C[i+j] + A[i] \times B[j]$
return C

Running time: $O(n^2)$

Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x + 2)x^2 + (-5x + 4)$$
$$q(x) = 2x^3 - 3x^2 + 6x - 5 = (2x - 3)x^2 + (6x - 5)$$

- $p(x)$: degree of $n - 1$ (assume n is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

$$\begin{aligned} pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\ &= p_H q_H x^n + \left(p_H q_L + p_L q_H\right)x^{n/2} + p_L q_L \end{aligned}$$

Divide-and-Conquer for Polynomial Multiplication

$$\begin{aligned} pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\ &= p_H q_H x^n + \left(p_H q_L + p_L q_H\right) x^{n/2} + p_L q_L \end{aligned}$$

$$\begin{aligned} \text{multiply}(p, q) = {} & \text{multiply}(p_H, q_H) \times x^n \\ & + \left(\text{multiply}(p_H, q_L) + \text{multiply}(p_L, q_H)\right) \times x^{n/2} \\ & + \text{multiply}(p_L, q_L) \end{aligned}$$

- Recurrence: $T(n) = 4T(n/2) + O(n)$
- $T(n) = O(n^2)$

Reduce Number from 4 to 3

$$
\begin{aligned}
pq &= \big(p_H x^{n/2} + p_L\big)\big(q_H x^{n/2} + q_L\big) \\
&= p_H q_H x^n + \big(p_H q_L + p_L q_H\big) x^{n/2} + p_L q_L
\end{aligned}
$$

- $p_H q_L + p_L q_H = (p_H + p_L)(q_H + q_L) - p_H q_H - p_L q_L$

Divide-and-Conquer for Polynomial Multiplication

$$r_H = \textsf{multiply}(p_H, q_H)$$
$$r_L = \textsf{multiply}(p_L, q_L)$$

$$\begin{aligned}\textsf{multiply}(p, q) = & \; r_H \times x^n \\ & + \big(\textsf{multiply}(p_H + p_L, q_H + q_L) - r_H - r_L\big) \times x^{n/2} \\ & + r_L\end{aligned}$$

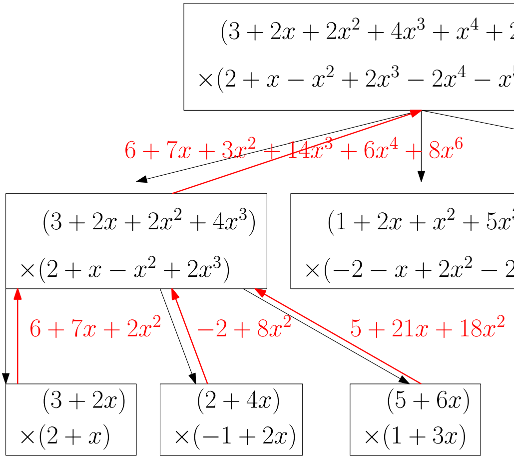
- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\log_2 3}) = O(n^{1.585})$

Assumption n is a power of 2. Arrays are 0-indexed.

multiply(A, B, n)

if $n = 1$ then return $(A[0]B[0])$
$A_L \leftarrow A[0 \mathinner{..} n/2-1], A_H \leftarrow A[n/2 \mathinner{..} n-1]$
$B_L \leftarrow B[0 \mathinner{..} n/2-1], B_H \leftarrow B[n/2 \mathinner{..} n-1]$
$C_L \leftarrow$ multiply$(A_L, B_L, n/2)$
$C_H \leftarrow$ multiply$(A_H, B_H, n/2)$
$C_M \leftarrow$ multiply$(A_L + A_H, B_L + B_H, n/2)$
$C \leftarrow$ array of $(2n-1)$ 0's
for $i \leftarrow 0$ to $n-2$ **do**
    $C[i] \leftarrow C[i] + C_L[i]$
    $C[i+n] \leftarrow C[i+n] + C_H[i]$
    $C[i+n/2] \leftarrow C[i+n/2] + C_M[i] - C_L[i] - C_H[i]$
return C

Example



0	1	2	3	4	5	6
6	7	2		−2	0	8
		1	14	8		
6	7	3	14	6	0	8

$$(5+21x+18x^2)-(6+7x+2x^2)-(-2+8x^2)=1+14x+8x^2$$

$$(6+7x+2x^2)+(1+14x+8x^2)x^2+(-2+8x^2)x^4$$
$$=6+7x+3x^2+14x^3+6x^4+8x^6$$

Outline

Matrix Multiplication

Input: two $n \times n$ matrices A and B

Output: $C = AB$

Naive Algorithm: matrix-multiplication(A, B, n)

for $i \leftarrow 1$ to n **do**
    for $j \leftarrow 1$ to n **do**
        $C[i, j] \leftarrow 0$
        for $k \leftarrow 1$ to n **do**
            $C[i, j] \leftarrow C[i, j] + A[i, k] \times B[k, j]$
return C

- running time $= O(n^3)$

Try to Use Divide-and-Conquer

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

- $C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$
- matrix_multiplication(A, B) recursively calls matrix_multiplication(A_{11}, B_{11}), matrix_multiplication(A_{12}, B_{21}), $\ldots$
- Recurrence for running time: $T(n) = 8T(n/2) + O(n^2)$
- $T(n) = O(n^3)$
- Strassen's Algorithm: $T(n) = 7T(n/2) + O(n^2)$
- Solving Recurrence $T(n) = O(n^{\log_2 7}) = O(n^{2.808})$

Strassen's Algorithm

$$A = \overbrace{\begin{array}{|c|c|}\hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array}}^{n/2} \; n/2 \qquad B = \overbrace{\begin{array}{|c|c|}\hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array}}^{n/2} \; n/2$$

- $C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$

- $M_1 \leftarrow (A_{11} + A_{22}) \times (B_{11} + B_{22})$
- $M_2 \leftarrow (A_{21} + A_{22}) \times B_{11}$
- $M_3 \leftarrow A_{11} \times (B_{12} - B_{22})$
- $M_4 \leftarrow A_{22} \times (B_{21} - B_{11})$
- $M_5 \leftarrow (A_{11} + A_{12}) \times B_{22}$
- $M_6 \leftarrow (A_{21} - A_{11}) \times (B_{11} + B_{12})$
- $M_7 \leftarrow (A_{12} - A_{22}) \times (B_{21} + B_{22})$

- $C_{11} \leftarrow M_1 + M_4 - M_5 + M_7$
- $C_{12} \leftarrow M_3 + M_5$
- $C_{21} \leftarrow M_2 + M_4$
- $C_{22} \leftarrow M_1 - M_2 + M_3 + M_6$

Outline

Interpolation of Polynomials

- $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$
- Known: given the value of $p(x)$ for n different values of x, p is uniquely determined
- $p(x) = 1 - x + 2x^2 : p(0) = 1, p(1) = 2, p(2) = 7$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

- Given $p(0) = 1, p(1) = 2, p(2) = 7$, to recover p:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

- $p(x) = 1 - x + 2x^2$

Using Interpolation for Polynomial Multiplication

- $p(x) = 1 - x + 2x^2, \qquad q(x) = 3 - x^2$
- Interpolation on 5 points $\{0, 1, 2, 3, 4\}$:

$$\text{interpolation for } p : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 7 \\ 16 \\ 29 \end{pmatrix}$$

$$\text{interpolation for } q : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \\ -6 \\ -13 \end{pmatrix}$$

Interpolation of pq:

$$\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 1\\ 1 & 2 & 4 & 8 & 16\\ 1 & 3 & 9 & 27 & 81\\ 1 & 4 & 16 & 64 & 256\end{pmatrix}\begin{pmatrix}c_0\\ c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = \begin{pmatrix}3\\ 4\\ -7\\ -102\\ -377\end{pmatrix}$$

$$\begin{pmatrix}c_0\\ c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 1\\ 1 & 2 & 4 & 8 & 16\\ 1 & 3 & 9 & 27 & 81\\ 1 & 4 & 16 & 64 & 256\end{pmatrix}^{-1}\begin{pmatrix}3\\ 4\\ -7\\ -96\\ -377\end{pmatrix}$$

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{25}{12} & 4 & -3 & \frac{4}{3} & -\frac{1}{4} \\ \frac{35}{24} & -\frac{13}{3} & \frac{19}{4} & -\frac{7}{3} & \frac{11}{24} \\ -\frac{5}{12} & \frac{3}{2} & -2 & \frac{7}{6} & -\frac{1}{4} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -7 \\ -96 \\ -377 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 5 \\ 1 \\ -2 \end{pmatrix}$$

$$pq = (1 - x + 2x^2)(3 - x^2) = 3 - 3x + 5x^2 + x^3 - 2x^4$$

Multiplication of two polynomials of degree $n-1$

- Choose $2n-1$ distinct values $x_0, x_1, x_2, \cdots, x_{m-1}$ carefully, $m = 2n-1$
- Compute the interpolation of p and q:

$$M := \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{m-1} & x_{m-1}^2 & x_m^3 & \cdots & x_{m-1}^{n-1} \end{pmatrix}$$

$$M \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{m-1} \end{pmatrix} \qquad M \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{m-1} \end{pmatrix}$$

Multiplication of two polynomials of degree $n-1$

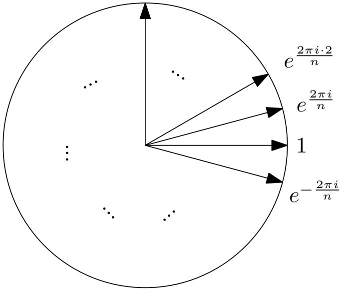
$$M \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{m-1} \end{pmatrix} = \begin{pmatrix} y_0 z_0 \\ y_1 z_1 \\ y_2 z_2 \\ \vdots \\ y_{m-1} z_{m-1} \end{pmatrix} \qquad \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{m-1} \end{pmatrix} = M^{-1} \begin{pmatrix} y_0 z_0 \\ y_1 z_1 \\ y_2 z_2 \\ \vdots \\ y_{m-1} z_{m-1} \end{pmatrix}$$

$$\begin{aligned} &(a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}) \\ \times\ &(b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}) \\ =\ &(c_0 + c_1 x + c_2 x^2 + \cdots + c_{2n-2} x^{2n-2}) \end{aligned}$$

Q: How should we set $x_0, x_1, \cdots, x_{n-1}$ so that we can compute Ma and $M^{-1}y$ fast (for any $a, y \in \mathbb{R}^{\{0,1,\cdots,n-1\}}$)?

A: Use the n complex roots of the equation $x^n = 1$

- $e^{\frac{2\pi i \cdot k}{n}} = \cos\left(\frac{2\pi \cdot k}{n}\right) + i \cdot \sin\left(\frac{2\pi \cdot k}{n}\right), k \in \{0, 1, \cdots, n-1\}$
- $\omega := e^{\frac{2\pi i}{n}}$, n-th roots are $1, \omega, \omega^2, \cdots, \omega^{n-1}$



$$F_n := \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \end{pmatrix}$$

- Interpolation and Inverse-Interpolation:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} = F_n \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} \qquad \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} = F_n^{-1} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

- Interpolation: Fast Fourier Transform (FFT)
- Invert-Interpolation: Inverse Fast Fourier Transform (iFFT)

Fast Fourier Transform: Divide and Conquer

- Assume n is even.

Breaking polynomial into even and odd parts

- $p_{\text{even}}(x) := a_0 + a_2 x + a_4 x^2 + \cdots + a_{n-2} x^{n/2-1}$
- $p_{\text{old}}(x) := a_1 + a_3 x + a_5 x^2 + \cdots + a_{n-1} x^{n/2-1}$
- $p(x) = p_{\text{even}}(x^2) + p_{\text{odd}}(x^2) \cdot x$

$$p(\omega^k) = p_{\text{even}}(\omega^{2k}) + p_{\text{odd}}(\omega^{2k}) \cdot \omega^k, \qquad k = 0, 1, \cdots, \frac{n}{2} - 1$$
$$p(\omega^{n/2+k}) = p_{\text{even}}(\omega^{2k}) - p_{\text{odd}}(\omega^{2k}) \cdot \omega^k, \qquad k = 0, 1, \cdots, \frac{n}{2} - 1$$

- Assume n is an integer power of 2

FFT$(n, a_0, a_1, \cdots, a_{n-1})$

if $n = 1$ **then return** (a_0)

$(e_0, e_1, \cdots, e_{n/2-1}) \leftarrow \mathrm{FFT}(n/2, a_0, a_2, \cdots, a_{n-2})$

$(o_0, o_1, \cdots, o_{n/2-1}) \leftarrow \mathrm{FFT}(n/2, a_1, a_3, \cdots, a_{n-1})$

for $k \leftarrow 0, 1, 2, \cdots n/2 - 1$ **do**

  $y_k \leftarrow e_k + o_k \cdot \omega^k$

  $y_{n/2+k} \leftarrow e_k - o_k \cdot \omega^k$

return $(y_0, y_1, \cdots, y_{n-1})$

- Recurrence for running time: $T(n) = 2T(n/2) + O(n)$
- $T(n) = O(n \log n)$

Example for one recursion of FFT

- $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (3, 2, 1, 2, 5, 6, 1, 4)$

$$\begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 6 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} o_0 \\ o_1 \\ o_2 \\ o_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ -4-2i \\ 2 \\ -4+2i \end{pmatrix}$$

- $\omega = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}$
- $y_0 = e_0 + o_0 = 10 + 14 = 24$
- $y_1 = e_1 + o_1\omega = -2 + (-4-2i)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}\right) = -2 - 2\sqrt{2} - 3\sqrt{2}i$
- $y_6 = e_2 - o_2\omega^2 = 6 - 2i \qquad y_7 = e_3 - o_3\omega^3$

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}$$
$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1}x^{n-1}$$

multiplying p and q, ▷ assuming n is a power of 2

$y \leftarrow \mathrm{FFT}(2n, a_0, a_1, \cdots, a_{n-1}, 0, 0, \cdots, 0)$
$z \leftarrow \mathrm{FFT}(2n, b_0, b_1, \cdots, b_{n-1}, 0, 0, \cdots, 0)$
$c \leftarrow \mathrm{iFFT}(2n, y_0 z_0, y_1 z_1, \cdots, y_{2n-1} z_{2n-1})$
return $(c_0, c_1, \cdots, c_{2n-2})$

- iFFT$(n, y_0, y_1, \cdots, y_{n-1})$: inverse FFT procedure: multiplying input vector y by the inverse of F_n, which is

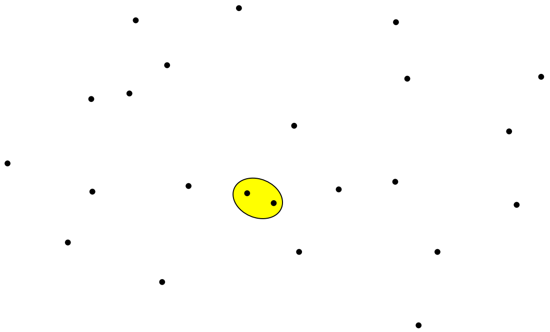
$$\frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \end{pmatrix}$$

Outline

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

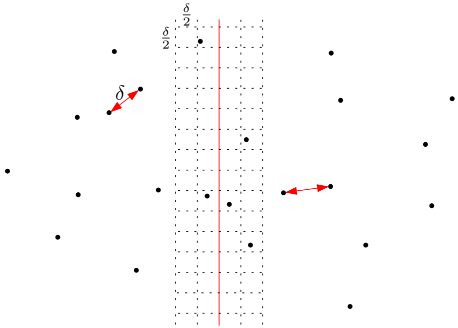
Output: the pair of points that are closest



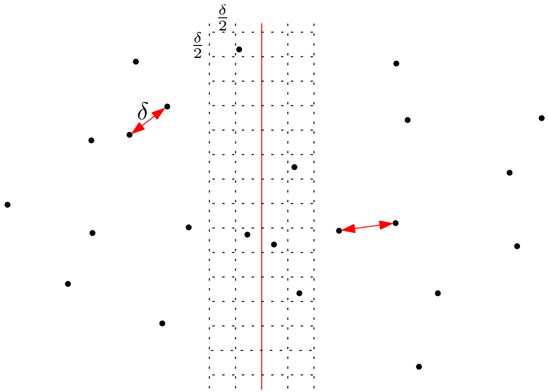
- Trivial algorithm: $O(n^2)$ running time

Divide-and-Conquer Algorithm for Closest Pair

- **Divide**: Divide the points into two halves via a vertical line
- **Conquer**: Solve two sub-instances recursively
- **Combine**: Check if there is a closer pair between left-half and right-half



Divide-and-Conquer Algorithm for Closest Pair



- Each box contains at most one pair
- For each point, only need to consider $O(1)$ boxes nearby
- Implementation: Sort points inside the stripe according to y-coordinates
- For every point, consider $O(1)$ points around it in the order

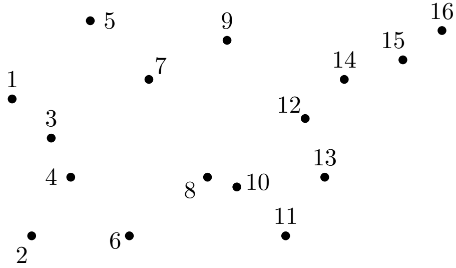
- time for combine step $= O(n \log n)$
- recurrence: $T(n) = 2T(n/2) + O(n \log n)$
- solving recurrence: $T(n) = O(n \log^2 n)$

Improve the running time of combine step to $O(n)$

- also sort the points in ascending order of y values at the beginning
- pass the sequence to the root recursion
- constructing two sub-sequences from the sequence, and pass them to the two sub-recursions respectively

- $T(n) = 2T(n/2) + O(n) \Longrightarrow T(n) = O(n \log n)$

Example for Closest Pair



- CP$(1, 16, (5, 16, 9, 15, 7, 14, 1, 12, 3, 4, 8, 13, 10, 11, 2, 6))$
- CP$(1, 8, (5, 7, 1, 3, 4, 8, 2, 6))$
- CP$(1, 4, (1, 3, 4, 2))$
- CP$(5, 8, (5, 7, 8, 6))$
- CP$(9, 16, (16, 9, 15, 14, 12, 13, 10, 11))$

Outline

Fibonacci Numbers

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

n-th Fibonacci Number

Input: integer $n > 0$

Output: F_n

Computing F_n : Stupid Divide-and-Conquer Algorithm

Fib(n)

```
if n = 0 return 0
if n = 1 return 1
return Fib(n − 1) + Fib(n − 2)
```

Q: Is the running time of the algorithm polynomial or exponential in n?

A: Exponential

- Running time is at least $\Omega(F_n)$
- F_n is exponential in n

Computing F_n: Reasonable Algorithm

Fib(n)

$F[0] \leftarrow 0$
$F[1] \leftarrow 1$
for $i \leftarrow 2$ to n **do**
    $F[i] \leftarrow F[i-1] + F[i-2]$
return $F[n]$

- Dynamic Programming
- Running time $= O(n)$

Computing F_n: Even Better Algorithm

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$\cdots$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

power(n)

if $n = 0$ then return $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$R \leftarrow$ power($\lfloor n/2 \rfloor$)
$R \leftarrow R \times R$
if n is odd then $R \leftarrow R \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
return R

Fib(n)

if $n = 0$ then return 0
$M \leftarrow$ power($n - 1$)
return $M[1][1]$

- Recurrence for running time? $T(n) = T(n/2) + O(1)$
- $T(n) = O(\log n)$

Running time $= O(\log n)$: We Cheated!

Q: How many bits do we need to represent $F(n)$?

A: $\Theta(n)$

- We can not add (or multiply) two integers of $\Theta(n)$ bits in $O(1)$ time
- Even printing $F(n)$ requires time much larger than $O(\log n)$

Fixing the Problem

To compute F_n, we need $O(\log n)$ basic arithmetic operations on integers

Summary: Divide-and-Conquer

- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance
- Write down recurrence for running time
- Solve recurrence using master theorem

Summary: Divide-and-Conquer

- Merge sort, quicksort, count-inversions, closest pair, FFT, $\cdots$:
  $T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$
- Polynomial Multiplication:
  $T(n) = 3T(n/2) + O(n) \Rightarrow T(n) = O(n^{\log_2 3})$
- Matrix Multiplication:
  $T(n) = 7T(n/2) + O(n^2) \Rightarrow T(n) = O(n^{\log_2 7})$

- To improve running time, design better algorithm for "combine" step, or reduce number of recursions, ...