算法设计与分析(2025年春季学期)
Introduction and Syllabus

授课老师: 栗师

南京大学计算机学院

Outline

1. Syllabus

2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort

3. Asymptotic Notations

4. Common Running times

- Course Webpage:
  `https://tcs.nju.edu.cn/shili/courses/2025spring-algo`

Course Information

- **Time**: Tuesdays and Thursdays, 10:10am - 12:00pm
- **Location**: 仙II-319
- **Instructor**: Shi Li (栗师)
- **Email**: [first name][last name][at][nju][dot][edu][dot][cn]

Logistics

- **Instructor's Office Hours**: Wednesdays 11:00am-12:00pm
- **Location**: 计算机系楼605
- **TA**: 梁梓豪(zhliang[at]smail[dot]nju[dot]edu[dot]cn)

What You Will Learn

- How to analyze the correctness and running time of an algorithm.
- Classic algorithms for classic problems
  - sorting, minimum spanning tree, shortest paths
- Algorithm design paradigms
  - greedy algorithms, divide and conquer, dynamic programming
- Network flow, linear programming, and problem reductions.
- NP-completeness.
- Advanced topics
  - randomized algorithms, approximation algorithms, fixed-parameter tractability, online algorithms

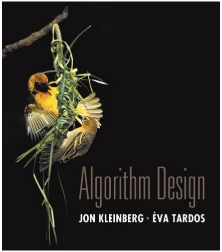
Prerequisites

- Basic skills in formulating mathematical proofs.
- Courses on data structures covering:
  - Linked lists, arrays, stacks, queues, priority queues, trees, graphs.
- Some programming experience using Python, C, C++, or Java.

Textbook

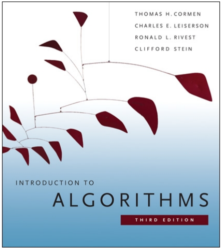
Required Textbook:

- Jon Kleinberg and Eva Tardos, *Algorithm Design*, 1st Edition, 2005, Pearson.



Reference Book:

- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, *Introduction to Algorithms*, 3rd Edition, 2009, MIT Press.



Grading

Your final grade will be calculated as follows:

- **5 Homework Assignments**: 20%.
- **Midterm Exam**: 20% or 30%.
- **Final Exam**: 60% or 50%.

Overall Score: The highest of the following weighting schemes:

- 20% Homework + 20% Midterm + 60% Final
- 20% Homework + 30% Midterm + 50% Final

Note: Both exams are closed-book.

Policies for Assignments

- No late submissions will be accepted.
- Do not search online for solutions or use AI tools to generate solutions.
- **Allowed Materials**: Textbook, reference book, course slides, and instructor-distributed materials.
- **Collaboration**:
  - You may discuss with classmates but must write solutions independently.
  - Write down the names of collaborators.

Use of AI Tools

- AI tools (e.g., ChatGPT, DeepSeek) are **allowed as learning tools** but prohibited for solving homework problems.
- AI-generated content may contain errors; you are responsible to verify correctness
- **Rule**: Once you begin working on an assignment, you must complete it without searching for solutions online or using AI tools.

Tentative Schedule

Topic	**Time**
Introduction	4 hours
Graph Basics	4 hours
Greedy Algorithms	6 hours
Divide and Conquer	6 hours
Dynamic Programming	6 hours
Graph Algorithms	6 hours
Midterm Exam	2 hours
Network Flow	6 hours
NP-Completeness	6 hours
Linear Programming	4 hours
Advanced Topics	10 hours
Final Review	2 hours

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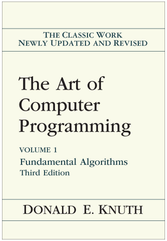
What is an Algorithm?



THE CLASSIC WORK
NEWLY UPDATED AND REVISED

The Art of
Computer
Programming

VOLUME 1
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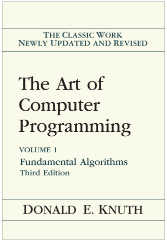
Third Edition

DONALD E. KNUTH

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.

What is an Algorithm?



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- finite: description is finite, (stronger requirement: terminate in finite number of steps)
- definite: clearly defined, no ambiguity
- effective: must be realizable using a finite amount of resources
- input: take 0 or some inputs
- output: produce 1 or more outputs

What is an Algorithm?

- Computational problem: specifies the input/output relationship.
- An algorithm solves a computational problem if it produces the correct output for any given input.

Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of a and b

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- Algorithm: Euclidean algorithm
- $\gcd(270, 210) = \gcd(210, 270 \bmod 210) = \gcd(210, 60)$

Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of a and b

Example:

- Input: 210, 270
- Output: 30

- Algorithm: Euclidean algorithm
- $\gcd(270, 210) = \gcd(210, 270 \bmod 210) = \gcd(210, 60)$
- $(270, 210) \to (210, 60) \to (60, 30) \to (30, 0)$

Examples

Sorting

Input: sequence of n numbers $(a_1, a_2, \cdots, a_n)$

Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

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- Algorithms: insertion sort, merge sort, quicksort, . . .

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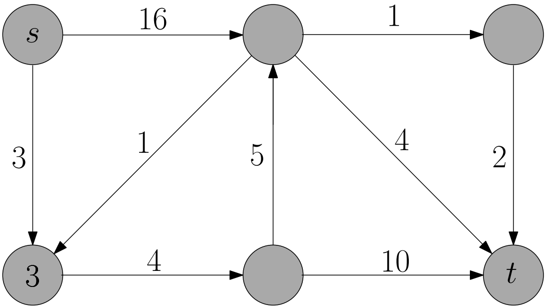

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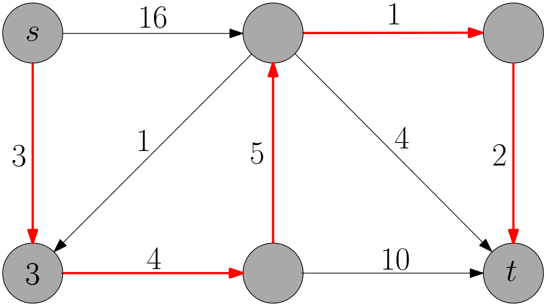

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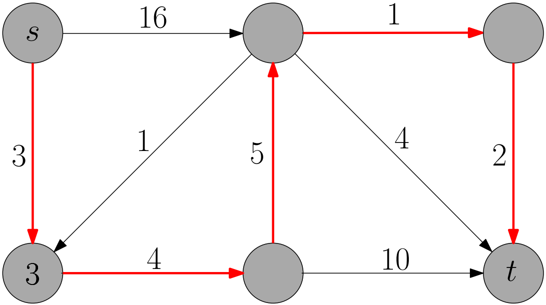

- Algorithm: Dijkstra's algorithm

Algorithm = Computer Program?

- Algorithm: "abstract", can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: "concrete", implementation of algorithm, using a particular programming language

Pseudo-Code

Pseudo-Code:

```
Euclidean($a, b$)
while $b > 0$ do
    $(a, b) \leftarrow (b, a \bmod b)$
return $a$
```

Python program:

```
def gcd(a, b):
    while b != 0:
        a, b = b, a % b
    return a
```

C++ program:

```
int Euclidean(int a, int b){
    int c;
     while (b > 0){
        c = b;
        b = a % b;
        a = c;
    }
    return a;
}
```

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3. fundamental
4. it is fun!

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- Input: $53, 12, 35, 21, 59, 15$
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Insertion-Sort

- At the end of j-th iteration, the first j numbers are sorted.

iteration 1: $53, 12, 35, 21, 59, 15$

iteration 2: $12, 53, 35, 21, 59, 15$

iteration 3: $12, 35, 53, 21, 59, 15$

iteration 4: $12, 21, 35, 53, 59, 15$

iteration 5: $12, 21, 35, 53, 59, 15$

iteration 6: $12, 15, 21, 35, 53, 59$

Example:

- Input: $53, 12, 35, 21, 59, 15$
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insertion-sort(A, n)

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for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
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- $j = 6$
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12 21 35 53 59 15

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Analysis of Insertion Sort

- Correctness
- Running time

Correctness of Insertion Sort

- Invariant: after iteration j of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

$$\text{after } j = 1 : 53, 12, 35, 21, 59, 15$$

$$\text{after } j = 2 : 12, 53, 35, 21, 59, 15$$

$$\text{after } j = 3 : 12, 35, 53, 21, 59, 15$$

$$\text{after } j = 4 : 12, 21, 35, 53, 59, 15$$

$$\text{after } j = 5 : 12, 21, 35, 53, 59, 15$$

$$\text{after } j = 6 : 12, 15, 21, 35, 53, 59$$

Analyzing Running Time of Insertion Sort

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  - For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.

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- Q2: Which input?
  - For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.
- A2: Worst-case analysis:
  - Running time for size n = worst running time over all possible arrays of length n

Analyzing Running Time of Insertion Sort

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- Q4: Programming language?

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Important idea: asymptotic analysis

- Focus on growth of running-time as a function, not any particular value.

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

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- $n^2/100 - 3n + 10 \Rightarrow n^2/100 \Rightarrow n^2$

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- programming language
- how we measure the running time: seconds or # instructions?

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  - program 2 requires 2 instructions, or 10^{-9} seconds
  - they only change by a constant in the running time, which will be hidden by the $O(\cdot)$ notation

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$

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Asymptotic Analysis of Insertion Sort

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**for** $j \leftarrow 2$ to $n$ **do**
    $key \leftarrow A[j]$
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- Total running time $= \sum_{j=2}^{n} O(j) = O(\sum_{j=2}^{n} j)$ $= O(\frac{n(n+1)}{2} - 1) = O(n^2)$

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- Can we do better than insertion sort asymptotically?
- Yes: merge sort, quicksort and heap sort take $O(n \log n)$ time

Questions?

Outline

1. Syllabus

2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort

3. Asymptotic Notations

4. Common Running times

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

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- $n^2 - n - 30$ Yes
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- $100n - n^2/10 + 50?$ No
- We only consider asymptotically positive functions.

O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \left\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\right\}.$$

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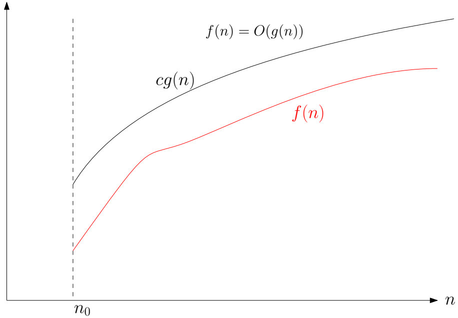

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Proof.

Let $c = 4$ and $n_0 = 50$, for every $n > n_0 = 50$, we have,

$$3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)$$

$$= -n^2 + 42n \le 0.$$

$$3n^2 + 2n \le c(n^2 - 10n)$$

□

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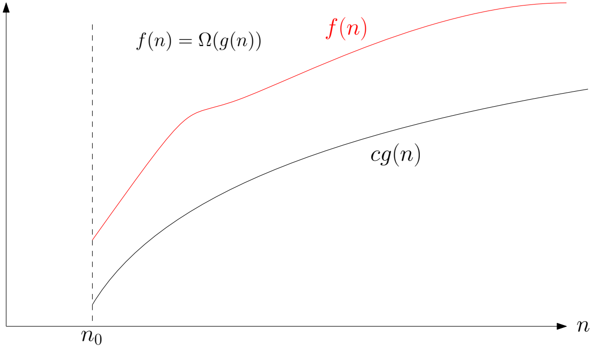

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Theorem $f(n) = O(g(n)) \;\Leftrightarrow\; g(n) = \Omega(f(n))$.

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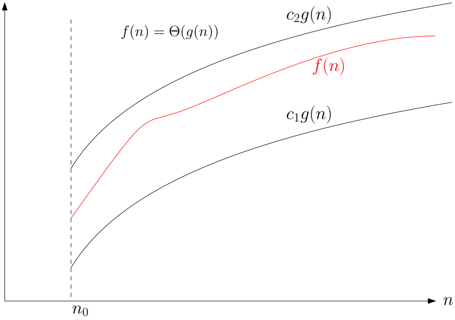

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Theorem $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

o and ω-Notations

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Example:

- $3n^2 + 5n + 10 = o(n^2 \log n)$.
- $3n^2 + 5n + 10 = \omega(n^2 / \log n)$.

Asymptotic Notations	O	Ω	Θ	o	ω
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For two constants $a, b \in \mathbb{R}$:

- $n^a = O(n^b)$ if and only if $a \leq b$
- $n^a = \Omega(n^b)$ if and only if $a \geq b$
- $n^a = \Theta(n^b)$ if and only if $a = b$
- $n^a = o(n^b)$ if and only if $a < b$
- $n^a = \omega(n^b)$ if and only if $a > b$

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Facts on Comparison Relations

- $a \leq b \iff b \geq a$
- $a = b \iff a \leq b$ and $a \geq b$
- $a < b \implies a \leq b$
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Correct Analogies

- $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
- $f(n) = \Theta(g(n)) \iff f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
- $f(n) = o(g(n)) \implies f(n) = O(g(n))$
- $f(n) = o(g(n)) \iff g(n) = \omega(f(n))$

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- $f(n) = O(g(n))$ or $f(n) = \Omega(g(n))$

$$f(n) = n^2$$

$$g(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ n^3 & \text{if } n \text{ is even} \end{cases}$$

Recall: Informal way to define O-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \to 3n^2$
- ignoring leading constant: $3n^2 \to n^2$
- $3n^2 - 10n - 5 = O(n^2)$
- In the formal definition of $O(\cdot)$, nothing tells us to ignore lower order terms and leading constant.

Recall: Informal way to define O-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \to 3n^2$
- ignoring leading constant: $3n^2 \to n^2$
- $3n^2 - 10n - 5 = O(n^2)$
- In the formal definition of $O(\cdot)$, nothing tells us to ignore lower order terms and leading constant.
- $3n^2 - 10n - 5 = O(5n^2 - 6n + 5)$ is correct, though weird
- $3n^2 - 10n - 5 = O(n^2)$ is the most natural since n^2 is the simplest term we can have inside $O(\cdot)$.

Notice that O denotes asymptotic upper bound

- $n^2 + 2n = O(n^3)$ is correct.
- The following sentence is correct: the running time of insertion sort is $O(n^4)$.
- Usually we say: The running time of insertion sort is $O(n^2)$ and the bound is tight.
- Also correct: the worst-case running time of insertion sort is $\Theta(n^2)$.

Outline

1. Syllabus
2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort
3. Asymptotic Notations
4. Common Running times

$O(n)$ (Linear) Running Time

Computing the sum of n numbers

sum(A, n)

```
$S \leftarrow 0$
for $i \leftarrow 1$ to $n$
    $S \leftarrow S + A[i]$
return $S$
```

$O(n)$ (Linear) Running Time

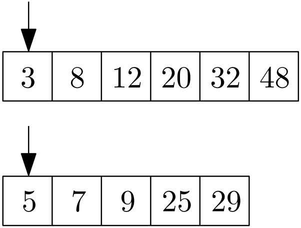
- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

$O(n)$ (Linear) Running Time

- Merge two sorted arrays



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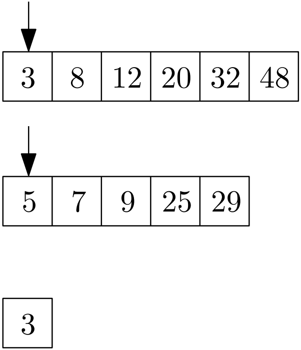

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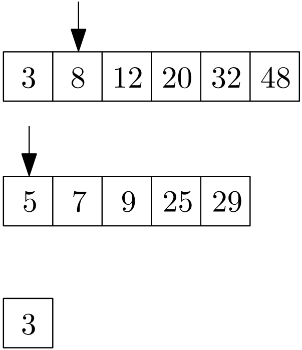

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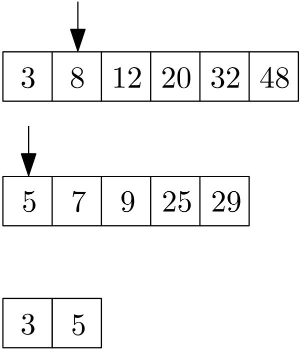

3	8	12	20	32	48

5	7	9	25	29

3	5

$O(n)$ (Linear) Running Time

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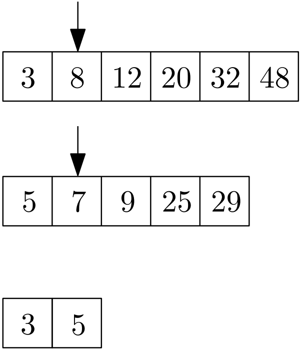

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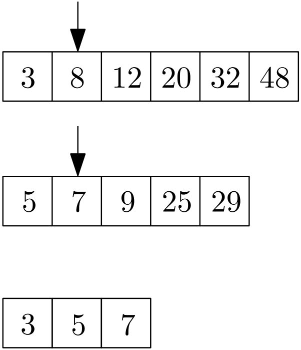

3 8 12 20 32 48

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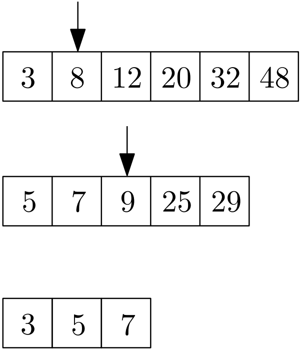

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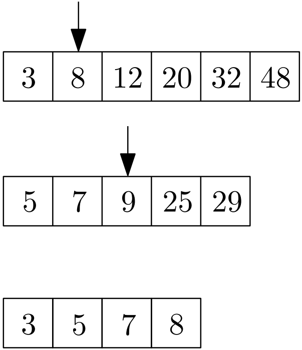

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3	5	7	8

$O(n)$ (Linear) Running Time

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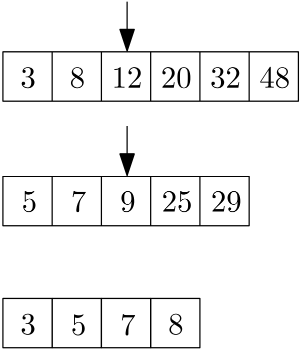

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$O(n)$ (Linear) Running Time

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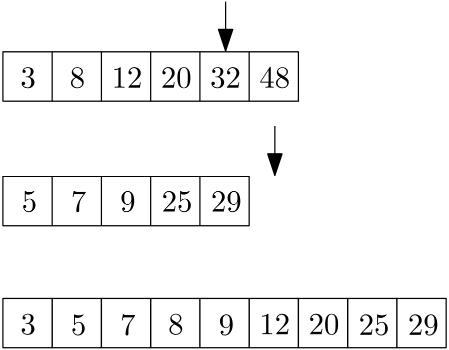

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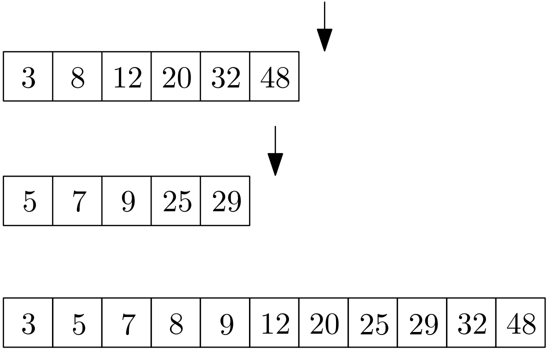

3	8	12	20	32	48

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$O(n)$ (Linear) Running Time

merge(B, C, n_1, n_2) \\ B and C are sorted, with length n_1 and n_2

```
A ← []; i ← 1; j ← 1
while $i \le n_1$ and $j \le n_2$ do
    if $B[i] \le C[j]$ then
        append $B[i]$ to $A$; $i \leftarrow i + 1$
    else
        append $C[j]$ to $A$; $j \leftarrow j + 1$
if $i \le n_1$ then append $B[i..n_1]$ to $A$
if $j \le n_2$ then append $C[j..n_2]$ to $A$
return $A$
```

$O(n)$ (Linear) Running Time

```
merge($B, C, n_1, n_2$)    \\ $B$ and $C$ are sorted, with length $n_1$ and $n_2$
A ← []; i ← 1; j ← 1
while $i \leq n_1$ and $j \leq n_2$ do
    if $B[i] \leq C[j]$ then
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if $i \leq n_1$ then append $B[i..n_1]$ to $A$
if $j \leq n_2$ then append $C[j..n_2]$ to $A$
return $A$
```

Running time $= O(n)$ where $n = n_1 + n_2$.

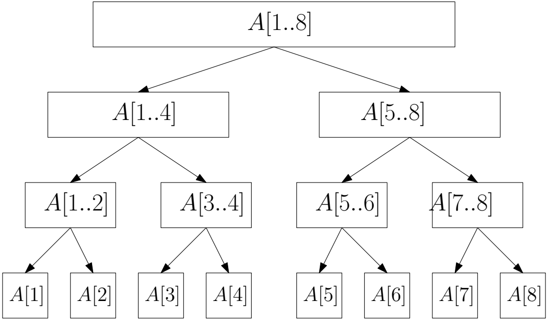
$O(n \log n)$ Running Time

merge-sort(A, n)

```
if $n = 1$ then
    return $A$
$B \leftarrow$ merge-sort$\Big(A\big[1..\lfloor n/2 \rfloor\big], \lfloor n/2 \rfloor\Big)$
$C \leftarrow$ merge-sort$\Big(A\big[\lfloor n/2 \rfloor + 1..n\big], n - \lfloor n/2 \rfloor\Big)$
return merge$(B, C, \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor)$
```

$O(n \log n)$ Running Time

- Merge-Sort



$O(n \log n)$ Running Time

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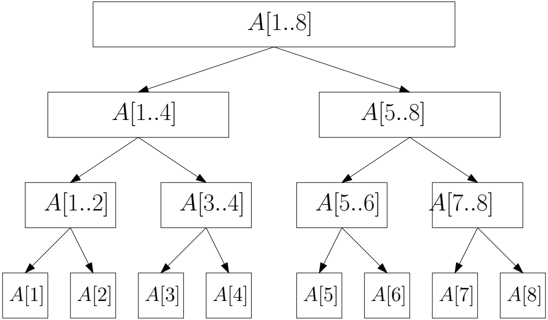

- Each level takes running time $O(n)$

$O(n \log n)$ Running Time

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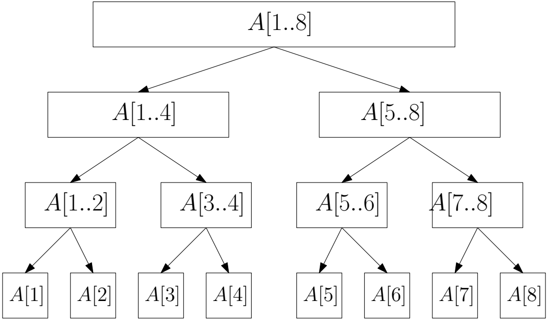
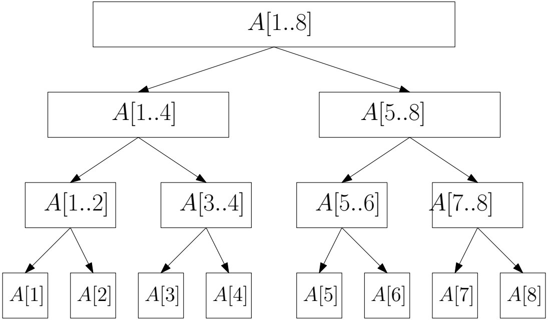


- Each level takes running time $O(n)$
- There are $O(\log n)$ levels

$O(n \log n)$ Running Time

- Merge-Sort



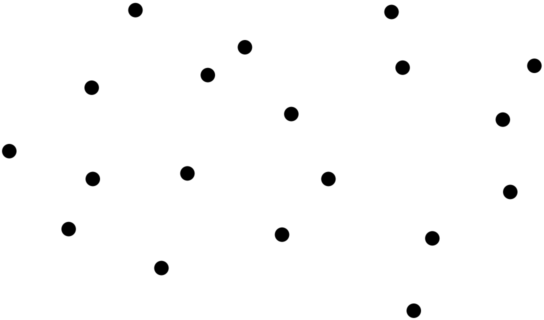
- Each level takes running time $O(n)$
- There are $O(\log n)$ levels
- Running time $= O(n \log n)$

$O(n^2)$ (Quadratic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest



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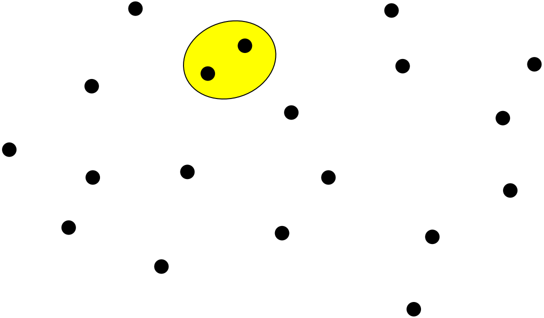

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest

closest-pair(x, y, n)

```
$bestd \leftarrow \infty$
for $i \leftarrow 1$ to $n - 1$ do
    for $j \leftarrow i + 1$ to $n$ do
        $d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$
        if $d < bestd$ then
            $besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d$
return $(besti, bestj)$
```

$O(n^2)$ (Quardatic) Running Time

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closest-pair(x, y, n)

```
bestd ← ∞
for i ← 1 to n − 1 do
    for j ← i + 1 to n do
        d ← √((x[i] − x[j])² + (y[i] − y[j])²)
        if d < bestd then
            besti ← i, bestj ← j, bestd ← d
return (besti, bestj)
```

Closest pair can be solved in $O(n \log n)$ time!

$O(n^3)$ (Cubic) Running Time

Multiply two matrices of size $n \times n$

matrix-multiplication(A, B, n)

```
C ← matrix of size $n \times n$, with all entries being $0$
for $i \leftarrow 1$ to $n$ do
    for $j \leftarrow 1$ to $n$ do
        for $k \leftarrow 1$ to $n$ do
            $C[i,k] \leftarrow C[i,k] + A[i,j] \times B[j,k]$
return $C$
```

Beyond Polynomial Time: 2^n

Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$.

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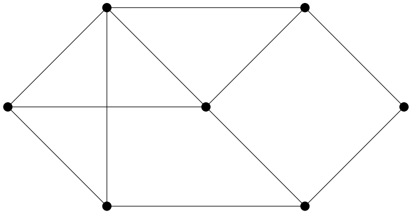

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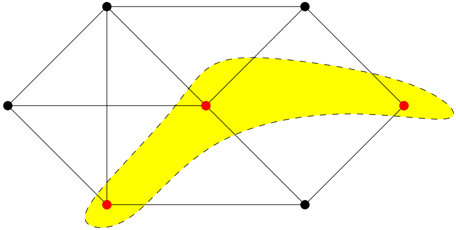


Beyond Polynomial Time: 2^n

Maximum Independent Set Problem

Input: graph $G = (V, E)$

Output: the maximum independent set of G

max-independent-set($G = (V, E)$)

```
$R \leftarrow \emptyset$
for every set $S \subseteq V$ do
    $b \leftarrow$ true
    for every $u, v \in S$ do
        if $(u, v) \in E$ then $b \leftarrow$ false
    if $b$ and $|S| > |R|$ then $R \leftarrow S$
return $R$
```

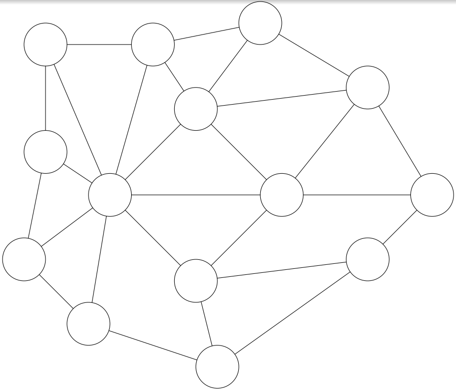
Running time $= O(2^n n^2)$.

Beyond Polynomial Time: $n!$

Hamiltonian Cycle Problem

Input: a graph with n vertices

Output: a cycle that visits each node exactly once,
or say no such cycle exists



Beyond Polynomial Time: $n!$

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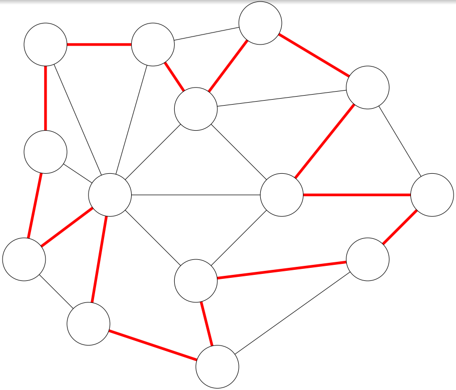

Beyond Polynomial Time: $n!$

Hamiltonian$(G = (V, E))$

```
for every permutation $(p_1, p_2, \cdots, p_n)$ of $V$ do
    $b \leftarrow$ true
    for $i \leftarrow 1$ to $n - 1$ do
        if $(p_i, p_{i+1}) \notin E$ then $b \leftarrow$ false
    if $(p_n, p_1) \notin E$ then $b \leftarrow$ false
    if $b$ then return $(p_1, p_2, \cdots, p_n)$
return "No Hamiltonian Cycle"
```

Running time $= O(n! \times n)$

$O(\log n)$ (Logarithmic) Running Time

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  - Input: sorted array A of size n, an integer t;
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- E.g, search 35 in the following array:

3	8	10	25	29	37	38	42	46	52	59	61	63	75	79

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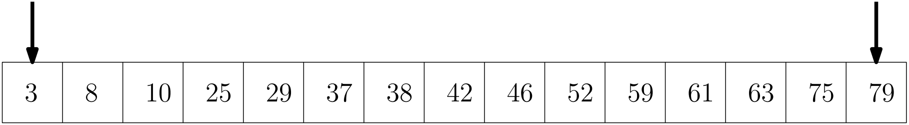

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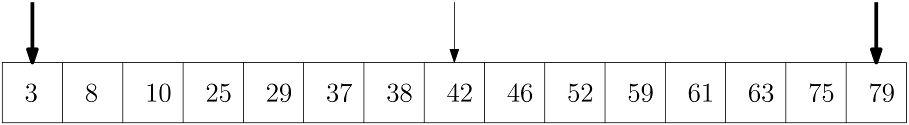

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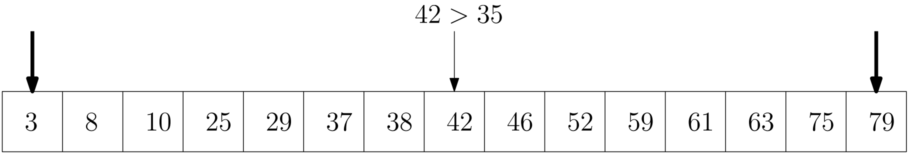

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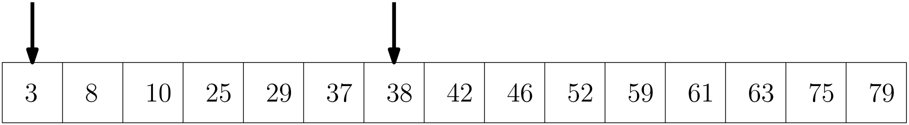

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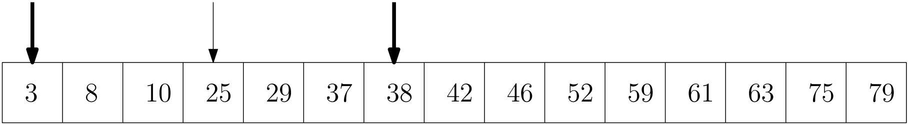

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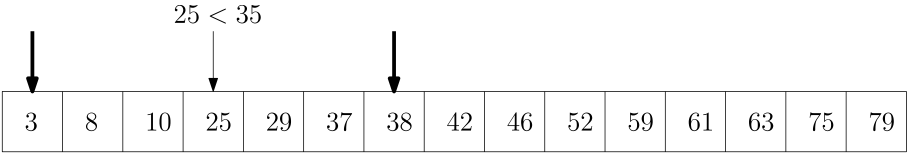

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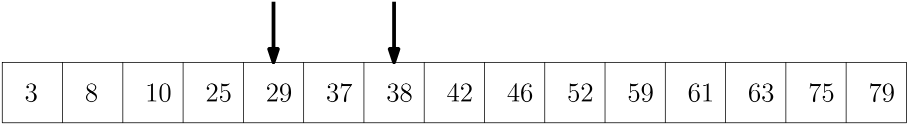

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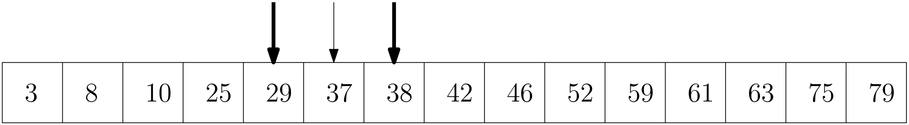

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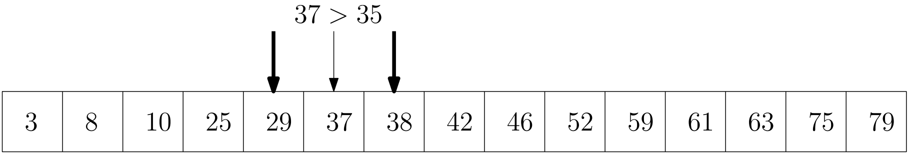

$37 > 35$

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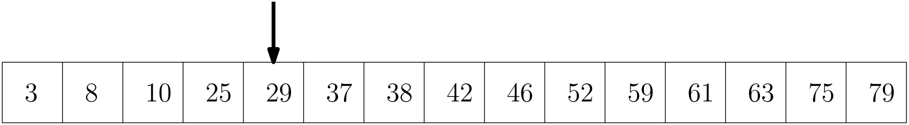

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Binary search

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binary-search(A, n, t)

```
$i \leftarrow 1, j \leftarrow n$
while $i \leq j$ do
    $k \leftarrow \lfloor (i+j)/2 \rfloor$
    if $A[k] = t$ return true
    if $t < A[k]$ then $j \leftarrow k - 1$ else $i \leftarrow k + 1$
return false
```

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**return** false
```

Running time $= O(\log n)$

Comparing the Orders of Running Times

- Sort the functions from smallest to largest asymptotically
  $\log n, \ \ n, \ \ n^2, \ \ n \log n, \ \ n!, \ \ 2^n, \ \ e^n, \ \ n^n, \ \ \log(n!)$

- $\log n$

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- $\log n \quad n$

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- $\log n \quad n \quad n^2$

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- $\log n \quad n \quad n \log n \quad n^2$

Comparing the Orders of Running Times

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- $\log n \quad n \quad \{n \log n, \, \log(n!)\} \quad n^2 \quad 2^n \quad e^n \quad n! \quad n^n$
- $\log n = o(n), \quad n = o(n \log n), \quad n \log n = \Theta(\log(n!))$
- $\log(n!) = o(n^2), \quad n^2 = o(2^n), \quad 2^n = o(e^n)$
- $e^n = o(n!), \quad n! = o(n^n)$

Terminologies

When we talk about upper bounds:

- Logarithmic time: $O(\lg n)$
- Linear time: $O(n)$
- Quadratic time: $O(n^2)$
- Cubic time: $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant k
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
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When we talk about lower bounds:

- Super-linear time: $\omega(n)$
- Super-quadratic time: $\omega(n^2)$
- Super-polynomial time: $\bigcap_{k>0} \omega(n^k) = n^{\omega(1)}$

Goal of Algorithm Design

- Design algorithms to minimize the order of the running time.

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- Using asymptotic analysis allows us to ignore the leading constants and lower order terms
- Makes our life much easier! (E.g., the leading constant depends on the implementation, complier and computer architecture of computer.)

Q: Can constants really be ignored?

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A:

- Sometimes no
- For most natural and simple algorithms, constants are not so big.
- Algorithm with lower order running time beats algorithm with higher order running time for reasonably large n.