算法设计与分析(2025年春季学期)
Divide-and-Conquer

授课老师: 栗师

南京大学计算机学院

Outline

1. Divide-and-Conquer
2. Counting Inversions
3. Solving Recurrences
4. Quicksort and Selection
    - Quicksort
    - Lower Bound for Comparison-Based Sorting Algorithms
    - Selection Problem
5. Polynomial Multiplication
6. Strassen's Algorithm for Matrix Multiplication
7. FFT(Fast Fourier Transform): Polynomial Multiplication in $O(n \log n)$ Time
8. Finding Closest Pair of Points in 2D Euclidean Space
9. Computing n-th Fibonacci Number

Greedy Algorithm

- mainly for combinatorial optimization problems
- trivial algorithm runs in exponential time
- greedy algorithm gives an efficient algorithm
- main focus of analysis: correctness of algorithm

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Divide-and-Conquer

- not necessarily for combinatorial optimization problems
- trivial algorithm already runs in polynomial time
- divide-and-conquer gives a more efficient algorithm
- main focus of analysis: running time

Divide-and-Conquer

- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance

merge-sort(A, n)

```
if $n = 1$ then
    return $A$
else
    $B \leftarrow$ merge-sort$\left(A\left[1..\lfloor n/2\rfloor\right], \lfloor n/2\rfloor\right)$
    $C \leftarrow$ merge-sort$\left(A\left[\lfloor n/2\rfloor + 1..n\right], \lceil n/2\rceil\right)$
    return merge$(B, C, \lfloor n/2\rfloor, \lceil n/2\rceil)$
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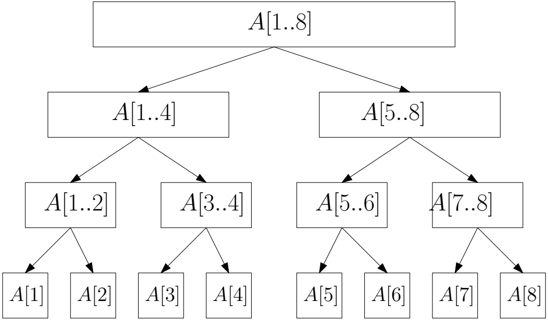
```

- Divide: trivial
- Conquer: 4, 5
- Combine: 6

Running Time for Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\log n)$ levels
- Running time $= O(n \log n)$
- Better than insertion sort

Running Time for Merge-Sort Using Recurrence

- $T(n) =$ running time for sorting n numbers,then

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) & \text{if } n \geq 2 \end{cases}$$

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- Solving this recurrence, we have $T(n) = O(n \log n)$ (we shall show how later)

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Example:

$$10 \qquad 8 \qquad 15 \qquad 9 \qquad 12$$

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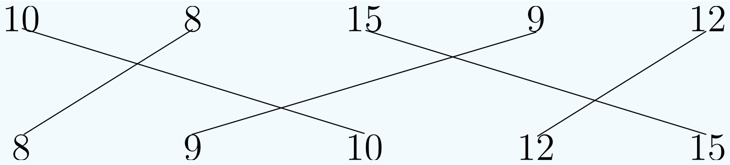

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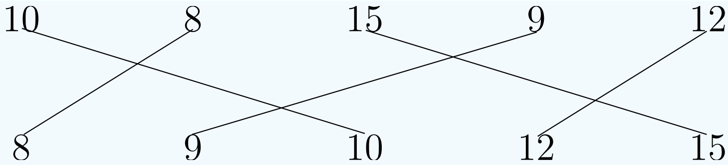


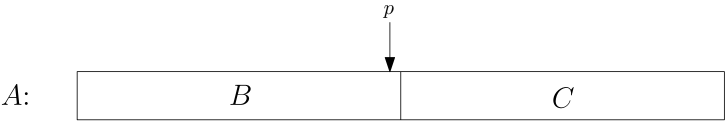
- 4 inversions (for convenience, using numbers, not indices): $(10, 8), (10, 9), (15, 9), (15, 12)$

Naive Algorithm for Counting Inversions

count-inversions(A, n)

```
$c \leftarrow 0$
for every $i \leftarrow 1$ to $n - 1$ do
    for every $j \leftarrow i + 1$ to $n$ do
        if $A[i] > A[j]$ then $c \leftarrow c + 1$
return $c$
```

Divide-and-Conquer



p

A: B C

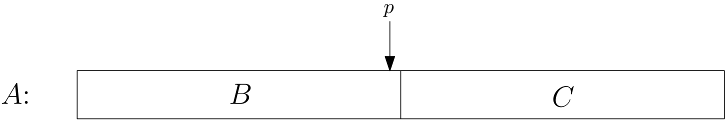
- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p+1..n]$
- $$\#\text{invs}(A) = \#\text{invs}(B) + \#\text{invs}(C) + m$$ $$m = \left|\left\{(i,j) : B[i] > C[j]\right\}\right|$$

Q: How fast can we compute m, via trivial algorithm?

A: $O(n^2)$

- Can not improve the $O(n^2)$ time for counting inversions.

Divide-and-Conquer



- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p+1..n]$
- $$\#\text{invs}(A) = \#\text{invs}(B) + \#\text{invs}(C) + m$$
  $$m = \big|\{(i,j) : B[i] > C[j]\}\big|$$

Lemma If both B and C are sorted, then we can compute m in $O(n)$ time!

Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:

B:

3	8	12	20	32	48

total$=$ 0

C:

5	7	9	25	29

Counting Inversions between B and C

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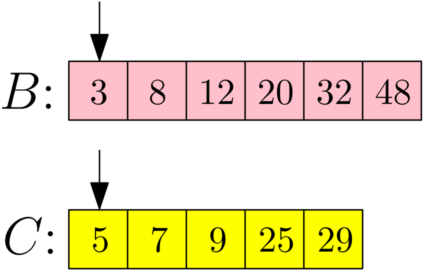

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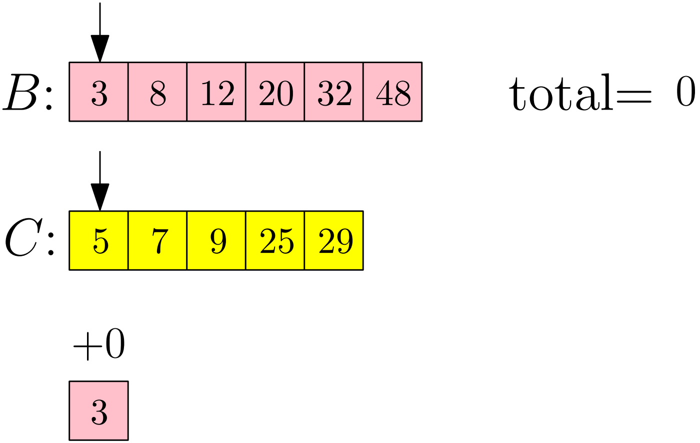

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$+0$

3

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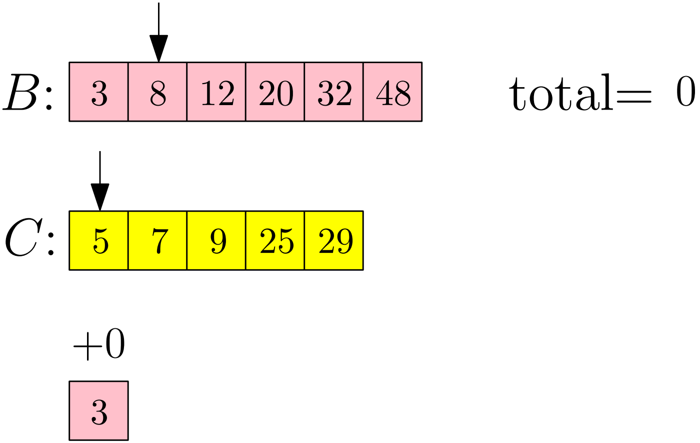

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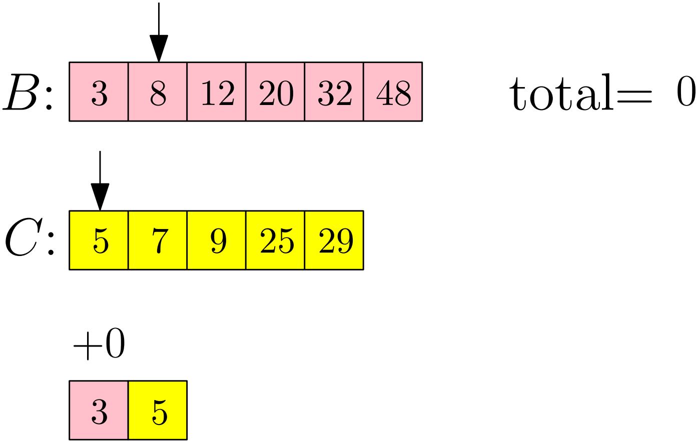

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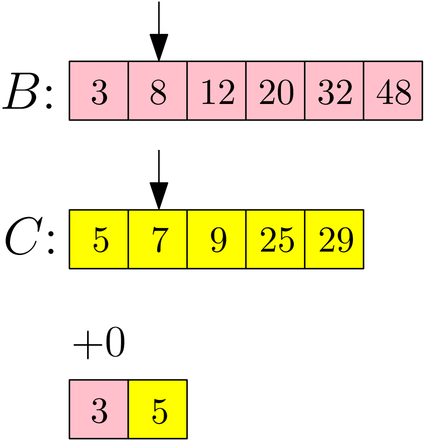

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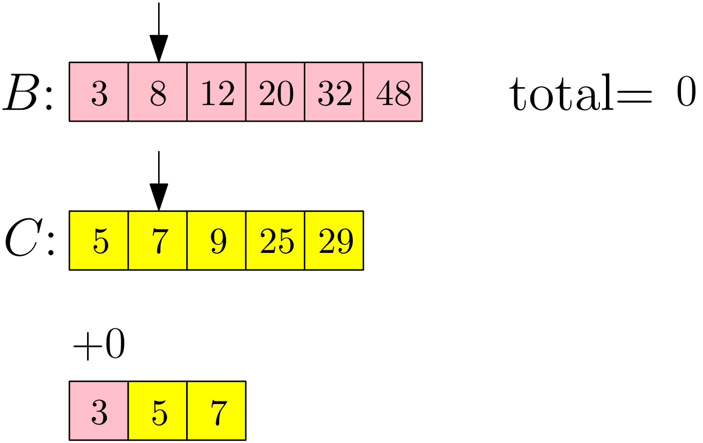

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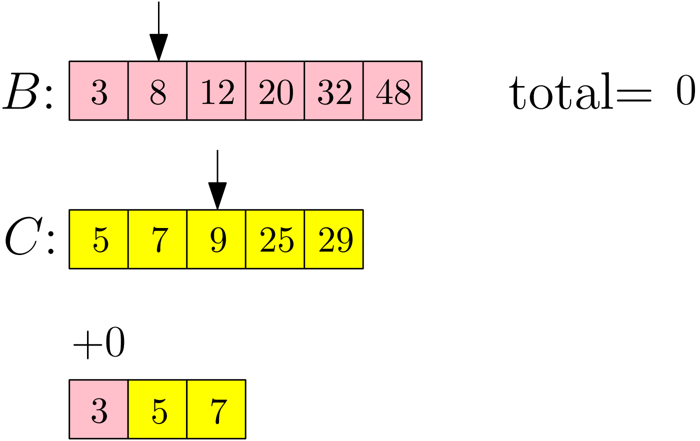

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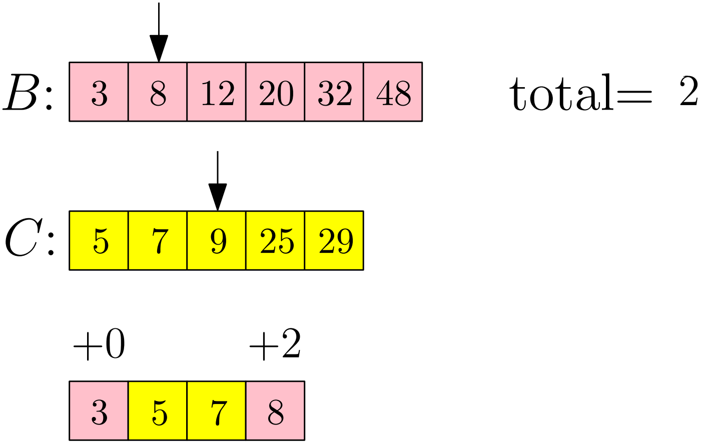

B: 3 | 8 | 12 | 20 | 32 | 48

total$=$ 2

C: 5 | 7 | 9 | 25 | 29

$+0$ $+2$

3 | 5 | 7 | 8

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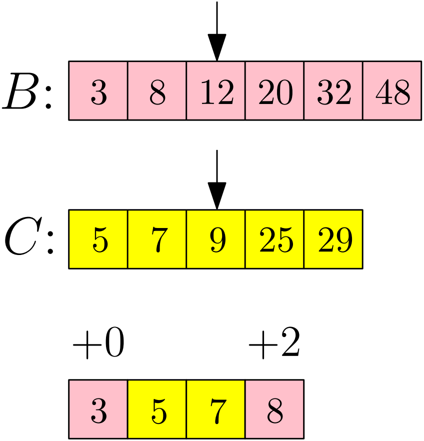

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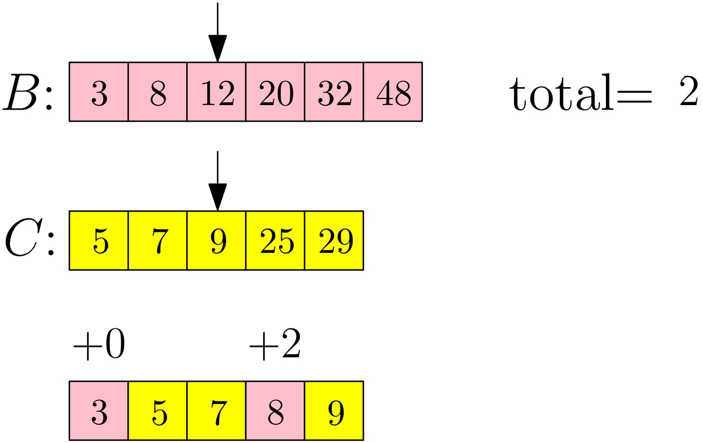

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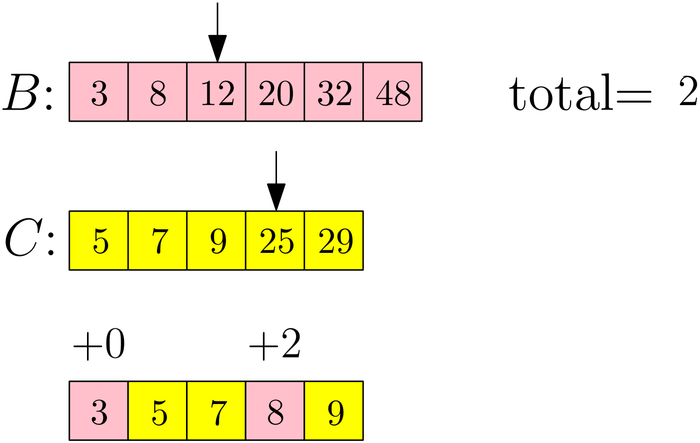

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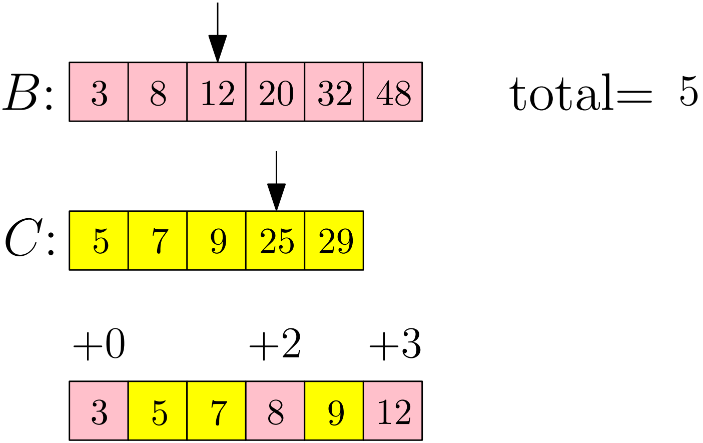

B: | 3 | 8 | 12 | 20 | 32 | 48 |

total$=$ 5

C: | 5 | 7 | 9 | 25 | 29 |

$+0$ $+2$ $+3$

| 3 | 5 | 7 | 8 | 9 | 12 |

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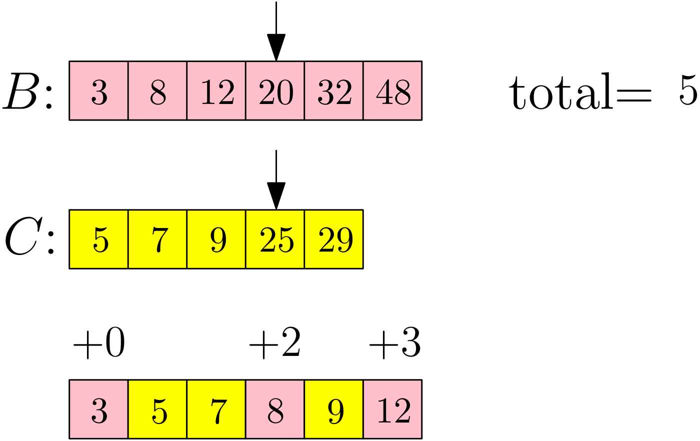

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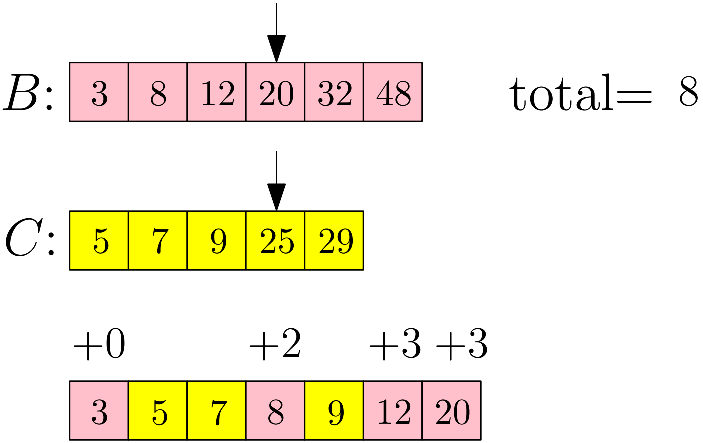

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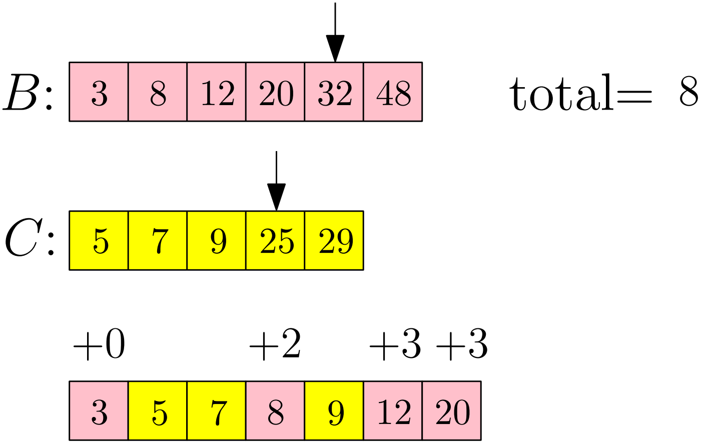

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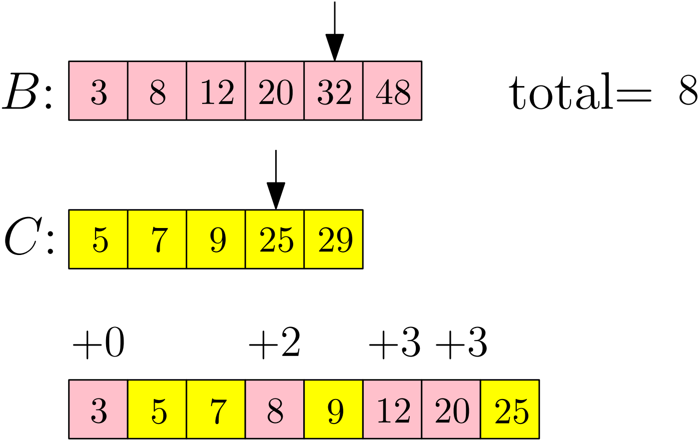

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+0 +2 +3 +3

3 | 5 | 7 | 8 | 9 | 12 | 20 | 25

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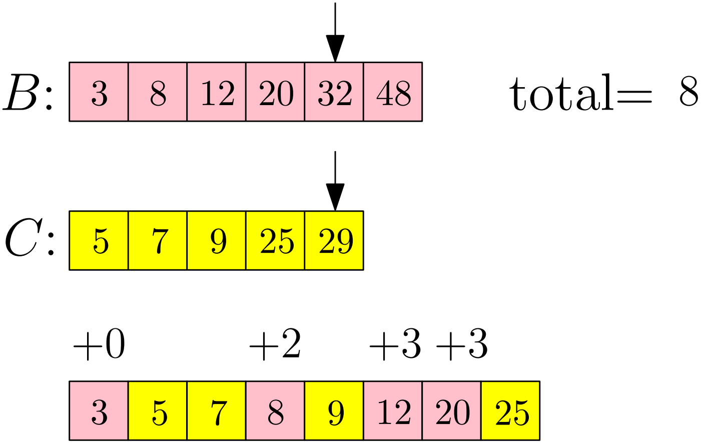

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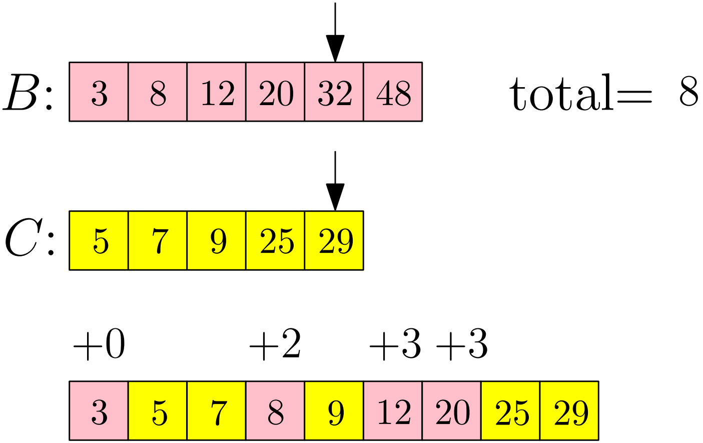

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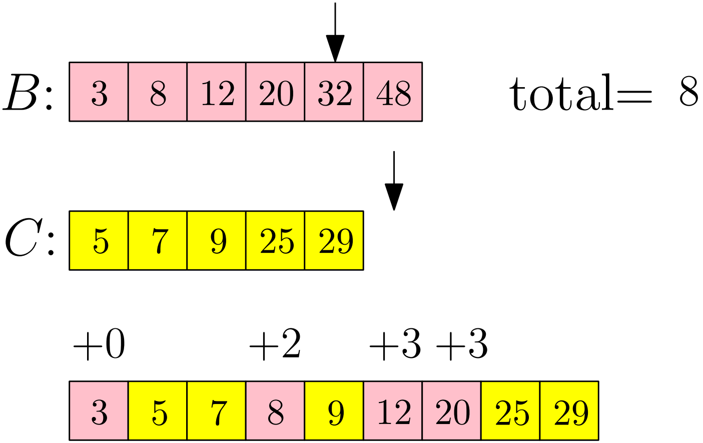

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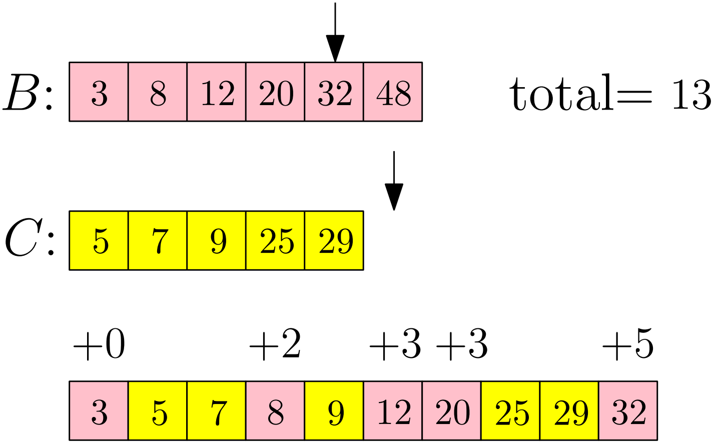

B: 3 8 12 20 32 48

total= 13

C: 5 7 9 25 29

+0 +2 +3 +3 +5

3 5 7 8 9 12 20 25 29 32

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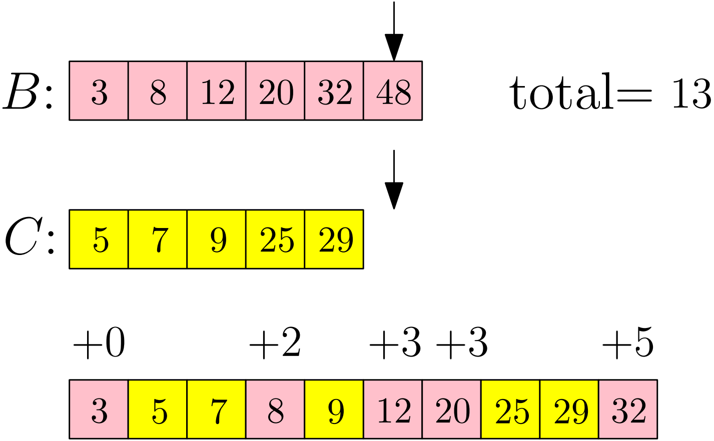

B:

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total $= 13$

C:

5	7	9	25	29

+0			+2		+3	+3			+5
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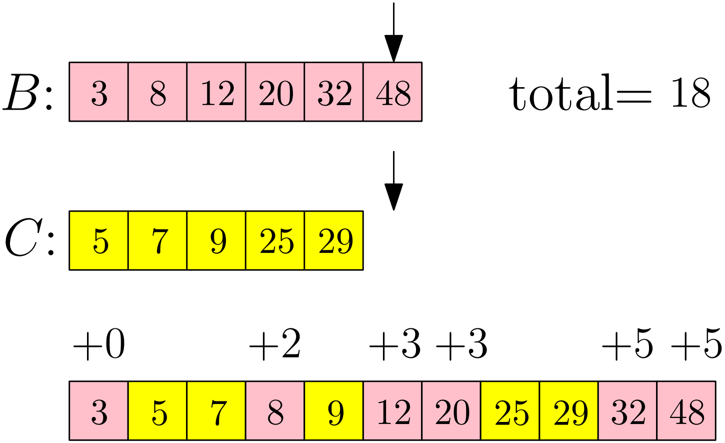

B:	3	8	12	20	32	48

$\text{total} = 18$

C:	5	7	9	25	29

$+0$			$+2$		$+3$	$+3$			$+5$	$+5$
3	5	7	8	9	12	20	25	29	32	48

Counting Inversions between B and C

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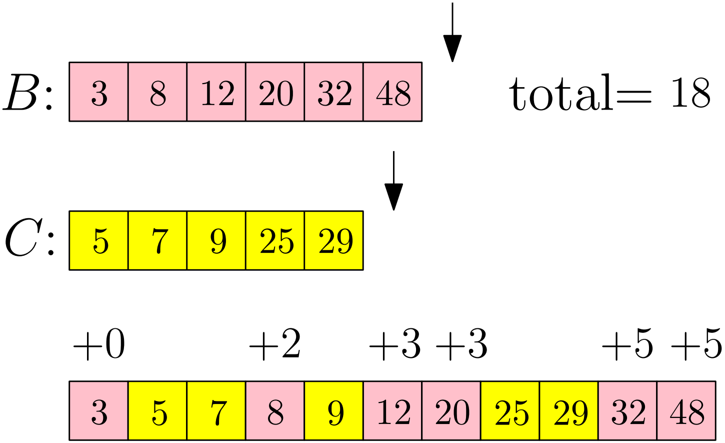

B: 3 8 12 20 32 48

total= 18

C: 5 7 9 25 29

$+0$ $+2$ $+3$ $+3$ $+5$ $+5$

3 5 7 8 9 12 20 25 29 32 48

Count Inversions between B and C

- Procedure that merges B and C and counts inversions between B and C at the same time

merge-and-count(B, C, n_1, n_2)

```
count ← 0;
A ← array of size $n_1 + n_2$; $i \leftarrow 1$; $j \leftarrow 1$
while $i \le n_1$ or $j \le n_2$ do
    if $j > n_2$ or ($i \le n_1$ and $B[i] \le C[j]$) then
        $A[i+j-1] \leftarrow B[i]$; $i \leftarrow i+1$
        $count \leftarrow count + (j-1)$
    else
        $A[i+j-1] \leftarrow C[j]$; $j \leftarrow j+1$
return $(A, count)$
```

Sort and Count Inversions in A

- A procedure that returns the sorted array of A and counts the number of inversions in A:

sort-and-count(A, n)

```
if $n = 1$ then
    return $(A, 0)$
else
    $(B, m_1) \leftarrow$ sort-and-count$\left(A\left[1..\lfloor n/2 \rfloor\right], \lfloor n/2 \rfloor\right)$
    $(C, m_2) \leftarrow$ sort-and-count$\left(A\left[\lfloor n/2 \rfloor + 1..n\right], \lceil n/2 \rceil\right)$
    $(A, m_3) \leftarrow$ merge-and-count$(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$
    return $(A, m_1 + m_2 + m_3)$
```

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```
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2:     return $(A, 0)$
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5:     $(C, m_2) \leftarrow$ sort-and-count$\left(A\left[\lfloor n/2 \rfloor + 1..n\right], \lceil n/2 \rceil\right)$
6:     $(A, m_3) \leftarrow$ merge-and-count$(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$
7:     return $(A, m_1 + m_2 + m_3)$
```

- Divide: trivial
- Conquer: 4, 5
- Combine: 6, 7

sort-and-count(A, n)

```
if $n = 1$ then
    return $(A, 0)$
else
    $(B, m_1) \leftarrow$ sort-and-count$\Big(A\big[1..\lfloor n/2\rfloor\big], \lfloor n/2\rfloor\Big)$
    $(C, m_2) \leftarrow$ sort-and-count$\Big(A\big[\lfloor n/2\rfloor + 1..n\big], \lceil n/2\rceil\Big)$
    $(A, m_3) \leftarrow$ merge-and-count$(B, C, \lfloor n/2\rfloor, \lceil n/2\rceil)$
    return $(A, m_1 + m_2 + m_3)$
```

- Recurrence for the running time: $T(n) = 2T(n/2) + O(n)$

sort-and-count(A, n)

```
if $n = 1$ then
    return $(A, 0)$
else
    $(B, m_1) \leftarrow$ sort-and-count$\Big(A\big[1..\lfloor n/2\rfloor\big], \lfloor n/2\rfloor\Big)$
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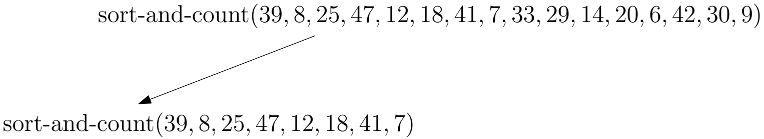
    return $(A, m_1 + m_2 + m_3)$
```

- Recurrence for the running time: $T(n) = 2T(n/2) + O(n)$
- Running time $= O(n \log n)$

Example

$$\text{sort-and-count}(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)$$

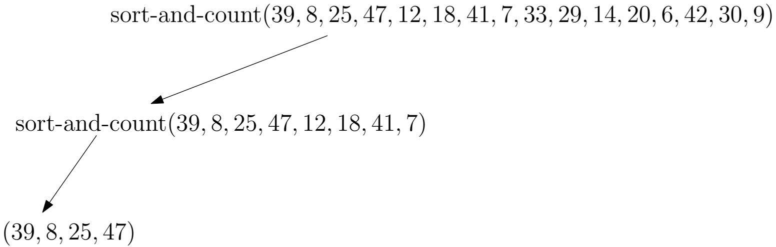
Example



sort-and-count(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)

sort-and-count(39, 8, 25, 47, 12, 18, 41, 7)

Example

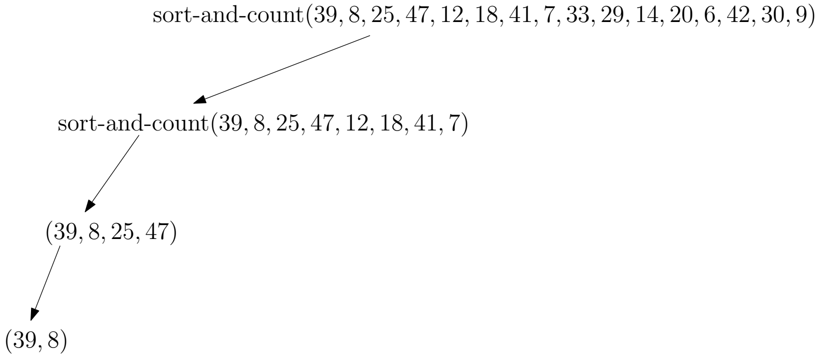


sort-and-count(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)

sort-and-count(39, 8, 25, 47, 12, 18, 41, 7)

(39, 8, 25, 47)

Example



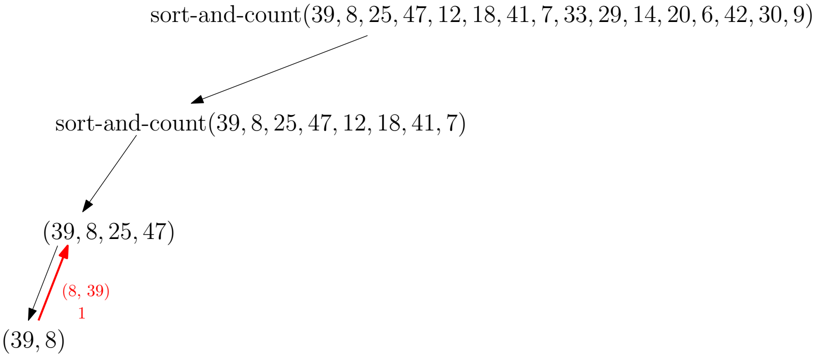
sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)$

sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7)$

$(39, 8, 25, 47)$

$(39, 8)$

Example



sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)$

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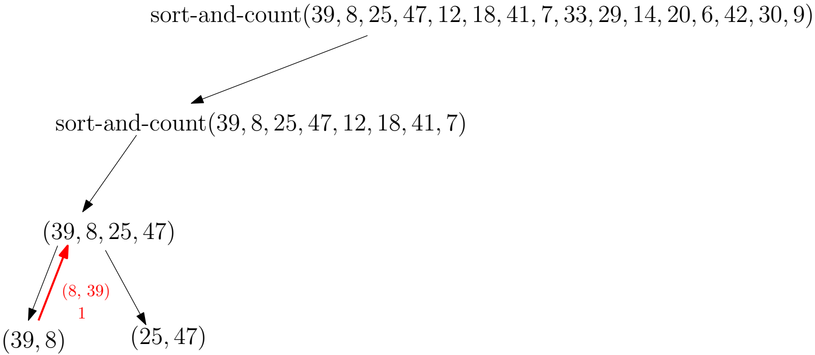
$(39, 8, 25, 47)$

$(39, 8)$

Example



sort-and-count(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)

sort-and-count(39, 8, 25, 47, 12, 18, 41, 7)

(39, 8, 25, 47)

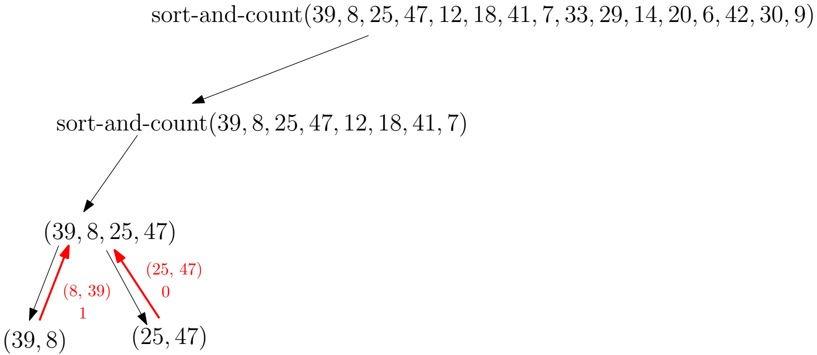
(39, 8)

(25, 47)

(8, 39)

1

Example



sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)$

sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7)$

$(39, 8, 25, 47)$

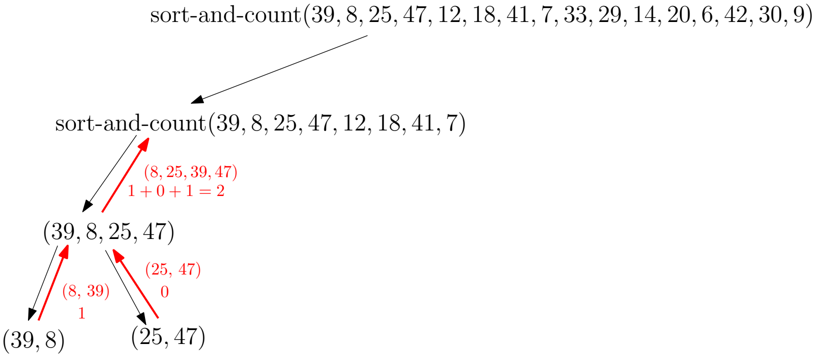
(8, 39)
1

(25, 47)
0

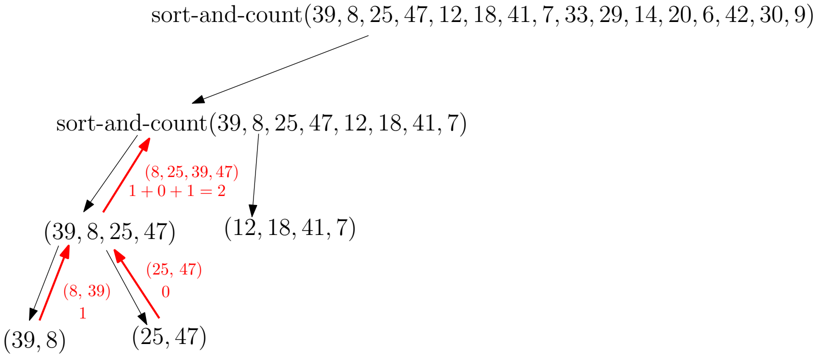
$(39, 8)$

$(25, 47)$

Example


sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)$
sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7)$
$(8, 25, 39, 47)$
$1 + 0 + 1 = 2$
$(39, 8, 25, 47)$
$(8,\ 39)$
1
$(25,\ 47)$
0
$(39, 8)$
$(25, 47)$

Example



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sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7)$

$(8, 25, 39, 47)$

$1 + 0 + 1 = 2$

$(39, 8, 25, 47)$

$(12, 18, 41, 7)$

$(25,\ 47)$

0

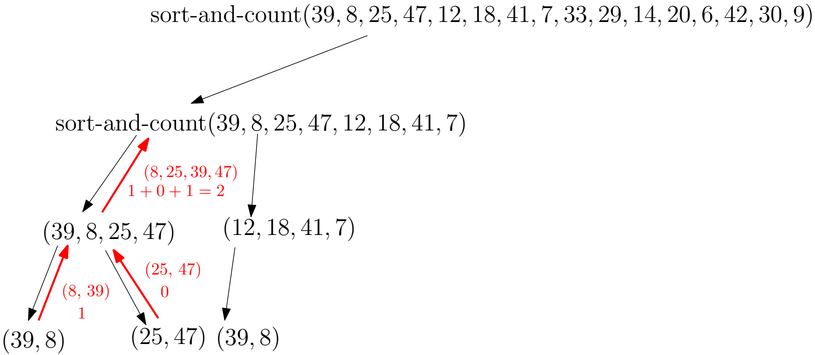
$(8,\ 39)$

1

$(39, 8)$

$(25, 47)$

Example



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sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7)$

$(8, 25, 39, 47)$

$1 + 0 + 1 = 2$

$(39, 8, 25, 47)$

$(12, 18, 41, 7)$

$(8,\ 39)$

1

$(25,\ 47)$

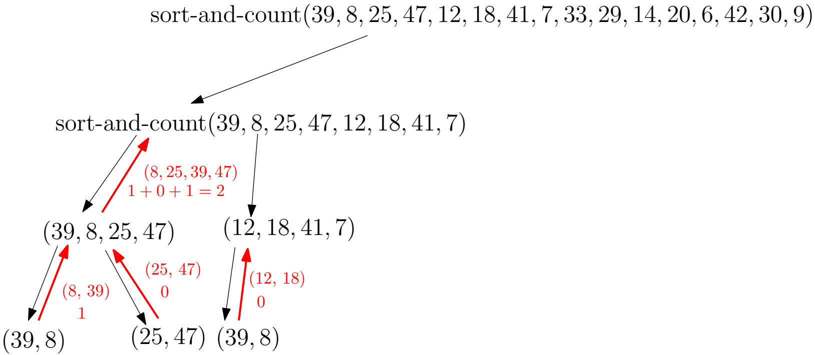
0

$(39, 8)$

$(25, 47)$

$(39, 8)$

Example



sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)$

sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7)$

$(8, 25, 39, 47)$

$1 + 0 + 1 = 2$

$(39, 8, 25, 47)$ $(12, 18, 41, 7)$

$(8,\ 39)$

1

$(25,\ 47)$

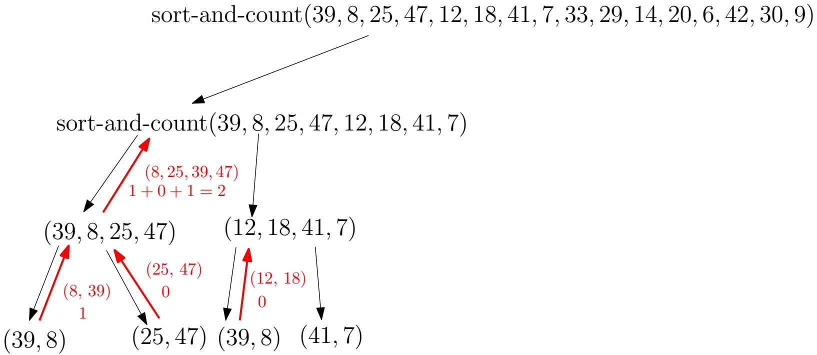
0

$(12,\ 18)$

0

$(39, 8)$ $(25, 47)$ $(39, 8)$

Example



sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)$

sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7)$

$(8, 25, 39, 47)$

$1 + 0 + 1 = 2$

$(39, 8, 25, 47)$

$(12, 18, 41, 7)$

$(8,\ 39)$

1

$(25,\ 47)$

0

$(12,\ 18)$

0

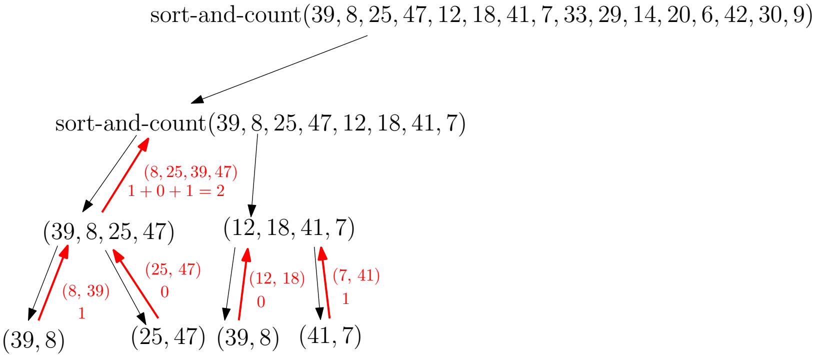
$(39, 8)$

$(25, 47)$

$(39, 8)$

$(41, 7)$

Example



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sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7)$

$(8, 25, 39, 47)$

$1 + 0 + 1 = 2$

$(39, 8, 25, 47)$

$(12, 18, 41, 7)$

(8, 39)

1

(25, 47)

0

(12, 18)

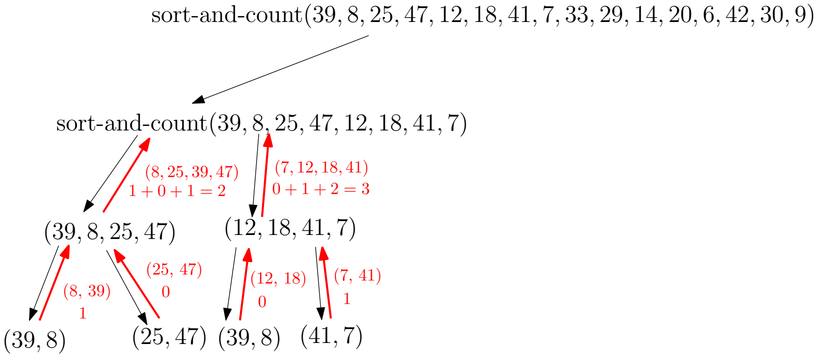
0

(7, 41)

1

$(39, 8)$ $(25, 47)$ $(39, 8)$ $(41, 7)$

Example



sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)$

sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7)$

$(8, 25, 39, 47)$

$1 + 0 + 1 = 2$

$(7, 12, 18, 41)$

$0 + 1 + 2 = 3$

$(39, 8, 25, 47)$

$(12, 18, 41, 7)$

$(25,\ 47)$

0

$(8,\ 39)$

1

$(12,\ 18)$

0

$(7,\ 41)$

1

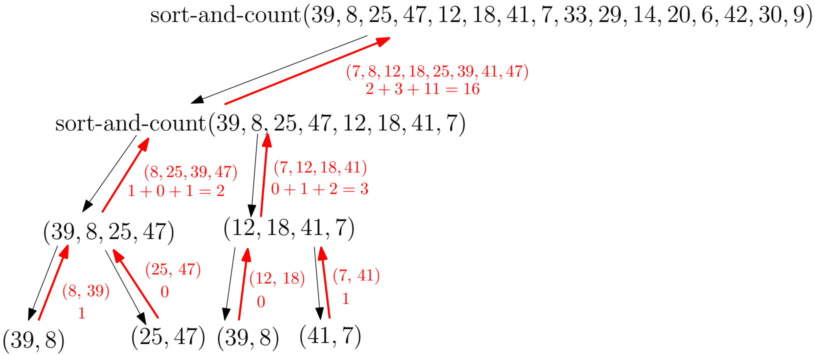
$(39, 8)$

$(25, 47)$

$(39, 8)$

$(41, 7)$

Example



sort-and-count(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)

(7, 8, 12, 18, 25, 39, 41, 47)
$2 + 3 + 11 = 16$

sort-and-count(39, 8, 25, 47, 12, 18, 41, 7)

(8, 25, 39, 47)
$1 + 0 + 1 = 2$

(7, 12, 18, 41)
$0 + 1 + 2 = 3$

(39, 8, 25, 47)

(12, 18, 41, 7)

(8, 39)
1

(25, 47)
0

(12, 18)
0

(7, 41)
1

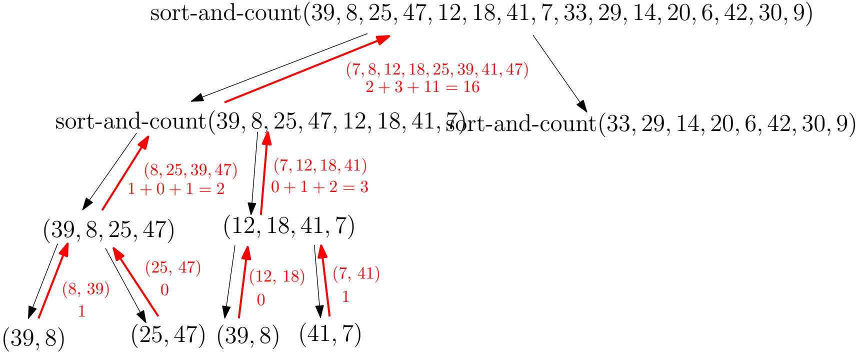
(39, 8)

(25, 47)

(39, 8)

(41, 7)

Example



sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)$

$(7, 8, 12, 18, 25, 39, 41, 47)$
$2 + 3 + 11 = 16$

sort-and-count$(39, 8, 25, 47, 12, 18, 41, 7)$

sort-and-count$(33, 29, 14, 20, 6, 42, 30, 9)$

$(8, 25, 39, 47)$
$1 + 0 + 1 = 2$

$(7, 12, 18, 41)$
$0 + 1 + 2 = 3$

$(39, 8, 25, 47)$

$(12, 18, 41, 7)$

$(8,\ 39)$
1

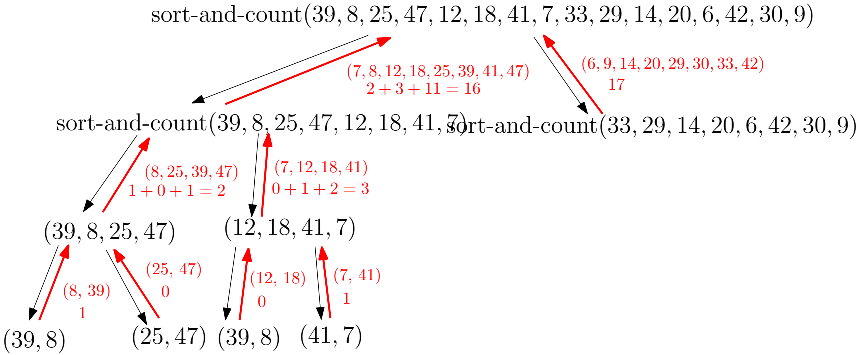
$(25,\ 47)$
0

$(12,\ 18)$
0

$(7,\ 41)$
1

$(39, 8)$ $(25, 47)$ $(39, 8)$ $(41, 7)$

Example



$\text{sort-and-count}(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)$

$(7, 8, 12, 18, 25, 39, 41, 47)$

$2 + 3 + 11 = 16$

$(6, 9, 14, 20, 29, 30, 33, 42)$

17

$\text{sort-and-count}(39, 8, 25, 47, 12, 18, 41, 7)$

$\text{sort-and-count}(33, 29, 14, 20, 6, 42, 30, 9)$

$(8, 25, 39, 47)$

$1 + 0 + 1 = 2$

$(7, 12, 18, 41)$

$0 + 1 + 2 = 3$

$(39, 8, 25, 47)$

$(12, 18, 41, 7)$

$(8,\ 39)$

1

$(25,\ 47)$

0

$(12,\ 18)$

0

$(7,\ 41)$

1

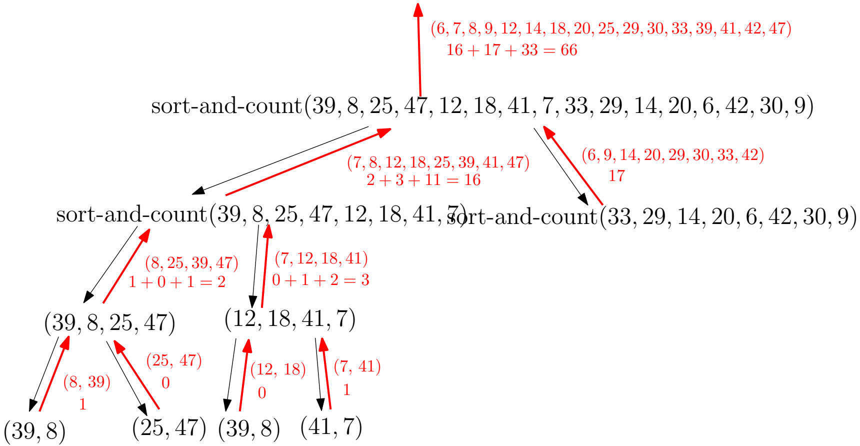
$(39, 8)$

$(25, 47)$

$(39, 8)$

$(41, 7)$

Example



$(6,7,8,9,12,14,18,20,25,29,30,33,39,41,42,47)$

$16+17+33=66$

sort-and-count$(39,8,25,47,12,18,41,7,33,29,14,20,6,42,30,9)$

$(7,8,12,18,25,39,41,47)$

$2+3+11=16$

$(6,9,14,20,29,30,33,42)$

17

sort-and-count$(39,8,25,47,12,18,41,7)$

sort-and-count$(33,29,14,20,6,42,30,9)$

$(8,25,39,47)$

$1+0+1=2$

$(7,12,18,41)$

$0+1+2=3$

$(39,8,25,47)$

$(12,18,41,7)$

$(8,\ 39)$

1

$(25,\ 47)$

0

$(12,\ 18)$

0

$(7,\ 41)$

1

$(39,8)$ $(25,47)$ $(39,8)$ $(41,7)$

Outline

1. Divide-and-Conquer
2. Counting Inversions
3. Solving Recurrences
4. Quicksort and Selection
    - Quicksort
    - Lower Bound for Comparison-Based Sorting Algorithms
    - Selection Problem
5. Polynomial Multiplication
6. Strassen's Algorithm for Matrix Multiplication
7. FFT(Fast Fourier Transform): Polynomial Multiplication in $O(n\log n)$ Time
8. Finding Closest Pair of Points in 2D Euclidean Space
9. Computing n-th Fibonacci Number

Methods for Solving Recurrences

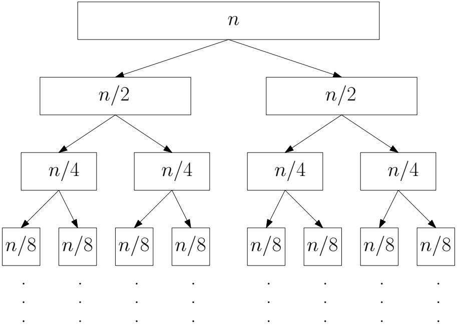
- The recursion-tree method
- The master theorem

Recursion-Tree Method

- $T(n) = 2T(n/2) + O(n)$

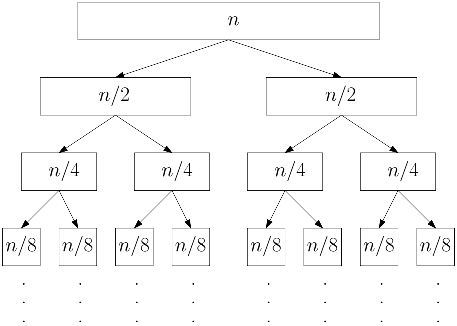
Recursion-Tree Method

- $T(n) = 2T(n/2) + O(n)$



Recursion-Tree Method

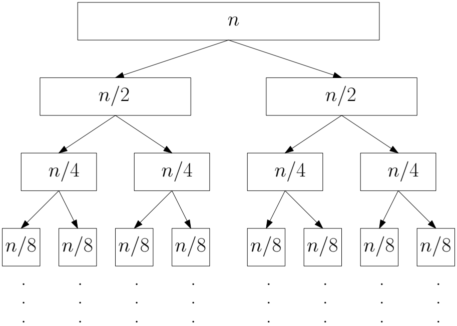
- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$

Recursion-Tree Method

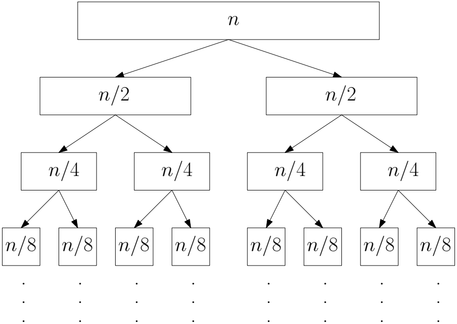
- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$
- There are $O(\log n)$ levels

Recursion-Tree Method

- $T(n) = 2T(n/2) + O(n)$



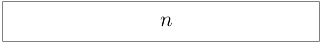
- Each level takes running time $O(n)$
- There are $O(\log n)$ levels
- Running time $= O(n \log n)$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$

Recursion-Tree Method

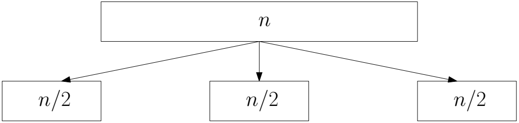
- $T(n) = 3T(n/2) + O(n)$



n

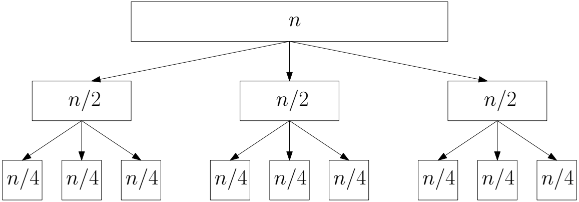
Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$



Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$



Recursion-Tree Method

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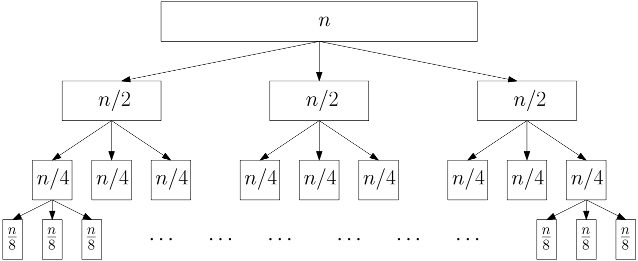
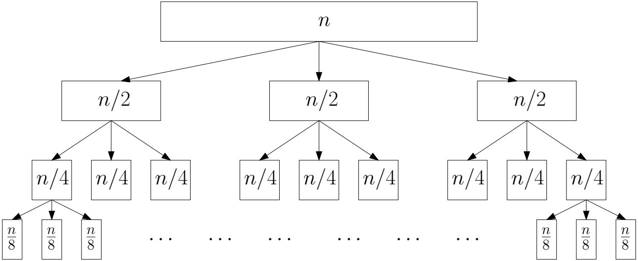


Recursion-Tree Method

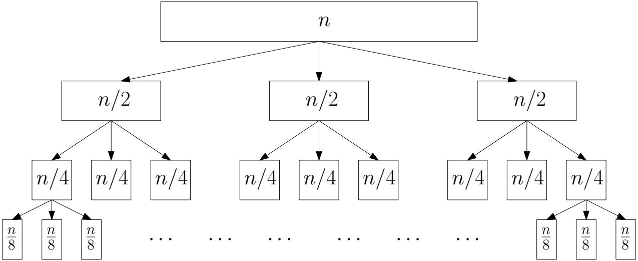
- $T(n) = 3T(n/2) + O(n)$



- Total running time at level i?

Recursion-Tree Method

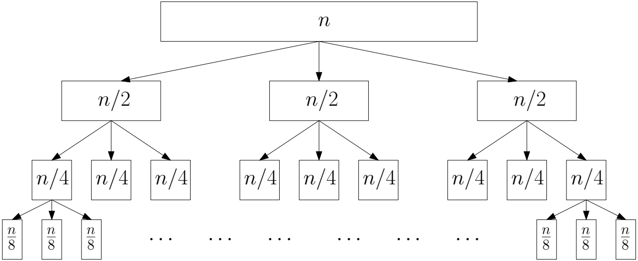
- $T(n) = 3T(n/2) + O(n)$



- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$

Recursion-Tree Method

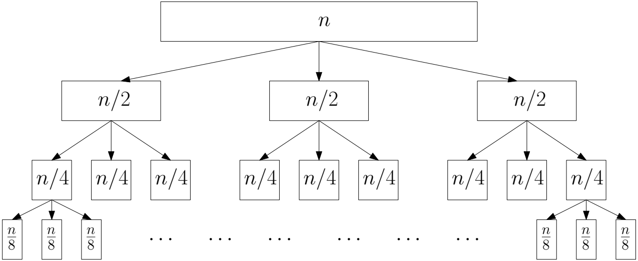
- $T(n) = 3T(n/2) + O(n)$



- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
- Index of last level?

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$



- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
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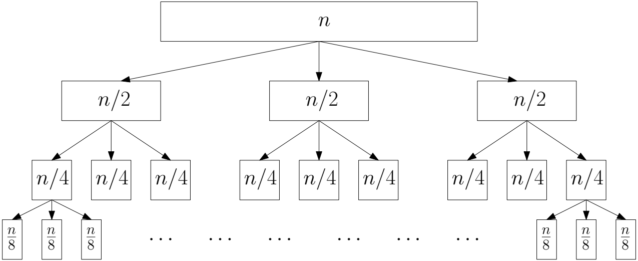
Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$



- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
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- Total running time?

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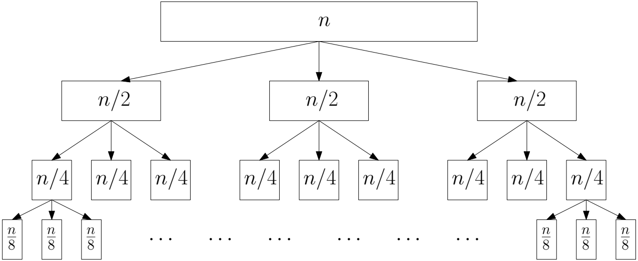


- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
- Index of last level? $\log_2 n$
- Total running time?

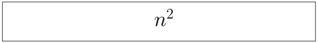
$$\sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i n = O\left(n\left(\frac{3}{2}\right)^{\log_2 n}\right) = O(3^{\log_2 n}) = O(n^{\log_2 3}).$$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$

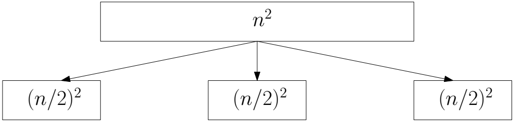
Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$



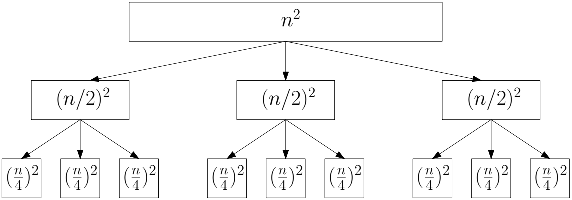
Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$



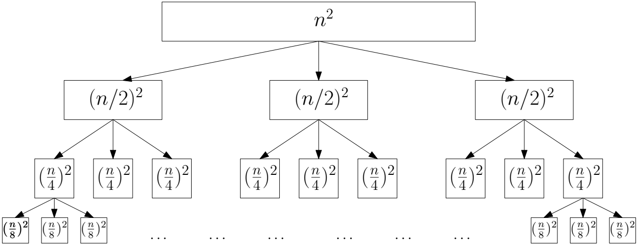
Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$



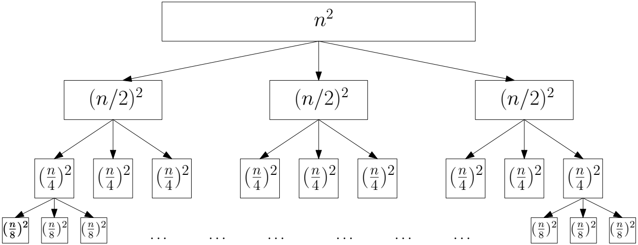
Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$



Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$



- Total running time at level i?

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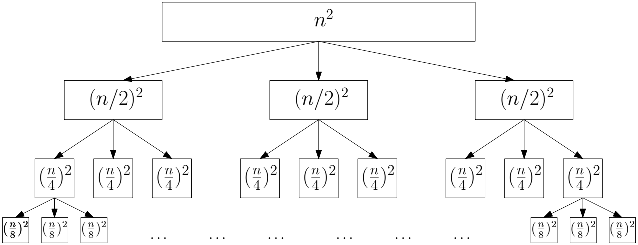

- Total running time at level i? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$

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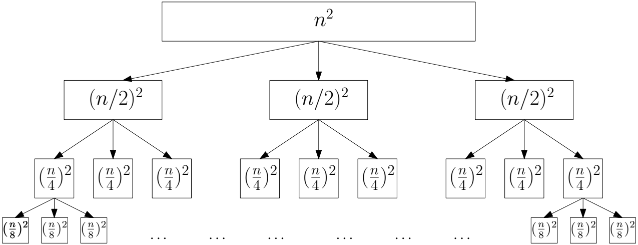

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- Index of last level?

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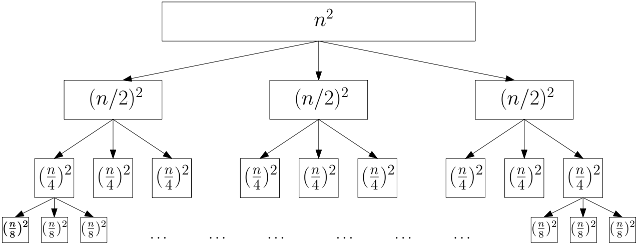

- Total running time at level i? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$
- Index of last level? $\log_2 n$

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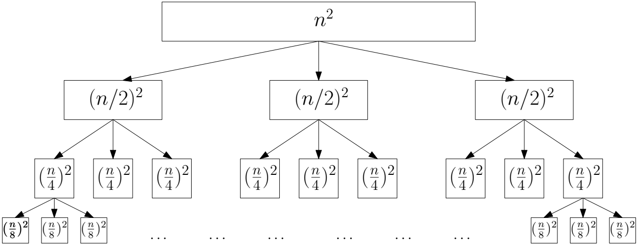

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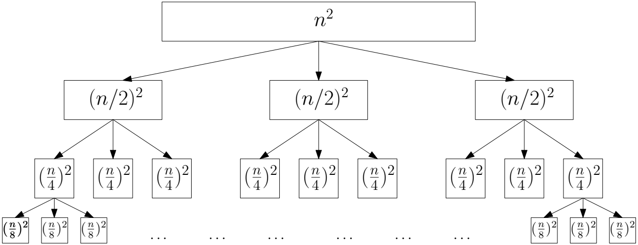

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$$\sum_{i=0}^{\log_2 n} \left(\frac{3}{4}\right)^i n^2 =$$

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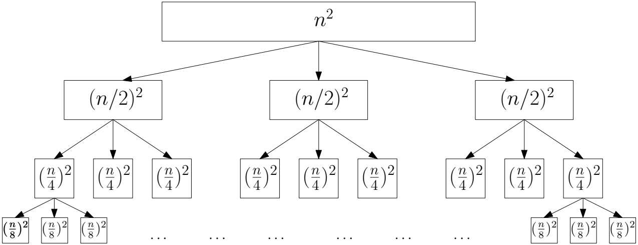

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$$\sum_{i=0}^{\log_2 n} \left(\frac{3}{4}\right)^i n^2 = O(n^2).$$

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$				$O(n \log n)$
$T(n) = 3T(n/2) + O(n)$				$O(n^{\log_2 3})$
$T(n) = 3T(n/2) + O(n^2)$				$O(n^2)$

Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \log n)$
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- Ex: $T(n) = 4T(n/2) + O(n^2)$. Which Case?

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- Ex: $T(n) = 4T(n/2) + O(n^2)$. Case 2.

Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

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Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

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Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

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- Ex: $T(n) = T(n/2) + O(1)$. Case 2. $T(n) = O(\log n)$

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- Ex: $T(n) = T(n/2) + O(1)$. Case 2. $T(n) = O(\log n)$
- Ex: $T(n) = 2T(n/2) + O(n^2)$. Which Case?

Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

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- Ex: $T(n) = 4T(n/2) + O(n^2)$. Case 2. $T(n) = O(n^2 \log n)$
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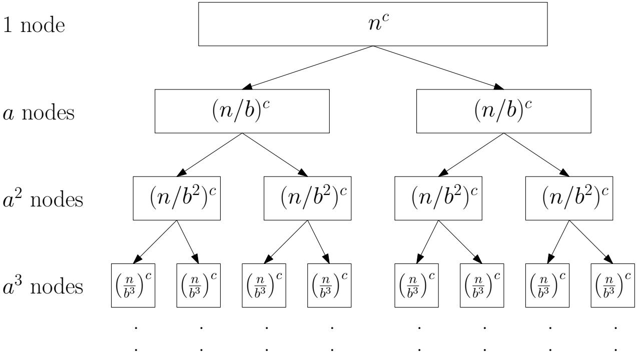
Proof of Master Theorem Using Recursion Tree

$$T(n) = aT(n/b) + O(n^c)$$


1 node
n^c
a nodes
$(n/b)^c$
$(n/b)^c$
a^2 nodes
$(n/b^2)^c$
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$(n/b^2)^c$
$(n/b^2)^c$
a^3 nodes
$\left(\frac{n}{b^3}\right)^c$
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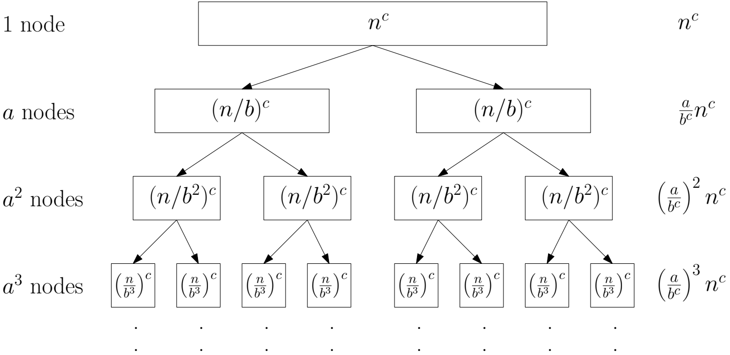

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n^c
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a^3 nodes
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Proof of Master Theorem Using Recursion Tree

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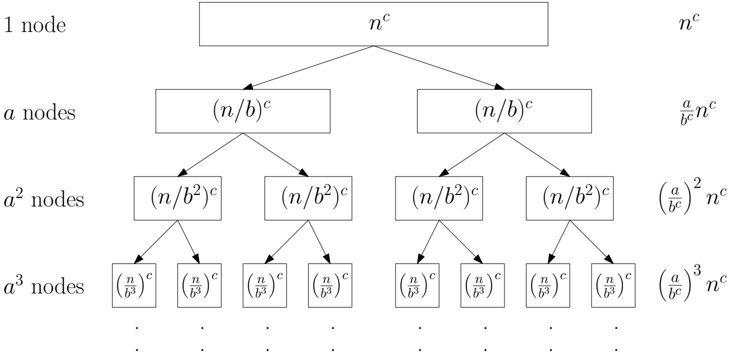
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- $c < \log_b a$: bottom-level dominates: $\left(\frac{a}{b^c}\right)^{\log_b n} n^c = n^{\log_b a}$

Proof of Master Theorem Using Recursion Tree

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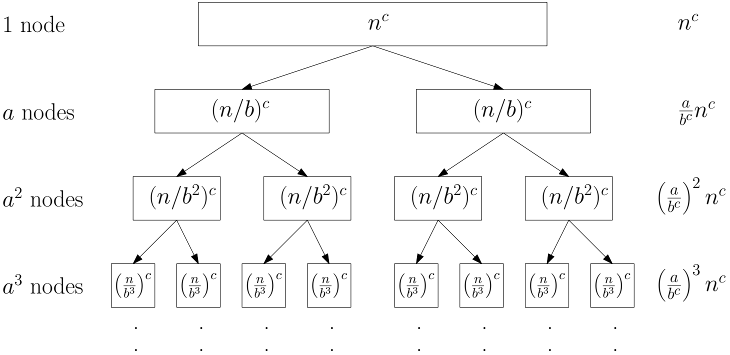

- $c < \log_b a$: bottom-level dominates: $\left(\frac{a}{b^c}\right)^{\log_b n} n^c = n^{\log_b a}$
- $c = \log_b a$: all levels have same time: $n^c \log_b n = O(n^c \log n)$

Proof of Master Theorem Using Recursion Tree

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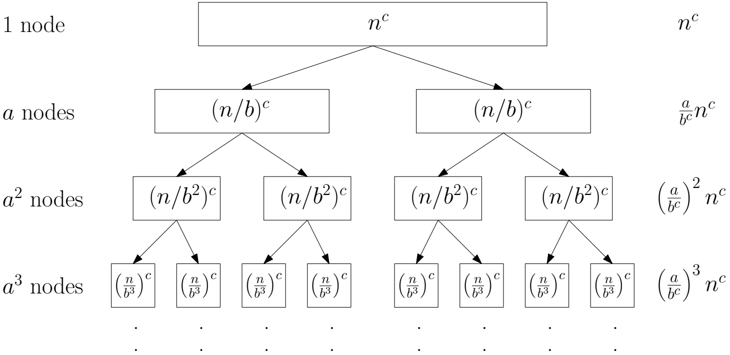
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$\left(\frac{a}{b^c}\right)^3 n^c$

- $c < \log_b a$: bottom-level dominates: $\left(\frac{a}{b^c}\right)^{\log_b n} n^c = n^{\log_b a}$
- $c = \log_b a$: all levels have same time: $n^c \log_b n = O(n^c \log n)$
- $c > \log_b a$: top-level dominates: $O(n^c)$

Outline

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2. Counting Inversions
3. Solving Recurrences
4. Quicksort and Selection
    - Quicksort
    - Lower Bound for Comparison-Based Sorting Algorithms
    - Selection Problem
5. Polynomial Multiplication
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Quicksort vs Merge-Sort

	Merge Sort	**Quicksort**
Divide	Trivial	Separate small and big numbers
Conquer	Recurse	Recurse
Combine	Merge 2 sorted arrays	Trivial

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

A:

29	82	75	64	38	45	94	69	25	76	15	92	37	17	85

$\text{quicksort}(A, 1, 15)$

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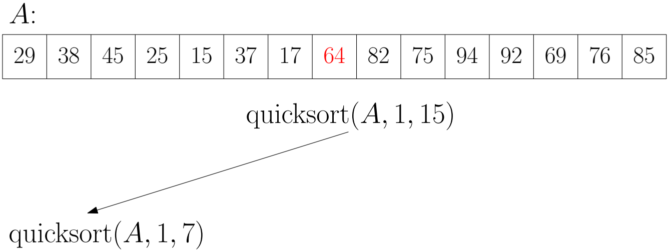

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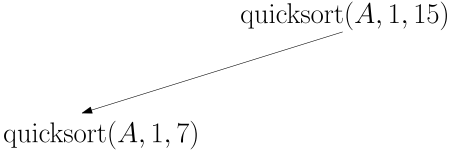

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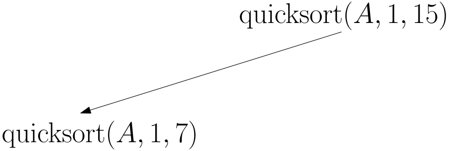
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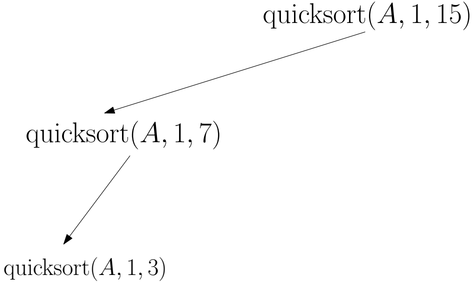

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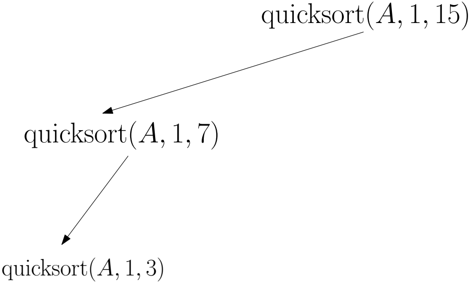

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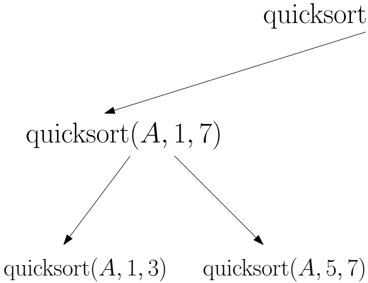

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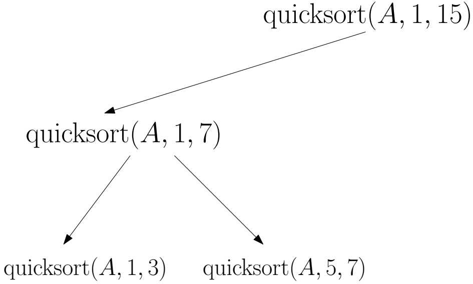

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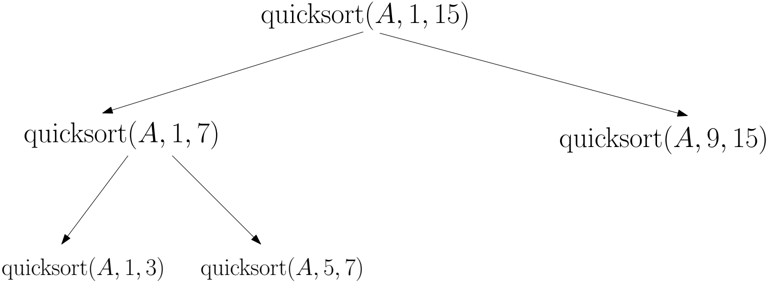

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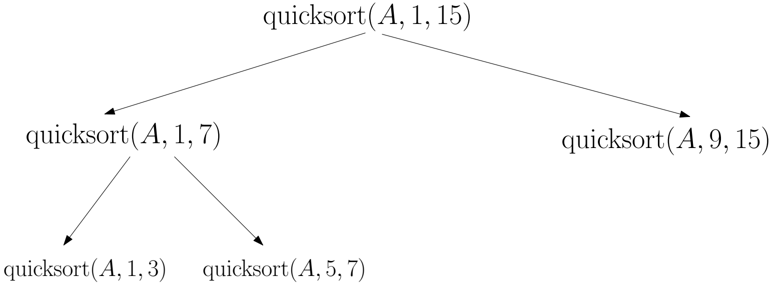

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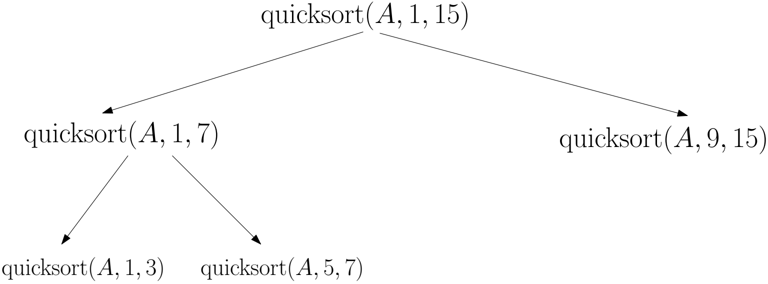

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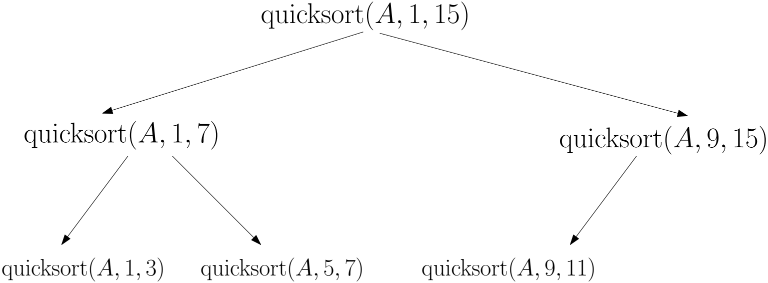

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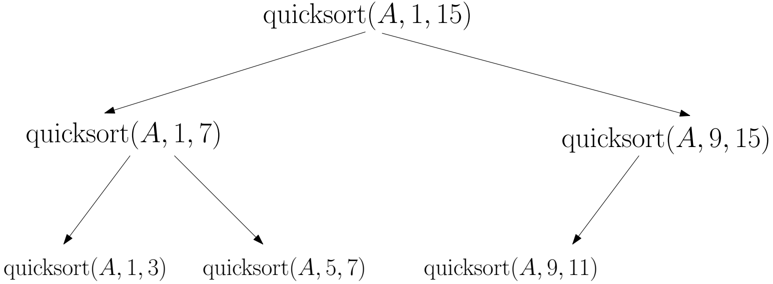

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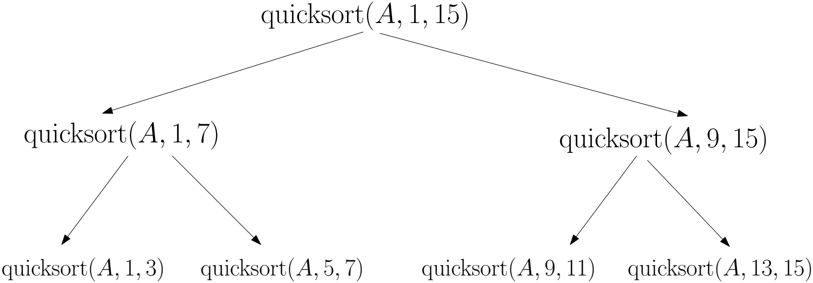

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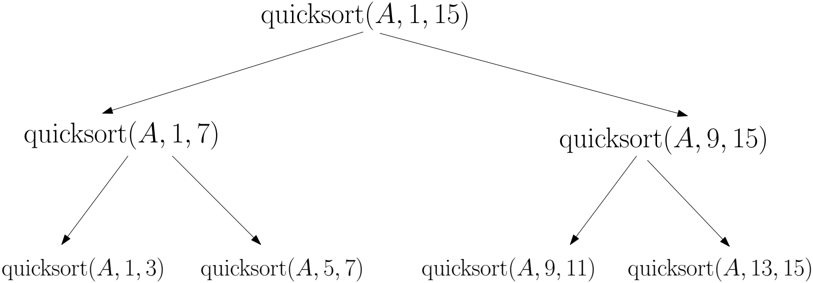

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quicksort$(A, 9, 11)$

quicksort$(A, 13, 15)$

Quicksort

quicksort(A, n)

```
if $n \le 1$ then return $A$
$x \leftarrow$ lower median of $A$
$A_L \leftarrow$ array of elements in $A$ that are less than $x$        \\ Divide
$A_R \leftarrow$ array of elements in $A$ that are greater than $x$     \\ Divide
$B_L \leftarrow$ quicksort$(A_L,$ length of $A_L)$                      \\ Conquer
$B_R \leftarrow$ quicksort$(A_R,$ length of $A_R)$                      \\ Conquer
$t \leftarrow$ number of times $x$ appear $A$
return concatenation of $B_L$, $t$ copies of $x$, and $B_R$
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Q: How to remove this assumption?

A:

1. There is an algorithm to find median in $O(n)$ time, using divide-and-conquer (we shall not talk about it; it is complicated and not practical)
2. Choose a pivot randomly and pretend it is the median (it is practical)

Quicksort Using A Random Pivot

quicksort(A, n)

```
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Assumption There is a procedure to produce a random real number in $[0, 1]$.

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- In practice: use pseudo-random-generator, a deterministic algorithm returning numbers that “look like” random
- In theory: assume they can.

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Lemma The expected running time of the algorithm is $O(n \log n)$.

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- In-Place Sorting Algorithm: an algorithm that only uses “small” extra space.

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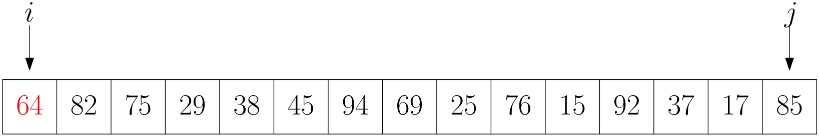

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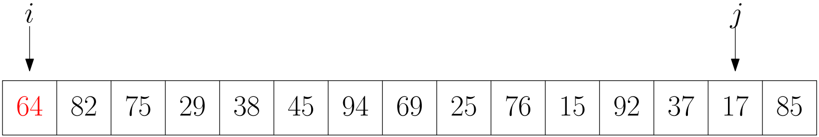

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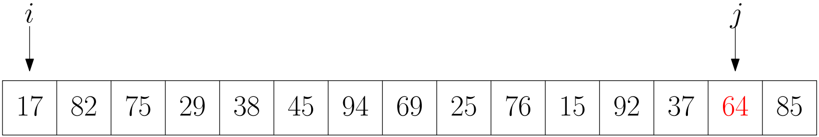

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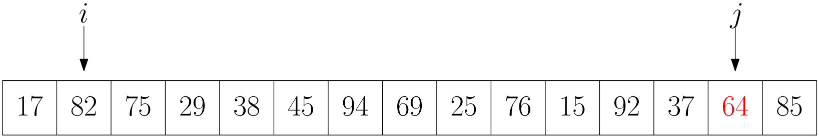

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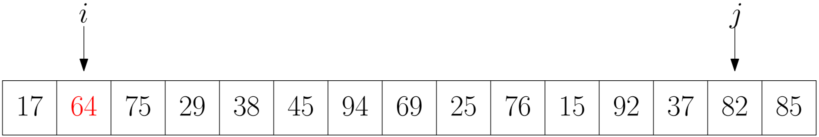

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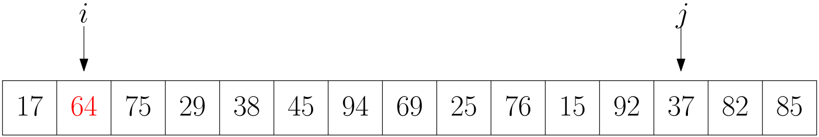

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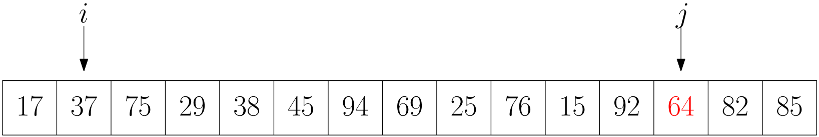

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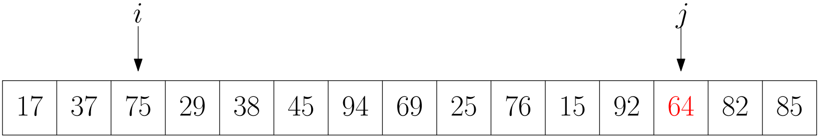

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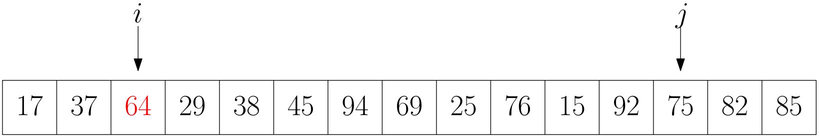

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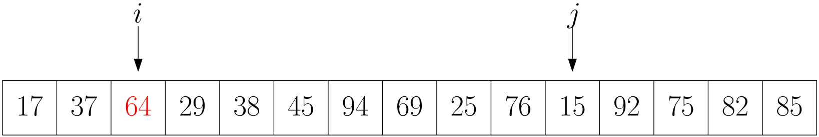

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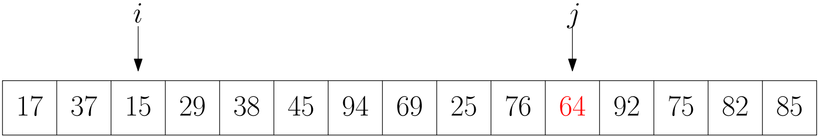

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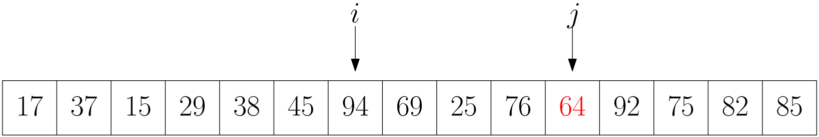

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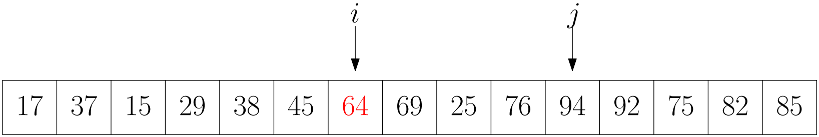

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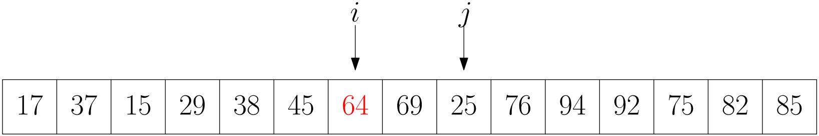

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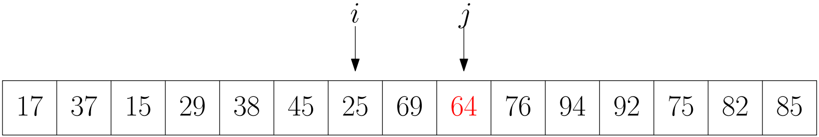

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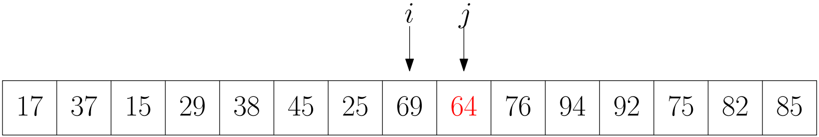

17	37	15	29	38	45	25	69	64	76	94	92	75	82	85

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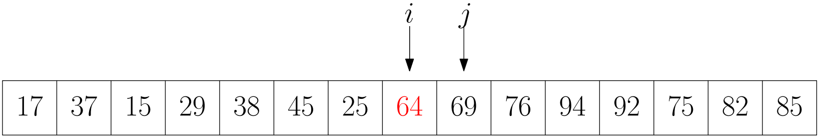

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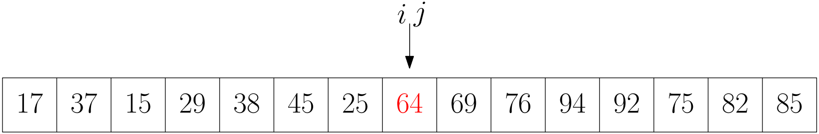

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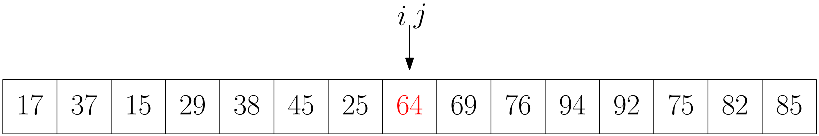

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- To partition the array into two parts, we only need $O(1)$ extra space.

partition(A, ℓ, r)

```
$p \leftarrow$ random integer between $\ell$ and $r$, swap $A[p]$ and $A[\ell]$
$i \leftarrow \ell, j \leftarrow r$
while true do
    while $i < j$ and $A[i] < A[j]$ do $j \leftarrow j - 1$
    if $i = j$ then break
    swap $A[i]$ and $A[j]$; $i \leftarrow i + 1$
    while $i < j$ and $A[i] < A[j]$ do $i \leftarrow i + 1$
    if $i = j$ then break
    swap $A[i]$ and $A[j]$; $j \leftarrow j - 1$
return $i$
```

In-Place Implementation of Quick-Sort

quicksort(A, ℓ, r)

```
if $\ell \geq r$ then return
$m \leftarrow$ patition$(A, \ell, r)$
quicksort$(A, \ell, m - 1)$
quicksort$(A, m + 1, r)$
```

- To sort an array A of size n, call quicksort$(A, 1, n)$.

Note: We pass the array A by reference, instead of by copying.

Merge-Sort is Not In-Place

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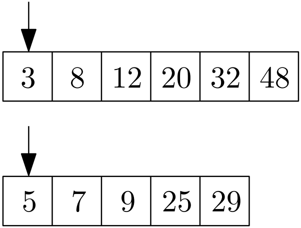

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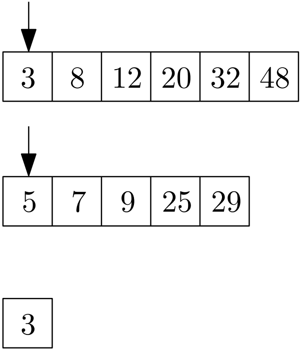

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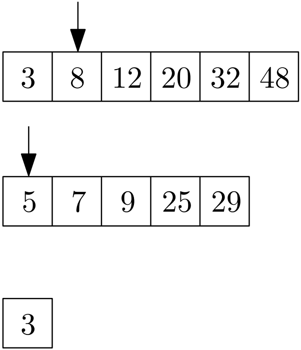

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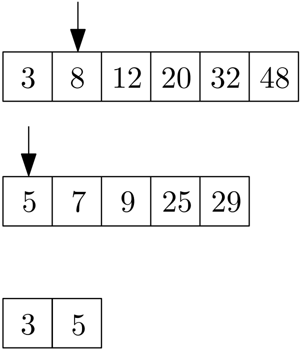

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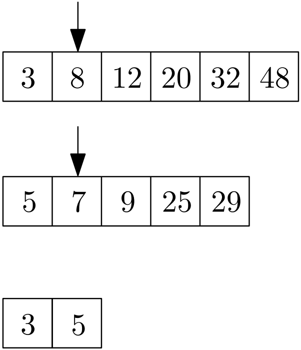

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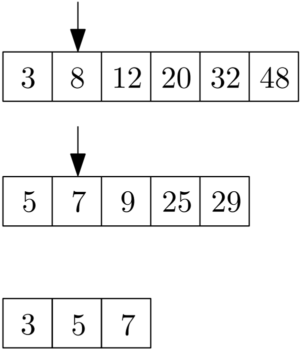

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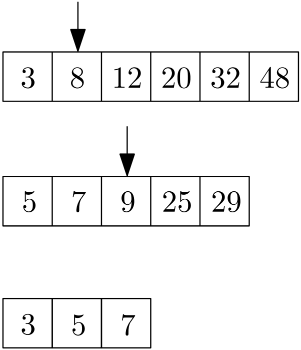

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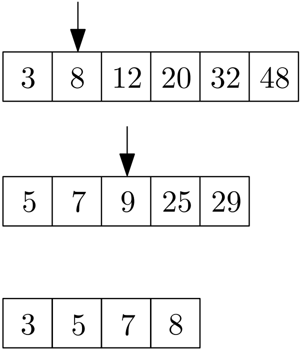

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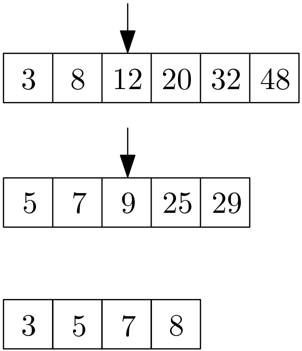

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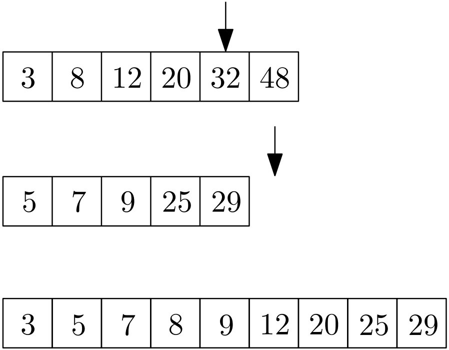

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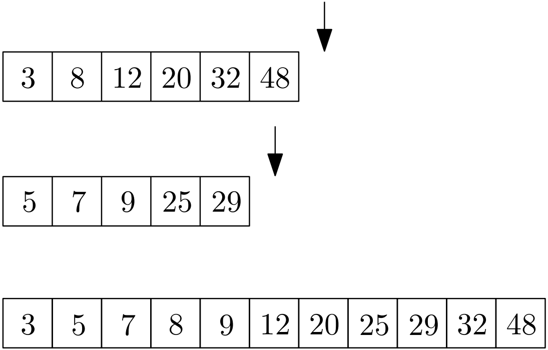

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Outline

1. Divide-and-Conquer
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3. Solving Recurrences
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Comparison-Based Sorting Algorithms

- To sort, we are only allowed to compare two elements
- We can not use "internal structures" of the elements

Lemma The (worst-case) running time of any comparison-based sorting algorithm is $\Omega(n \log n)$.

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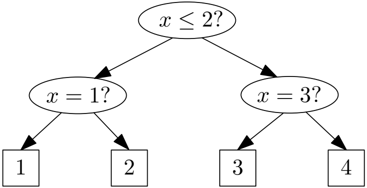

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A: At least $\log_2 n! = \Theta(n \log n)$

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Input: a set A of n numbers, and $1 \le i \le n$

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- Our goal: $O(n)$ running time

Recall: Quicksort with Median Finder

```
quicksort($A, n$)
if $n \leq 1$ then return $A$
$x \leftarrow$ lower median of $A$
$A_L \leftarrow$ elements in $A$ that are less than $x$        ▷ Divide
$A_R \leftarrow$ elements in $A$ that are greater than $x$     ▷ Divide
$B_L \leftarrow$ quicksort($A_L, A_L$.size)                     ▷ Conquer
$B_R \leftarrow$ quicksort($A_R, A_R$.size)                     ▷ Conquer
$t \leftarrow$ number of times $x$ appear $A$
return the array obtained by concatenating $B_L$, the array containing $t$ copies of $x$, and $B_R$
```

Selection Algorithm with Median Finder

selection(A, n, i)

```
if $n = 1$ then return $A$
$x \leftarrow$ lower median of $A$
$A_L \leftarrow$ elements in $A$ that are less than $x$                ▷ Divide
$A_R \leftarrow$ elements in $A$ that are greater than $x$             ▷ Divide
if $i \leq A_L$.size then
    return selection$(A_L, A_L.\text{size}, i)$                         ▷ Conquer
else if $i > n - A_R$.size then
    return selection$(A_R, A_R.\text{size}, i - (n - A_R.\text{size}))$  ▷ Conquer
else
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- Recurrence for selection: $T(n) = T(n/2) + O(n)$

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- Recurrence for selection: $T(n) = T(n/2) + O(n)$
- Solving recurrence: $T(n) = O(n)$

Randomized Selection Algorithm

```
selection($A, n, i$)
if $n = 1$ thenreturn $A$
$x \leftarrow$ random element of $A$ (called pivot)
$A_L \leftarrow$ elements in $A$ that are less than $x$          ▷ Divide
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if $i \leq A_L$.size then
    return selection($A_L, A_L$.size$, i$)                       ▷ Conquer
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- expected running time $= O(n)$

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Input: two polynomials of degree $n-1$

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Example:

$$(3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5)$$

Polynomial Multiplication

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Example:

$$
\begin{aligned}
&(3x^3+2x^2-5x+4)\times(2x^3-3x^2+6x-5)\\
={}&6x^6-9x^5+18x^4-15x^3\\
&+4x^5-6x^4+12x^3-10x^2\\
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\end{aligned}$$

- **Input**: $(4,-5,2,3),(-5,6,-3,2)$
- **Output**: $(-20,49,-52,20,2,-5,6)$

Discrete Convolution on Finite Domain

- $f : \{0, 1, \cdots, n-1\} \to \mathbb{R}, g : \{0, 1, \cdots, m-1\} \to \mathbb{R}$
- the convolution of f and g, denoted as $h := f \times g$, is defined as

$$h(k) := \sum_{i,j:i+j=k} f(i)g(j) \qquad \forall k \in \{0, 1, 2, \cdots, m+n-2\}$$

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$f \times g$	-20	49	-52	20	2	-5	6

Applications of Convolutions

- Polynomial and integer multiplication
- Signal and Image Processing
- Probability theory: Sum of two distributions
- Convolutional neural network

- Polynomial multiplication ⇔ Convolution
- We shall focus on multiplication.

Big Integer Multiplication Using Polynomial Multiplication

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Big Integer Multiplication Using Polynomial Multiplication

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Big Integer Multiplication Using Polynomial Multiplication

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- $(16x^3 + 103x^2 + 416x + 169) \times (424x^2 + 317x + 167)$

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- Polynomial multiplication $\Leftrightarrow$ Convolution
- We shall focus on multiplication.

Big Integer Multiplication Using Polynomial Multiplication

- $16103416169 \times 424317167$
- $(16x^3 + 103x^2 + 416x + 169) \times (424x^2 + 317x + 167)$
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- 6832955927852073223

Naïve Algorithm

polynomial-multiplication(A, B, n)

```
let $C[k] \leftarrow 0$ for every $k = 0, 1, 2, \cdots, 2n-2$
for $i \leftarrow 0$ to $n-1$ do
    for $j \leftarrow 0$ to $n-1$ do
        $C[i+j] \leftarrow C[i+j] + A[i] \times B[j]$
return $C$
```

Naïve Algorithm

```
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```

Running time: $O(n^2)$

Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x+2)x^2 + (-5x+4)$$
$$q(x) = 2x^3 - 3x^2 + 6x - 5 = (2x-3)x^2 + (6x-5)$$

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- $p(x)$: degree of $n - 1$ (assume n is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

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$$pq = \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right)$$

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$$\begin{aligned} pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\ &= p_H q_H x^n + \left(p_H q_L + p_L q_H\right)x^{n/2} + p_L q_L \end{aligned}$$

Divide-and-Conquer for Polynomial Multiplication

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$$\begin{aligned} \mathsf{multiply}(p, q) = {} & \mathsf{multiply}(p_H, q_H) \times x^n \\ & + \bigl(\mathsf{multiply}(p_H, q_L) + \mathsf{multiply}(p_L, q_H)\bigr) \times x^{n/2} \\ & + \mathsf{multiply}(p_L, q_L) \end{aligned}$$

Divide-and-Conquer for Polynomial Multiplication

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- Recurrence: $T(n) = 4T(n/2) + O(n)$

Divide-and-Conquer for Polynomial Multiplication

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- Recurrence: $T(n) = 4T(n/2) + O(n)$
- $T(n) = O(n^2)$

Reduce Number from 4 to 3

Reduce Number from 4 to 3

$$\begin{aligned} pq &= \big(p_H x^{n/2} + p_L\big)\big(q_H x^{n/2} + q_L\big) \\ &= p_H q_H x^n + \big(p_H q_L + p_L q_H\big) x^{n/2} + p_L q_L \end{aligned}$$

Reduce Number from 4 to 3

$$
\begin{aligned}
pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\
&= p_H q_H x^n + \left(p_H q_L + p_L q_H\right) x^{n/2} + p_L q_L
\end{aligned}
$$

- $p_H q_L + p_L q_H = (p_H + p_L)(q_H + q_L) - p_H q_H - p_L q_L$

Divide-and-Conquer for Polynomial Multiplication

Divide-and-Conquer for Polynomial Multiplication

$$r_H = \mathsf{multiply}(p_H, q_H)$$
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$$r_H = \mathsf{multiply}(p_H, q_H)$$
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$$\begin{aligned}\mathsf{multiply}(p, q) = {} & r_H \times x^n \\ & + \big(\mathsf{multiply}(p_H + p_L, q_H + q_L) - r_H - r_L\big) \times x^{n/2} \\ & + r_L\end{aligned}$$

Divide-and-Conquer for Polynomial Multiplication

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- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$

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- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\log_2 3}) = O(n^{1.585})$

Assumption n is a power of 2. Arrays are 0-indexed.

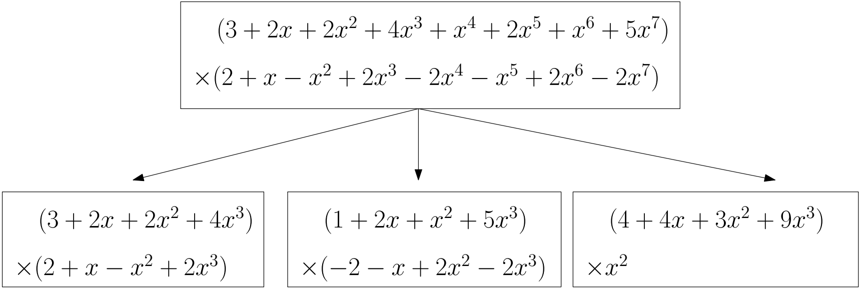
multiply(A, B, n)

```
if $n = 1$ then return $(A[0]B[0])$
$A_L \leftarrow A[0 \,..\, n/2 - 1], A_H \leftarrow A[n/2 \,..\, n - 1]$
$B_L \leftarrow B[0 \,..\, n/2 - 1], B_H \leftarrow B[n/2 \,..\, n - 1]$
$C_L \leftarrow$ multiply$(A_L, B_L, n/2)$
$C_H \leftarrow$ multiply$(A_H, B_H, n/2)$
$C_M \leftarrow$ multiply$(A_L + A_H, B_L + B_H, n/2)$
$C \leftarrow$ array of $(2n - 1)$ 0's
**for** $i \leftarrow 0$ to $n - 2$ **do**
    $C[i] \leftarrow C[i] + C_L[i]$
    $C[i + n] \leftarrow C[i + n] + C_H[i]$
    $C[i + n/2] \leftarrow C[i + n/2] + C_M[i] - C_L[i] - C_H[i]$
**return** $C$
```

Example

$$\boxed{\begin{aligned} &(3 + 2x + 2x^2 + 4x^3 + x^4 + 2x^5 + x^6 + 5x^7) \\ \times&(2 + x - x^2 + 2x^3 - 2x^4 - x^5 + 2x^6 - 2x^7) \end{aligned}}$$

Example



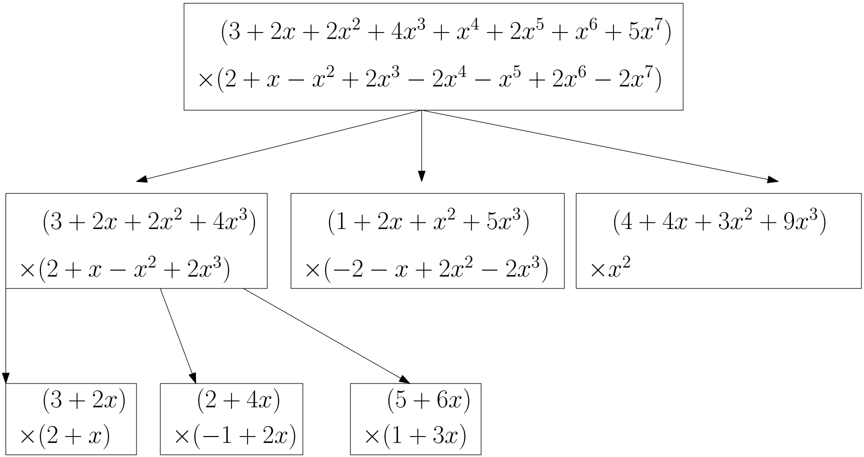
$$(3+2x+2x^2+4x^3+x^4+2x^5+x^6+5x^7)$$
$$\times(2+x-x^2+2x^3-2x^4-x^5+2x^6-2x^7)$$

$$(3+2x+2x^2+4x^3)$$
$$\times(2+x-x^2+2x^3)$$

$$(1+2x+x^2+5x^3)$$
$$\times(-2-x+2x^2-2x^3)$$

$$(4+4x+3x^2+9x^3)$$
$$\times x^2$$

Example



$(3+2x+2x^2+4x^3+x^4+2x^5+x^6+5x^7)$
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$\times(-1+2x)$

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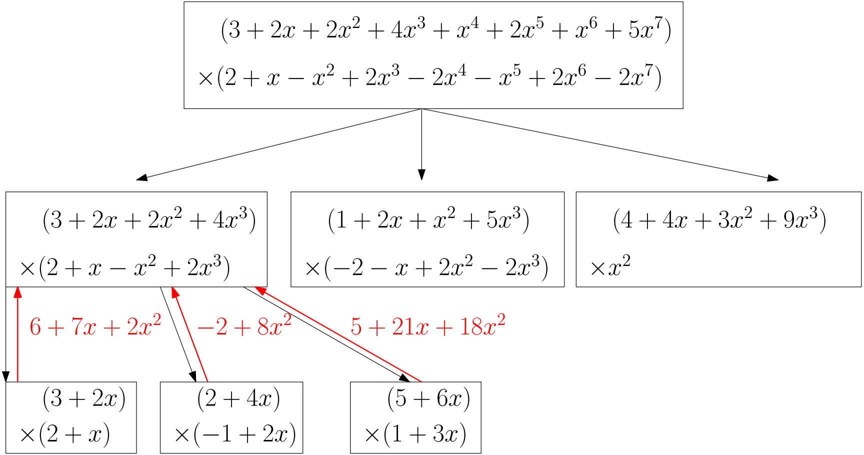

$$(3+2x+2x^2+4x^3+x^4+2x^5+x^6+5x^7)$$
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$6+7x+2x^2$

$-2+8x^2$

$5+21x+18x^2$

$$(3+2x)$$
$$\times(2+x)$$

$$(2+4x)$$
$$\times(-1+2x)$$

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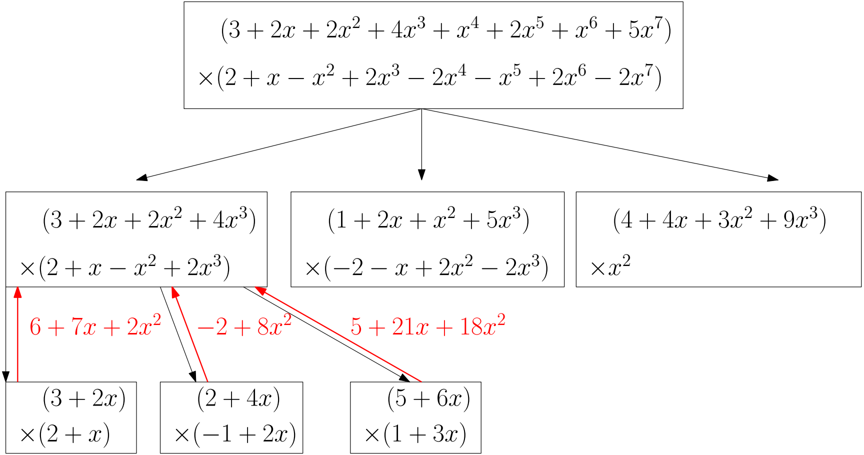
$$\times(1+3x)$$

Example



$(3 + 2x + 2x^2 + 4x^3 + x^4 + 2x^5 + x^6 + 5x^7)$
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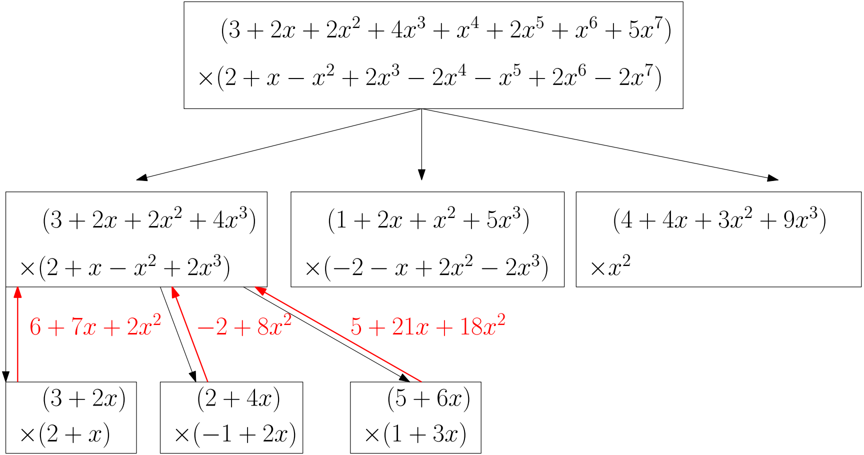
$$(5 + 21x + 18x^2) - (6 + 7x + 2x^2) - (-2 + 8x^2) = 1 + 14x + 8x^2$$

Example



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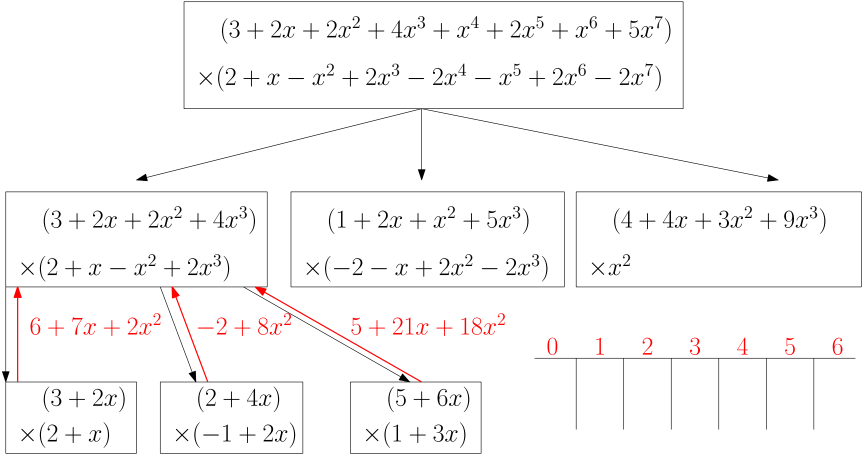
$$(6+7x+2x^2)+(1+14x+8x^2)x^2+(-2+8x^2)x^4$$
$$=6+7x+3x^2+14x^3+6x^4+8x^6$$

Example



$$(3+2x+2x^2+4x^3+x^4+2x^5+x^6+5x^7)$$
$$\times(2+x-x^2+2x^3-2x^4-x^5+2x^6-2x^7)$$

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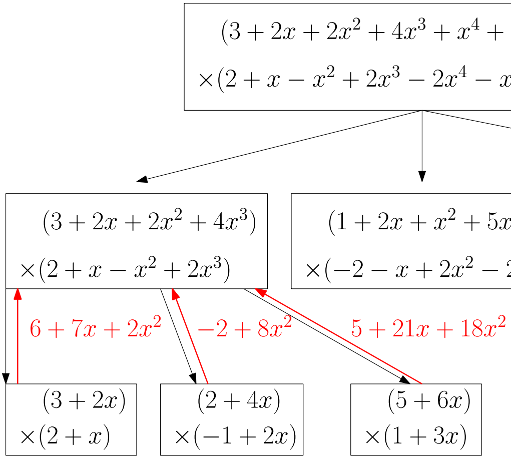
Example



$(3 + 2x + 2x^2 + 4x^3 + x^4 + 2x^5 + x^6 + 5x^7)$
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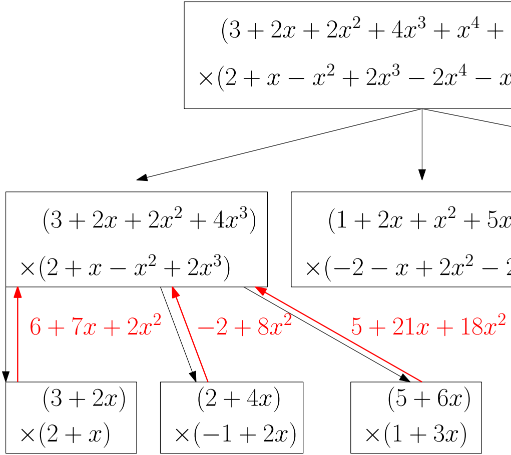
0	1	2	3	4	5	6
6	7	2				

$$(5 + 21x + 18x^2) - (6 + 7x + 2x^2) - (-2 + 8x^2) = 1 + 14x + 8x^2$$

$$\begin{aligned} &(6 + 7x + 2x^2) + (1 + 14x + 8x^2)x^2 + (-2 + 8x^2)x^4 \\ &= 6 + 7x + 3x^2 + 14x^3 + 6x^4 + 8x^6 \end{aligned}$$

Example



0	1	2	3	4	5	6
6	7	2		-2	0	8

$$(5+21x+18x^2)-(6+7x+2x^2)-(-2+8x^2)=1+14x+8x^2$$

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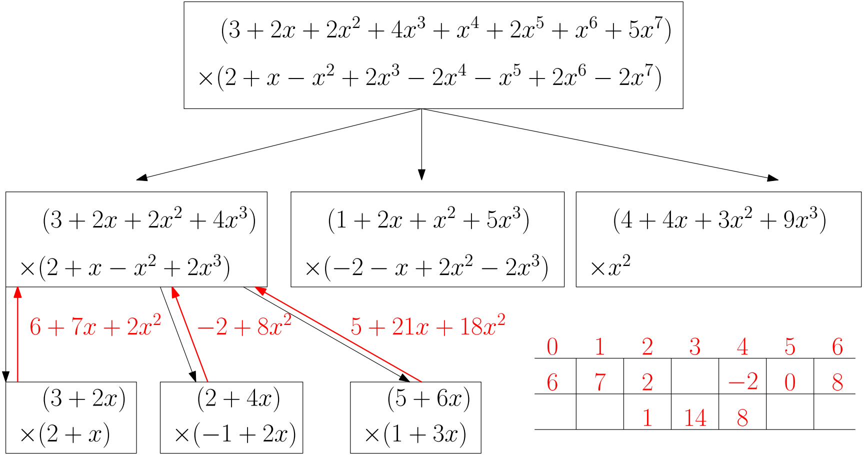
Example



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0	1	2	3	4	5	6
6	7	2		−2	0	8
		1	14	8		

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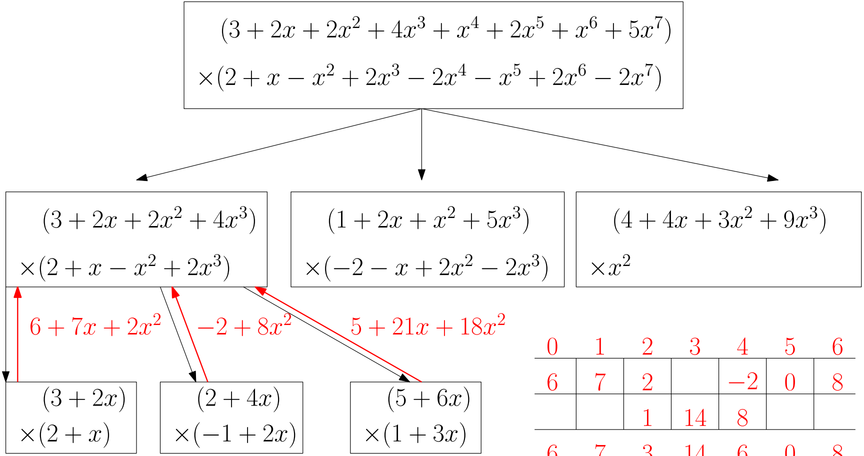
Example



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0	1	2	3	4	5	6
6	7	2		−2	0	8
		1	14	8		
6	7	3	14	6	0	8

$$(5+21x+18x^2)-(6+7x+2x^2)-(-2+8x^2)=1+14x+8x^2$$

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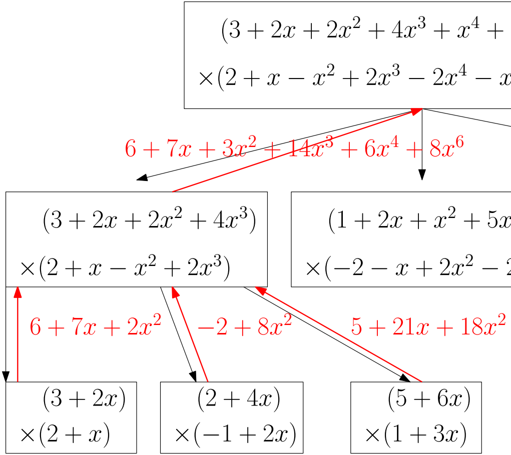
Example



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$\times(1+3x)$

0	1	2	3	4	5	6
6	7	2		−2	0	8
		1	14	8		
6	7	3	14	6	0	8

$$(5+21x+18x^2)-(6+7x+2x^2)-(-2+8x^2)=1+14x+8x^2$$

$$(6+7x+2x^2)+(1+14x+8x^2)x^2+(-2+8x^2)x^4$$
$$=6+7x+3x^2+14x^3+6x^4+8x^6$$

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Input: two $n \times n$ matrices A and B

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Naive Algorithm: matrix-multiplication(A, B, n)

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for $i \leftarrow 1$ to $n$ do
    for $j \leftarrow 1$ to $n$ do
        $C[i,j] \leftarrow 0$
        for $k \leftarrow 1$ to $n$ do
            $C[i,j] \leftarrow C[i,j] + A[i,k] \times B[k,j]$
return $C$
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- running time $= O(n^3)$

Try to Use Divide-and-Conquer

$$A = \begin{array}{|c|c|l} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\overbrace{\hphantom{A_{12}}}^{n/2}} & \\ \hline A_{11} & A_{12} & \left.\vphantom{A_{12}}\right\} n/2 \\ \hline A_{21} & A_{22} & \\ \hline \end{array} \qquad B = \begin{array}{|c|c|l} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\overbrace{\hphantom{B_{12}}}^{n/2}} & \\ \hline B_{11} & B_{12} & \left.\vphantom{B_{12}}\right\} n/2 \\ \hline B_{21} & B_{22} & \\ \hline \end{array}$$

- $C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$
- matrix_multiplication(A, B) recursively calls matrix_multiplication(A_{11}, B_{11}), matrix_multiplication(A_{12}, B_{21}), $\cdots$

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- Solving Recurrence $T(n) = O(n^{\log_2 7}) = O(n^{2.808})$

Strassen's Algorithm

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- $M_1 \leftarrow (A_{11} + A_{22}) \times (B_{11} + B_{22})$
- $M_2 \leftarrow (A_{21} + A_{22}) \times B_{11}$
- $M_3 \leftarrow A_{11} \times (B_{12} - B_{22})$
- $M_4 \leftarrow A_{22} \times (B_{21} - B_{11})$
- $M_5 \leftarrow (A_{11} + A_{12}) \times B_{22}$
- $M_6 \leftarrow (A_{21} - A_{11}) \times (B_{11} + B_{12})$
- $M_7 \leftarrow (A_{12} - A_{22}) \times (B_{21} + B_{22})$

- $C_{11} \leftarrow M_1 + M_4 - M_5 + M_7$
- $C_{12} \leftarrow M_3 + M_5$
- $C_{21} \leftarrow M_2 + M_4$
- $C_{22} \leftarrow M_1 - M_2 + M_3 + M_6$

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- $p(x) = 1 - x + 2x^2 : p(0) = 1, p(1) = 2, p(2) = 7$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

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Using Interpolation for Polynomial Multiplication

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Using Interpolation for Polynomial Multiplication

- $p(x) = 1 - x + 2x^2, \qquad q(x) = 3 - x^2$
- Interpolation on 5 points $\{0, 1, 2, 3, 4\}$:

$$\text{interpolation for } p: \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 7 \\ 16 \\ 29 \end{pmatrix}$$

$$\text{interpolation for } q: \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \\ -6 \\ -13 \end{pmatrix}$$

Interpolation of pq:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -7 \\ -102 \\ -377 \end{pmatrix}$$

Interpolation of pq:

$$\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 1\\ 1 & 2 & 4 & 8 & 16\\ 1 & 3 & 9 & 27 & 81\\ 1 & 4 & 16 & 64 & 256\end{pmatrix}\begin{pmatrix}c_0\\ c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = \begin{pmatrix}3\\ 4\\ -7\\ -102\\ -377\end{pmatrix}$$

$$\begin{pmatrix}c_0\\ c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 1\\ 1 & 2 & 4 & 8 & 16\\ 1 & 3 & 9 & 27 & 81\\ 1 & 4 & 16 & 64 & 256\end{pmatrix}^{-1}\begin{pmatrix}3\\ 4\\ -7\\ -96\\ -377\end{pmatrix}$$

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{25}{12} & 4 & -3 & \frac{4}{3} & -\frac{1}{4} \\ \frac{35}{24} & -\frac{13}{3} & \frac{19}{4} & -\frac{7}{3} & \frac{11}{24} \\ -\frac{5}{12} & \frac{3}{2} & -2 & \frac{7}{6} & -\frac{1}{4} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -7 \\ -96 \\ -377 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 5 \\ 1 \\ -2 \end{pmatrix}$$

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$$pq = (1 - x + 2x^2)(3 - x^2) = 3 - 3x + 5x^2 + x^3 - 2x^4$$

Multiplication of two polynomials of degree $n-1$

- Choose $2n-1$ distinct values $x_0, x_1, x_2, \cdots, x_{m-1}$ carefully, $m = 2n-1$
- Compute the interpolation of p and q:

$$M := \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{m-1} & x_{m-1}^2 & x_m^3 & \cdots & x_{m-1}^{n-1} \end{pmatrix}$$

$$M \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{m-1} \end{pmatrix} \qquad M \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{m-1} \end{pmatrix}$$

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$$\begin{aligned} &(a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}) \\ \times\ &(b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}) \\ =\ &(c_0 + c_1 x + c_2 x^2 + \cdots + c_{2n-2} x^{2n-2}) \end{aligned}$$

Q: How should we set $x_0, x_1, \cdots, x_{n-1}$ so that we can compute Ma and $M^{-1}y$ fast (for any $a, y \in \mathbb{R}^{\{0,1,\cdots,n-1\}}$)?

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A: Use the n complex roots of the equation $x^n = 1$

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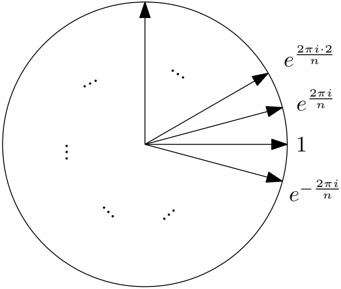
A: Use the n complex roots of the equation $x^n = 1$

- $e^{\frac{2\pi i \cdot k}{n}} = \cos\left(\frac{2\pi \cdot k}{n}\right) + i \cdot \sin\left(\frac{2\pi \cdot k}{n}\right), k \in \{0, 1, \cdots, n-1\}$
- $\omega := e^{\frac{2\pi i}{n}}$, n-th roots are $1, \omega, \omega^2, \cdots, \omega^{n-1}$



$$F_n := \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \end{pmatrix}$$

- Interpolation and Inverse-Interpolation:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} = F_n \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} \qquad \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} = F_n^{-1} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

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- Interpolation: Fast Fourier Transform (FFT)
- Invert-Interpolation: Inverse Fast Fourier Transform (iFFT)

Fast Fourier Transform: Divide and Conquer

- Assume n is even.

Breaking polynomial into even and odd parts

- $p_{\textsf{even}}(x) := a_0 + a_2 x + a_4 x^2 + \cdots + a_{n-2} x^{n/2-1}$
- $p_{\textsf{old}}(x) := a_1 + a_3 x + a_5 x^2 + \cdots + a_{n-1} x^{n/2-1}$
- $p(x) = p_{\textsf{even}}(x^2) + p_{\textsf{odd}}(x^2) \cdot x$

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$$\begin{aligned} p(\omega^k) &= p_{\mathsf{even}}(\omega^{2k}) + p_{\mathsf{odd}}(\omega^{2k}) \cdot \omega^k, \qquad & k = 0, 1, \cdots, \frac{n}{2} - 1 \\ p(\omega^{n/2+k}) &= p_{\mathsf{even}}(\omega^{2k}) - p_{\mathsf{odd}}(\omega^{2k}) \cdot \omega^k, \qquad & k = 0, 1, \cdots, \frac{n}{2} - 1 \end{aligned}$$

- Assume n is an integer power of 2

```
FFT$(n, a_0, a_1, \cdots, a_{n-1})$
if $n = 1$ then return $(a_0)$
$(e_0, e_1, \cdots, e_{n/2-1}) \leftarrow \mathrm{FFT}(n/2, a_0, a_2, \cdots, a_{n-2})$
$(o_0, o_1, \cdots, o_{n/2-1}) \leftarrow \mathrm{FFT}(n/2, a_1, a_3, \cdots, a_{n-1})$
for $k \leftarrow 0, 1, 2, \cdots n/2 - 1$ do
    $y_k \leftarrow e_k + o_k \cdot \omega^k$
    $y_{n/2+k} \leftarrow e_k - o_k \cdot \omega^k$
return $(y_0, y_1, \cdots, y_{n-1})$
```

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FFT$(n, a_0, a_1, \cdots, a_{n-1})$

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for $k \leftarrow 0, 1, 2, \cdots n/2 - 1$ do
    $y_k \leftarrow e_k + o_k \cdot \omega^k$
    $y_{n/2+k} \leftarrow e_k - o_k \cdot \omega^k$
return $(y_0, y_1, \cdots, y_{n-1})$
```

- Recurrence for running time: $T(n) = 2T(n/2) + O(n)$
- $T(n) = O(n \log n)$

Example for one recursion of FFT

- $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (3, 2, 1, 2, 5, 6, 1, 4)$

$$\begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 6 \\ -2 \end{pmatrix}$$

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- $y_6 = e_2 - o_2\omega^2 = 6 - 2i \qquad y_7 = e_3 - o_3\omega^3$

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$
$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

multiplying p and q, ▷ assuming n is a power of 2

```
1: $y \leftarrow \mathrm{FFT}(2n, a_0, a_1, \cdots, a_{n-1}, 0, 0, \cdots, 0)$
2: $z \leftarrow \mathrm{FFT}(2n, b_0, b_1, \cdots, b_{n-1}, 0, 0, \cdots, 0)$
3: $c \leftarrow \mathrm{iFFT}(2n, y_0 z_0, y_1 z_1, \cdots, y_{2n-1} z_{2n-1})$
4: return $(c_0, c_1, \cdots, c_{2n-2})$
```

- iFFT$(n, y_0, y_1, \cdots, y_{n-1})$: inverse FFT procedure: multiplying input vector y by the inverse of F_n, which is

$$\frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \end{pmatrix}$$

Outline

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3. Solving Recurrences
4. Quicksort and Selection
    - Quicksort
    - Lower Bound for Comparison-Based Sorting Algorithms
    - Selection Problem
5. Polynomial Multiplication
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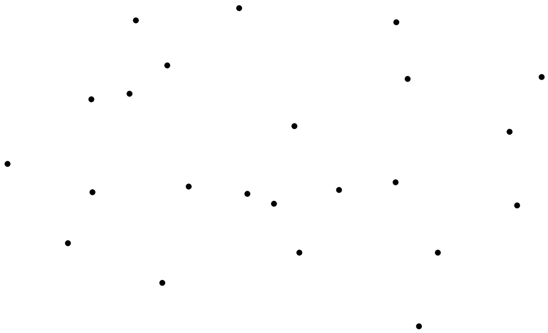
7. FFT(Fast Fourier Transform): Polynomial Multiplication in $O(n \log n)$ Time
8. **Finding Closest Pair of Points in 2D Euclidean Space**
9. Computing n-th Fibonacci Number

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest



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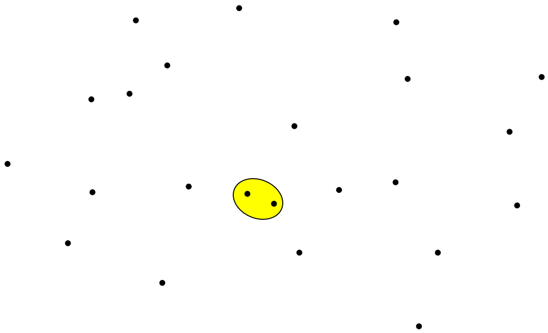

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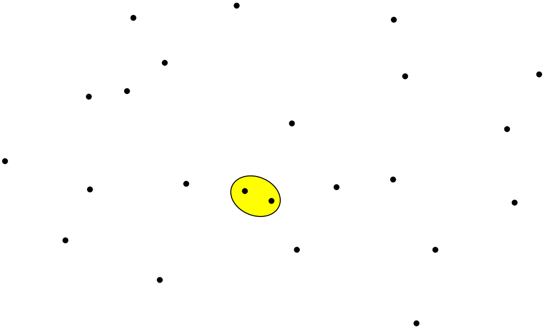
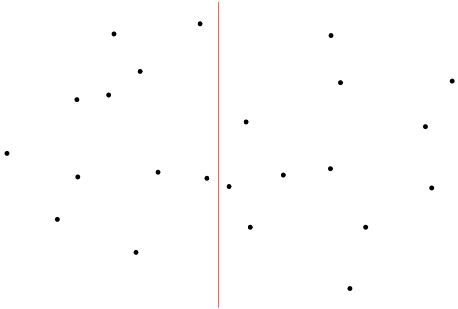


- Trivial algorithm: $O(n^2)$ running time

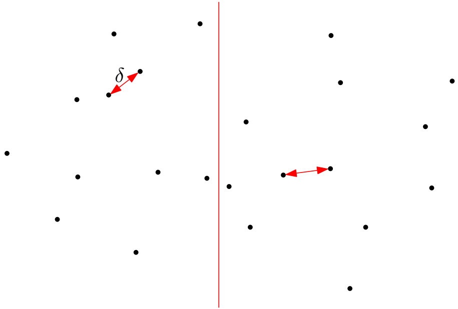
Divide-and-Conquer Algorithm for Closest Pair

- **Divide**: Divide the points into two halves via a vertical line



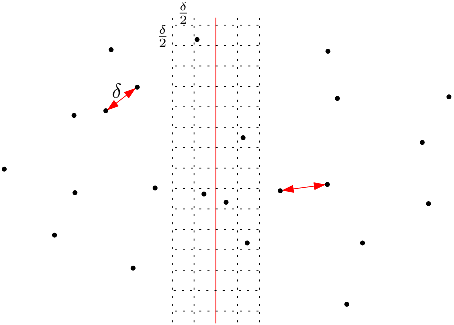
Divide-and-Conquer Algorithm for Closest Pair

- **Divide**: Divide the points into two halves via a vertical line
- **Conquer**: Solve two sub-instances recursively



Divide-and-Conquer Algorithm for Closest Pair

- **Divide**: Divide the points into two halves via a vertical line
- **Conquer**: Solve two sub-instances recursively
- **Combine**: Check if there is a closer pair between left-half and right-half



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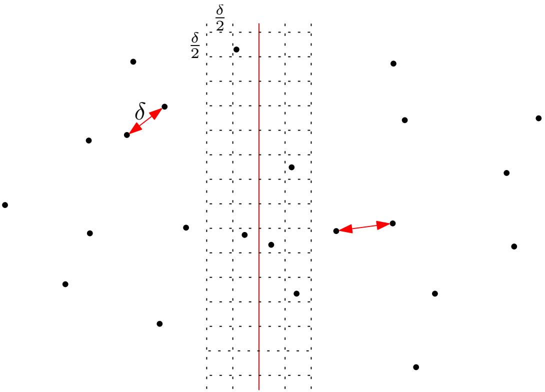

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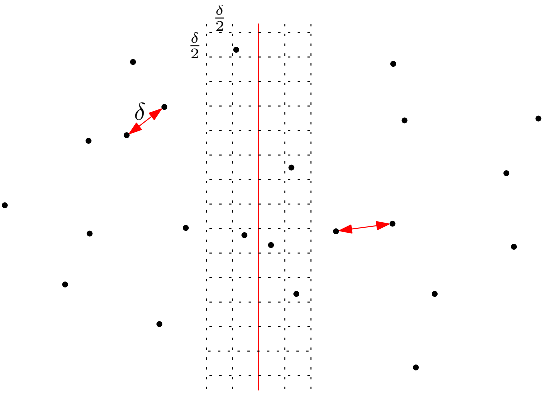
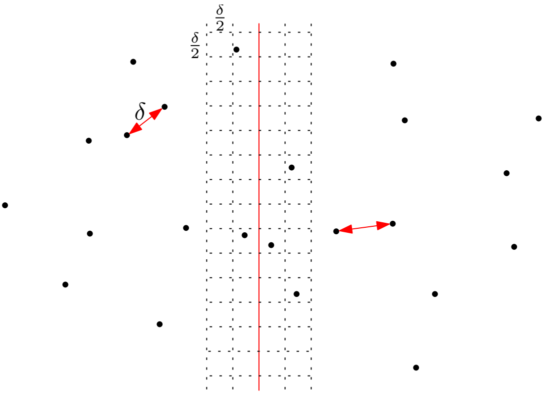


- Each box contains at most one pair

Divide-and-Conquer Algorithm for Closest Pair



- Each box contains at most one pair
- For each point, only need to consider $O(1)$ boxes nearby

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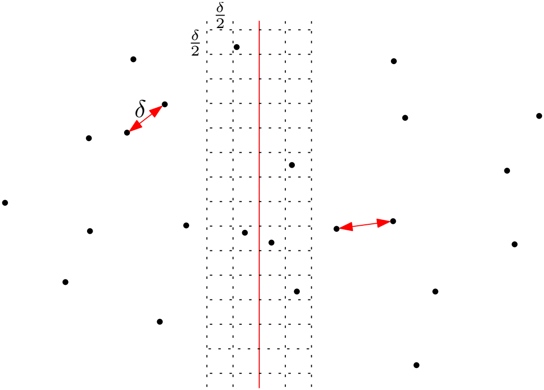

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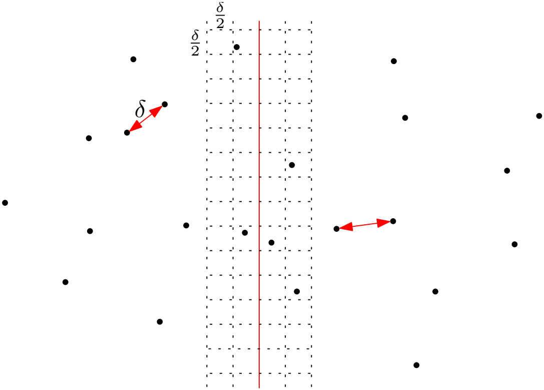

- Each box contains at most one pair
- For each point, only need to consider $O(1)$ boxes nearby
- Implementation: Sort points inside the stripe according to y-coordinates
- For every point, consider $O(1)$ points around it in the order

- time for combine step $= O(n \log n)$
- recurrence: $T(n) = 2T(n/2) + O(n \log n)$

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Improve the running time of combine step to $O(n)$

- also sort the points in ascending order of y values at the beginning
- pass the sequence to the root recursion
- constructing two sub-sequences from the sequence, and pass them to the two sub-recursions respectively

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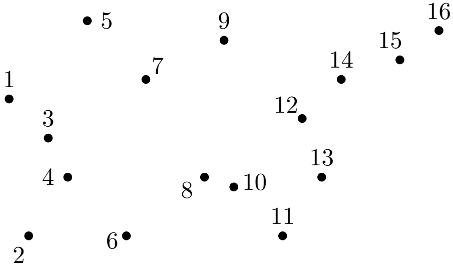
Example for Closest Pair



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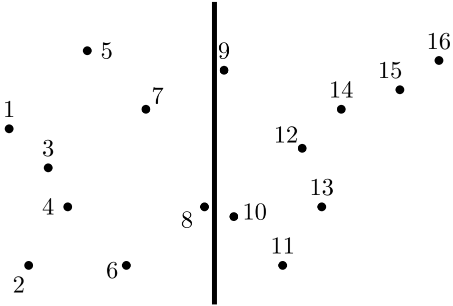

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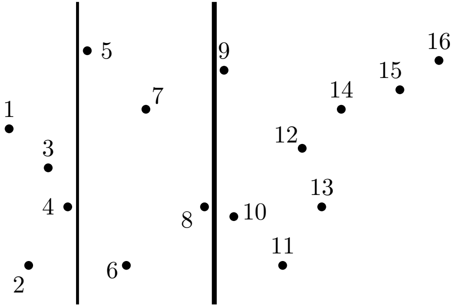

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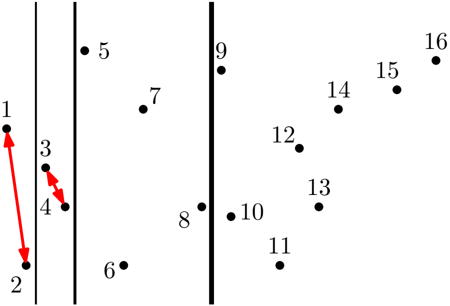

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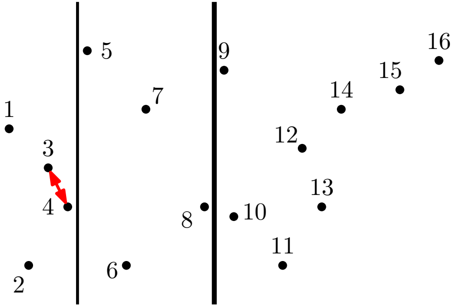

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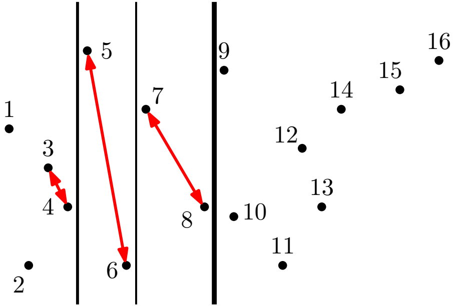

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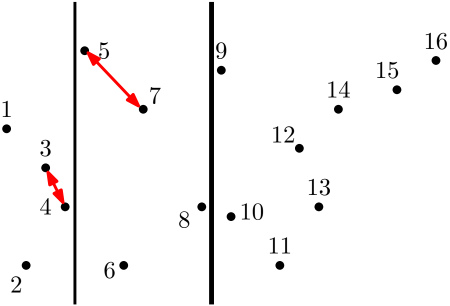

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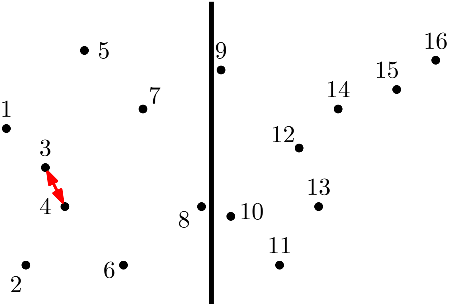

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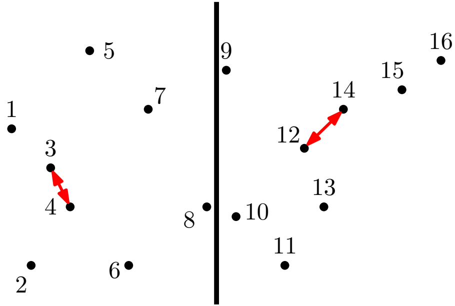

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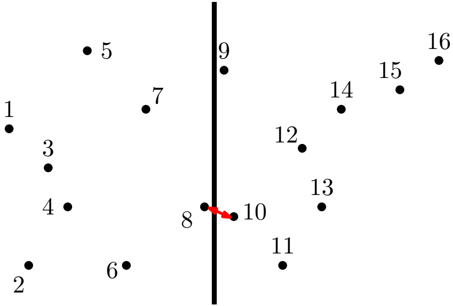

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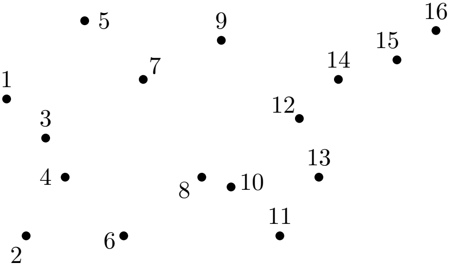

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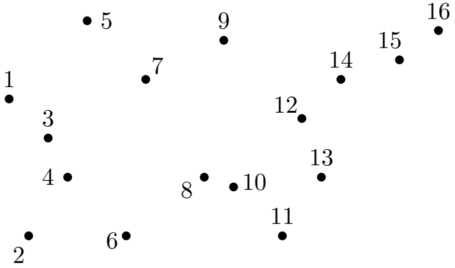

- CP$(1, 16, (5, 16, 9, 15, 7, 14, 1, 12, 3, 4, 8, 13, 10, 11, 2, 6))$

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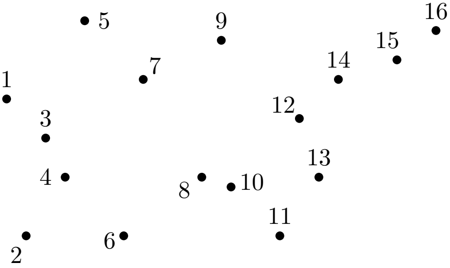

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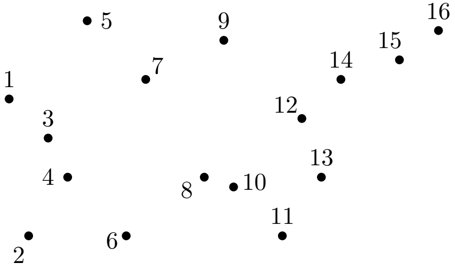

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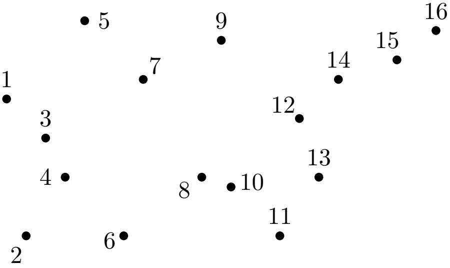

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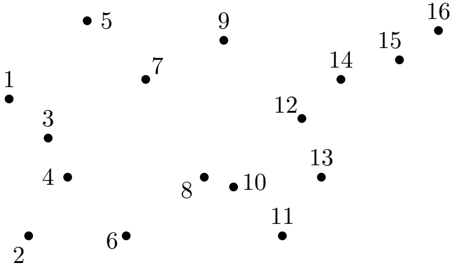

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Fibonacci Numbers

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

n-th Fibonacci Number

Input: integer $n > 0$

Output: F_n

Computing F_n : Stupid Divide-and-Conquer Algorithm

Fib(n)

```
if $n = 0$ return 0
if $n = 1$ return 1
return Fib($n - 1$) + Fib($n - 2$)
```

Q: Is the running time of the algorithm polynomial or exponential in n?

Computing F_n : Stupid Divide-and-Conquer Algorithm

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A: Exponential

- Running time is at least $\Omega(F_n)$
- F_n is exponential in n

Computing F_n: Reasonable Algorithm

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```
$F[0] \leftarrow 0$
$F[1] \leftarrow 1$
for $i \leftarrow 2$ to $n$ do
    $F[i] \leftarrow F[i-1] + F[i-2]$
return $F[n]$
```

- Dynamic Programming

Computing F_n: Reasonable Algorithm

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- Dynamic Programming
- Running time $=$?

Computing F_n: Reasonable Algorithm

Fib(n)

```
$F[0] \leftarrow 0$
$F[1] \leftarrow 1$
**for** $i \leftarrow 2$ to $n$ **do**
    $F[i] \leftarrow F[i-1] + F[i-2]$
**return** $F[n]$
```

- Dynamic Programming
- Running time $= O(n)$

Computing F_n: Even Better Algorithm

$$
\begin{aligned}
\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} \\
\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix} \\
& \cdots \\
\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}
\end{aligned}
$$

power(n)

```
if $n = 0$ then return $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$R \leftarrow$ power($\lfloor n/2 \rfloor$)
$R \leftarrow R \times R$
if $n$ is odd then $R \leftarrow R \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
**return** $R$
```

Fib(n)

```
if $n = 0$ then return $0$
$M \leftarrow$ power($n - 1$)
**return** $M[1][1]$
```

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Fixing the Problem

To compute F_n, we need $O(\log n)$ basic arithmetic operations on integers

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- **Divide**: Divide instance into many smaller instances
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- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance
- Write down recurrence for running time
- Solve recurrence using master theorem

Summary: Divide-and-Conquer

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  $T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$

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- To improve running time, design better algorithm for "combine" step, or reduce number of recursions, ...