算法设计与分析(2025年春季学期)
Linear Programming

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南京大学计算机学院

Outline

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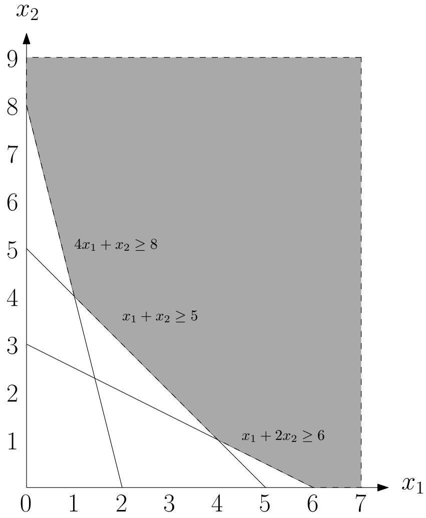
1. Linear Programming
   - Introduction
   - Preliminaries
   - Methods for Solving Linear Programs
2. Linear Programming Duality
3. Integral Polytopes: Exact Algorithms Using LP
   - Bipartite Matching Polytope
   - s-t Flow Polytope
   - Weighted Interval Scheduling Problem and Totally Unimodular Matrices

Example of Linear Programming

$$\begin{aligned} \min \qquad & 7x_1 + 4x_2 \\ x_1 + x_2 &\geq 5 \\ x_1 + 2x_2 &\geq 6 \\ 4x_1 + x_2 &\geq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$



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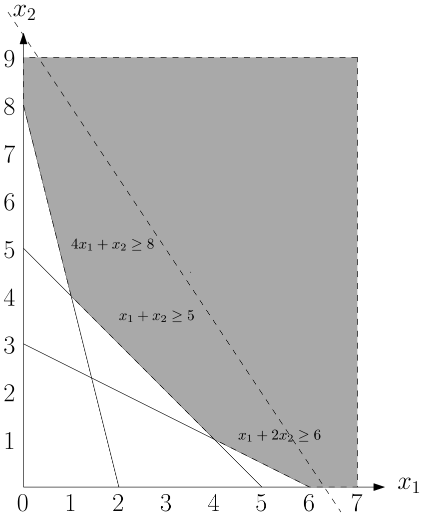

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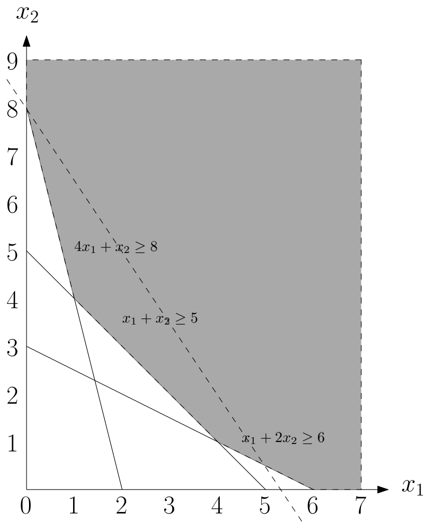

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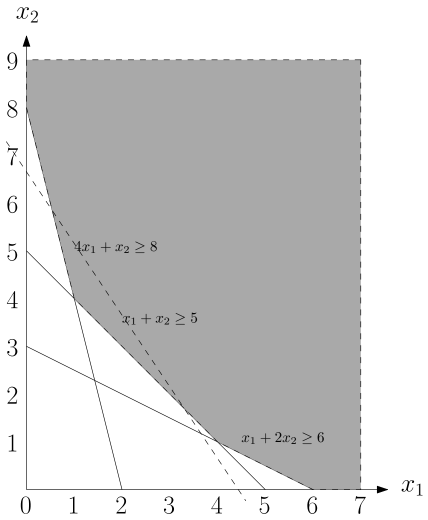

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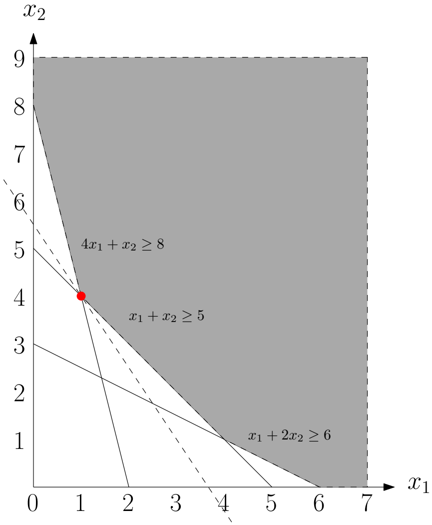

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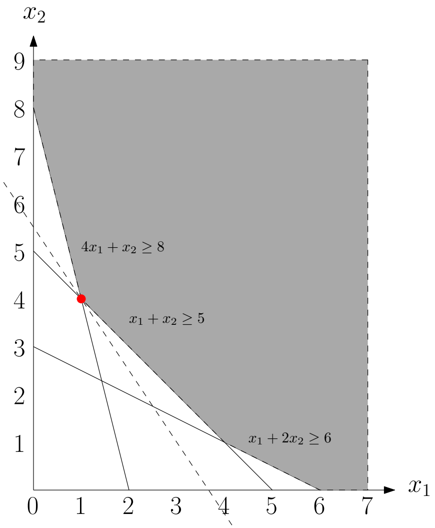
- optimum point: $x_1 = 1, x_2 = 4$



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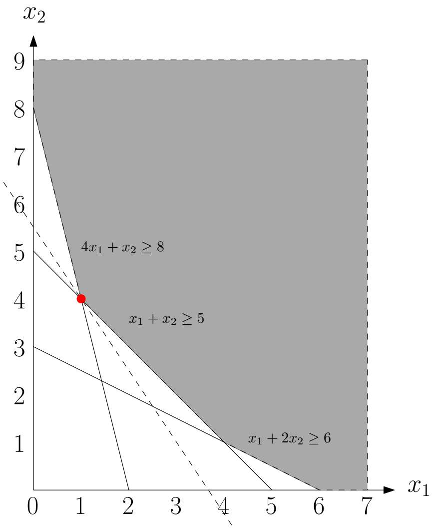
- optimum point: $x_1 = 1, x_2 = 4$
- value $= 7 \times 1 + 4 \times 4 = 23$



Standard Form of Linear Programming

$$\min \qquad c_1x_1 + c_2x_2 + \cdots + c_nx_n \qquad \textsf{s.t.}$$

$$\begin{aligned}
\sum A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n &\geq b_1 \\
\sum A_{2,1}x_1 + A_{2,2}x_2 + \cdots + A_{2,n}x_n &\geq b_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
\sum A_{m,1}x_1 + A_{m,2}x_2 + \cdots + A_{m,n}x_n &\geq b_m \\
x_1, x_2, \cdots, x_n &\geq 0
\end{aligned}$$

Standard Form of Linear Programming

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$,

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Then, LP becomes $\min \quad c^{\mathrm{T}}x$ s.t.

$$Ax \geq b$$
$$x \geq 0$$

- $\geq$ means coordinate-wise greater than or equal to

Standard Form of Linear Programming

$$\min \quad c^{\mathrm{T}}x \quad \text{s.t.}$$
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- Linear programmings can be solved in polynomial time

Algorithm	Theory	Practice
Simplex Method	Exponential Time	Works Well
Ellipsoid Method	Polynomial Time	Slow
Internal Point Methods	Polynomial Time	Works Well

History

- [Fourier, 1827]: Fourier-Motzkin elimination method
- [Kantorovich, Koopmans 1939]: formulated the general linear programming problem

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- [Kantorovich, Koopmans 1939]: formulated the general linear programming problem
- [Dantzig 1946]: simplex method
- [Khachiyan 1979]: ellipsoid method, polynomial time, proved linear programming is in P
- [Karmarkar, 1984]: interior-point method, polynomial time, algorithm is pratical

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Preliminaries

- feasible region: the set of x's satisfying
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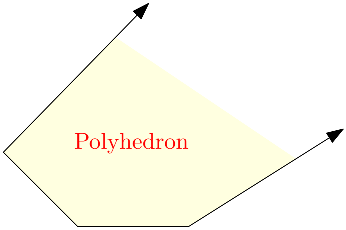

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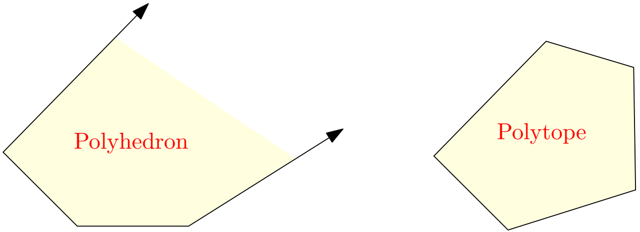
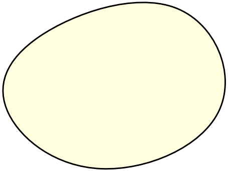
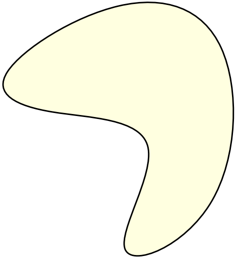
- feasible region: the set of x's satisfying $Ax \geq b, x \geq 0$
- feasible region is a polyhedron
- if every coordinate has an upper and lower bound in the polyhedron, then the polyhedron is a polytope



Def. A set of points $P \subseteq \mathbb{R}^n$ is said to be convex if for every $x, x' \in P$ and two reals $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, we have $\alpha x + \alpha x' \in P$.





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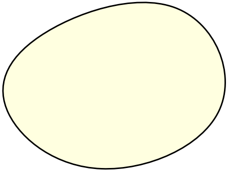

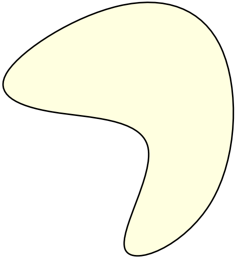

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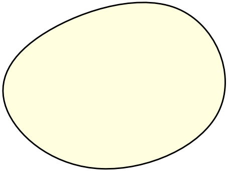
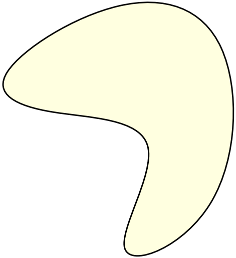


convex



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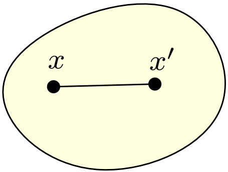

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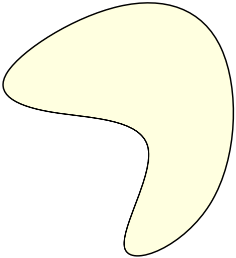

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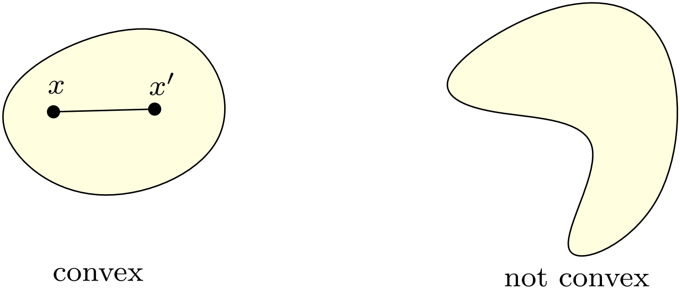

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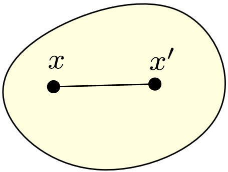

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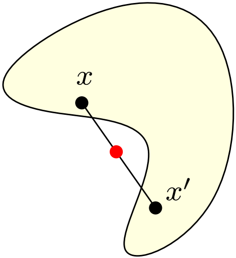

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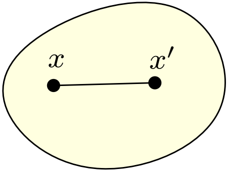

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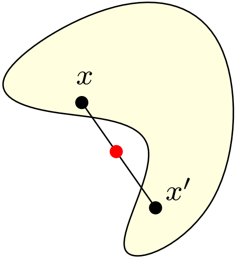

not convex

Obs. A polyhedron is convex.

Preliminaries

- We say x is a convex combination of $x^{(1)}, x^{(2)}, \cdots, x^{(t)}$ if the following condition holds: there exist $\lambda_1, \lambda_2, \cdots, \lambda_t \in [0, 1]$ such that

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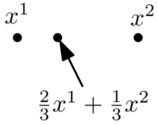

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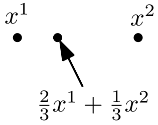

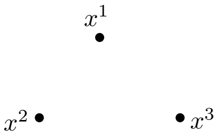

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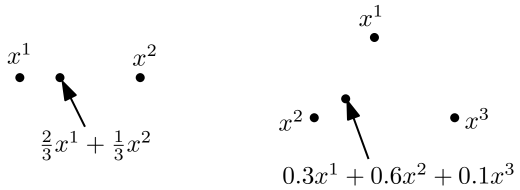

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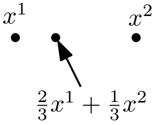

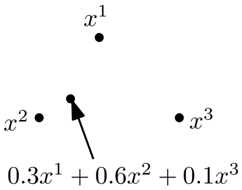

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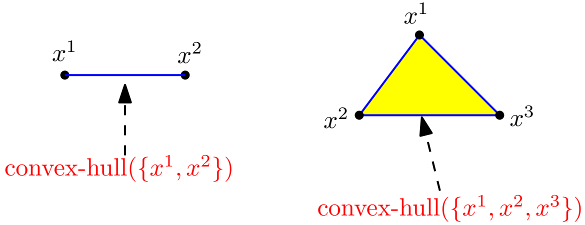

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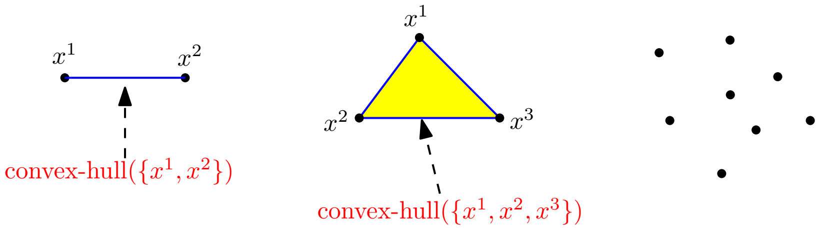

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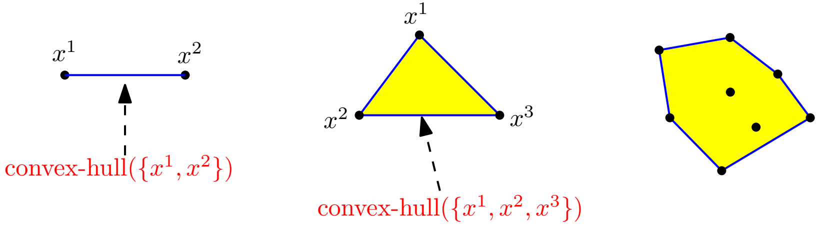

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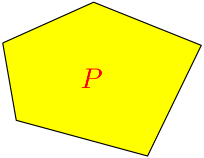

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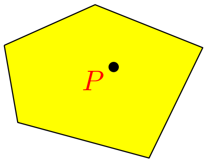

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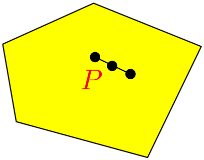

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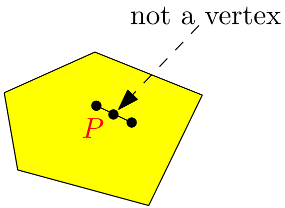

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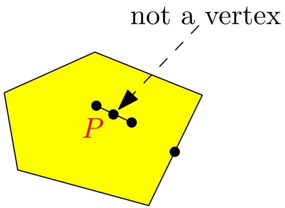

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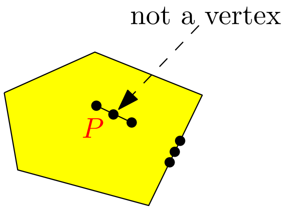

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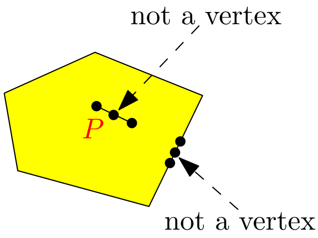

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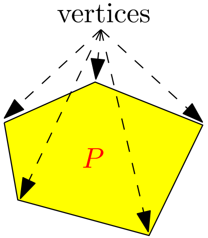

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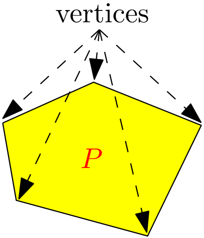
Lemma A polytope has finite number of vertices, and it is the convex hull of the vertices.



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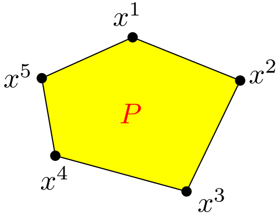
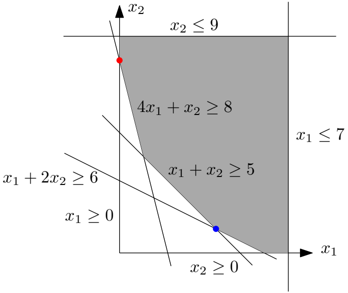


$P = \text{convex-hull}(\{x^1, x^2, x^3, x^4, x^5\})$

Preliminaries

Lemma Let $x \in \mathbb{R}^n$ be an extreme point in a n-dimensional polytope. Then, there are n constraints in the definition of the polytope, such that x is the unique solution to the linear system obtained from the n constraints by replacing inequalities to equalities.



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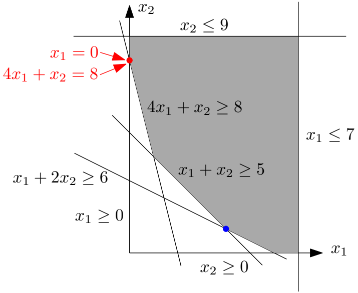

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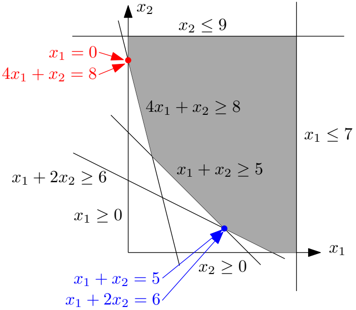

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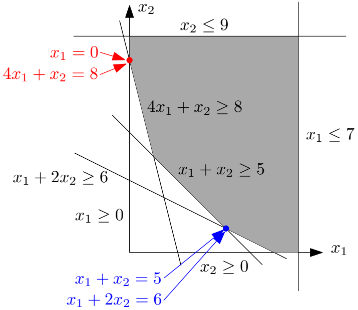

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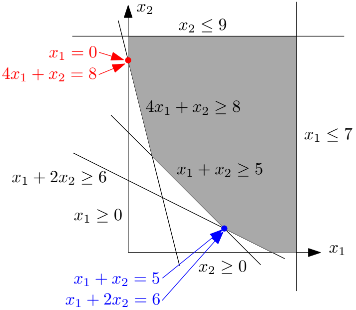

Lemma If the feasible region of a linear program is a polytope, then the opimum value can be attained at some vertex of the polytope.

Special cases (for minimization linear programs):

- if feasible region is empty, then its value is ∞
- if the feasible region is unbounded, then its value can be $-\infty$

Outline

Simplex Method

- [Dantzig, 1946]

- move from one vertex to another, so as to improve the objective
- repeat until we reach an optimum vertex

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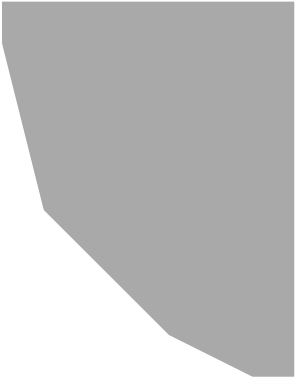

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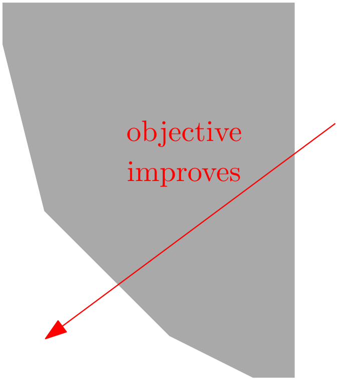

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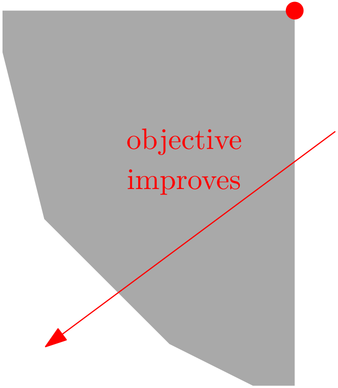

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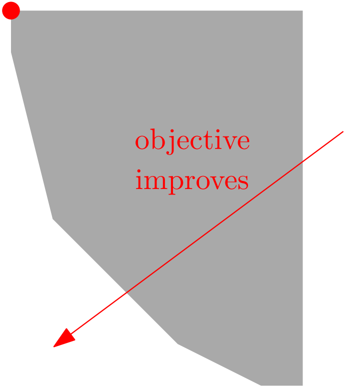

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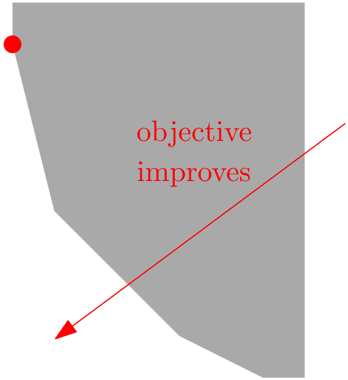

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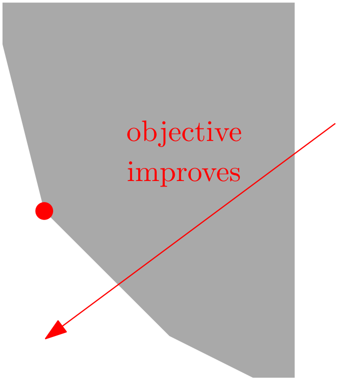

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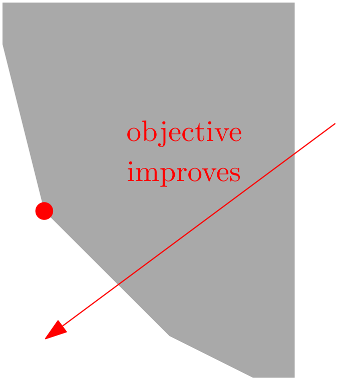

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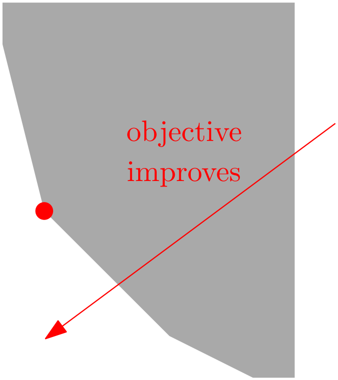

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- [Spielman-Teng,2002]: smoothed analysis

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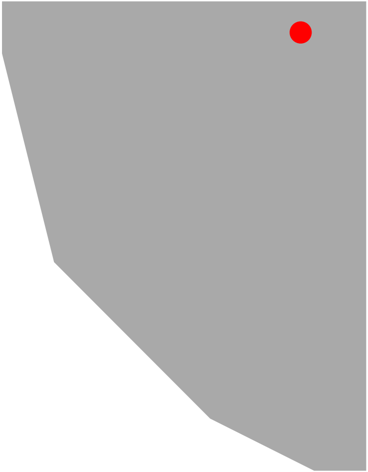

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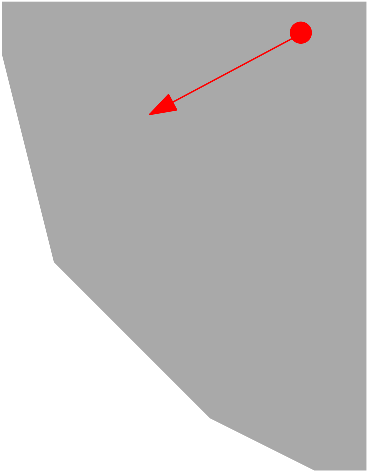

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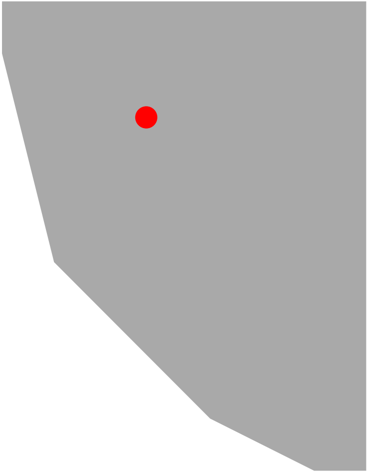

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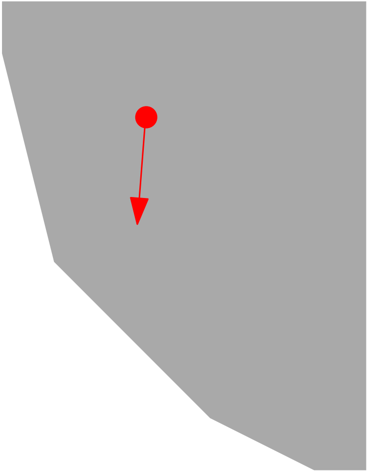

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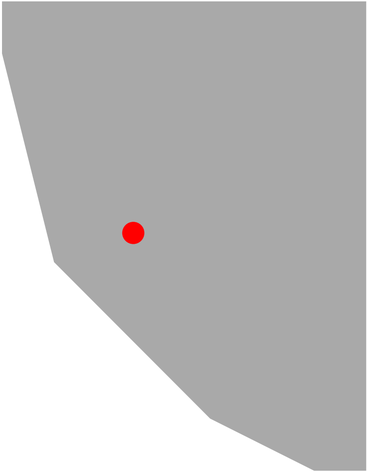

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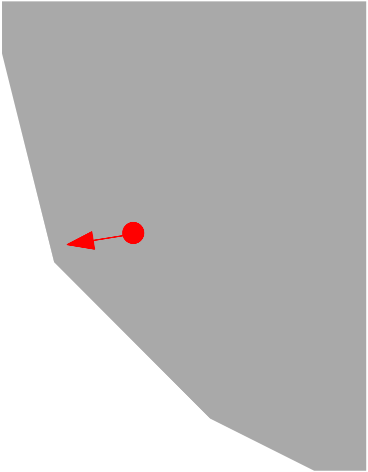

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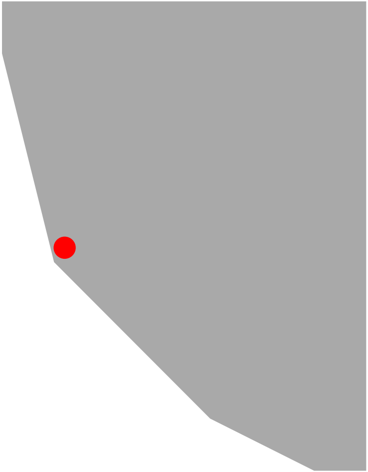

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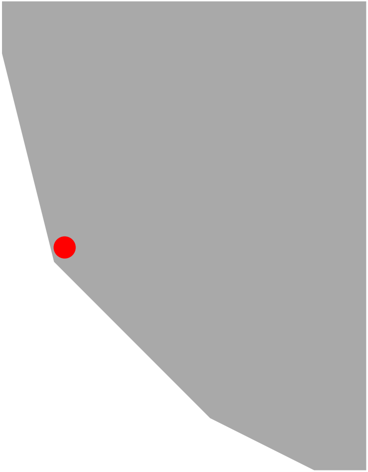

- polynomial time

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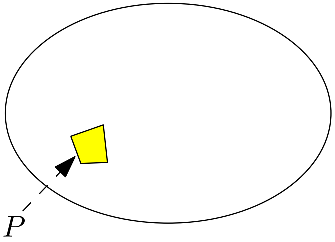

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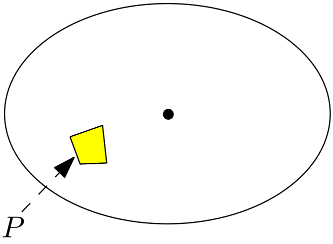

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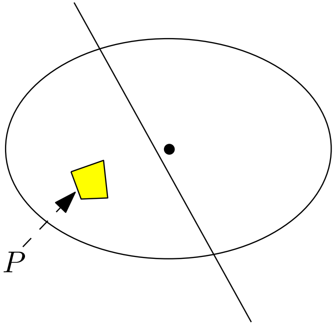

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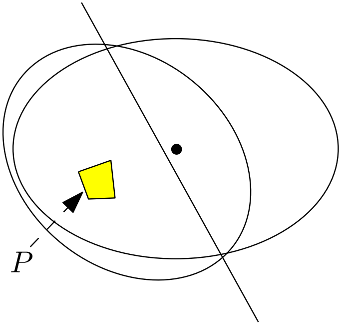

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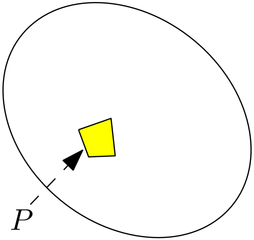

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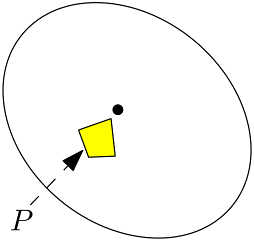

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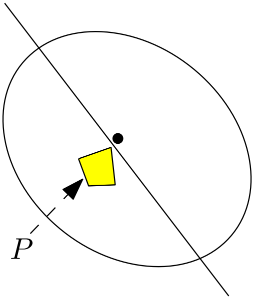

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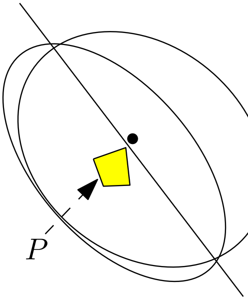

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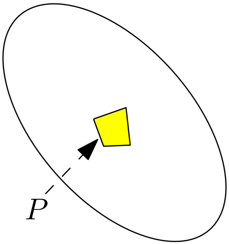

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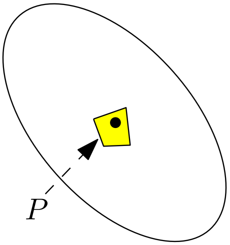

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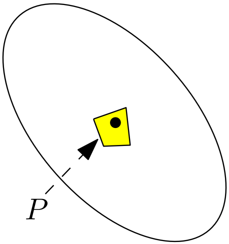

- polynomial time, but impractical

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Open Problem

Can linear programming be solved in strongly polynomial time algorithm?

Applications of Linear Programming

- domain: computer science, mathematics, operations research, economics
- types of problems: transportation, scheduling, clustering, network routing, resource allocation, facility location

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Research Directions

- polynomial time exact algorithm
- polynomial time approximation algorithm
- sub-routines for the branch-and-bound method for integer programming
- other algorithmic models: online algorithm, distributed algorithms, dynamic algorithms, fast algorithms

Outline

$$
\begin{aligned}
\min \qquad & 7x_1 + 4x_2 \\
& x_1 + x_2 \geq 5 \\
& x_1 + 2x_2 \geq 6 \\
& 4x_1 + x_2 \geq 8 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

- optimum point: $x_1 = 1, x_2 = 4$
- value $= 7 \times 1 + 4 \times 4 = 23$

Q: How can we prove a lower bound for the value?

$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ x_1 + x_2 &\geq 5 \\ x_1 + 2x_2 &\geq 6 \\ 4x_1 + x_2 &\geq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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- value $= 7 \times 1 + 4 \times 4 = 23$

Q: How can we prove a lower bound for the value?

- $7x_1 + 4x_2 \geq 2(x_1 + x_2) + (x_1 + 2x_2) \geq 2 \times 5 + 6 = 16$
- $7x_1 + 4x_2 \geq (x_1 + 2x_2) + 1.5(4x_1 + x_2) \geq 6 + 1.5 \times 8 = 18$
- $7x_1 + 4x_2 \geq (x_1 + x_2) + (x_1 + 2x_2) + (4x_1 + x_2) \geq 5 + 6 + 8 = 19$
- $7x_1 + 4x_2 \geq 4(x_1 + x_2) \geq 4 \times 5 = 20$
- $7x_1 + 4x_2 \geq 3(x_1 + x_2) + (4x_1 + x_2) \geq 3 \times 5 + 8 = 23$

Primal LP

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A way to prove lower bound on the value of primal LP

$$\begin{aligned} & 7x_1 + 4x_2 \qquad \text{(if } 7 \geq y_1 + y_2 + 4y_3 \text{ and } 4 \geq y_1 + 2y_2 + y_3\text{)} \\ \geq\ & y_1(x_1 + x_2) + y_2(x_1 + 2x_2) + y_3(4x_1 + x_2) \quad \text{(if } y_1, y_2, y_3 \geq 0\text{)} \\ \geq\ & 5y_1 + 6y_2 + 8y_3. \end{aligned}$$

- Goal: need to maximize $5y_1 + 6y_2 + 8y_3$

Primal LP

$$\min \quad 7x_1 + 4x_2$$
$$x_1 + x_2 \geq 5$$
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Dual LP

$$\max \quad 5y_1 + 6y_2 + 8y_3 \quad \text{s.t.}$$
$$y_1 + y_2 + 4y_3 \leq 7$$
$$y_1 + 2y_2 + y_3 \leq 4$$
$$y_1, y_2 \geq 0$$

A way to prove lower bound on the value of primal LP

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$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & 5y_1 + 6y_2 + 8y_3 \quad \text{s.t.} \\ & y_1 + y_2 + 4y_3 \leq 7 \\ & y_1 + 2y_2 + y_3 \leq 4 \\ & y_1, y_2 \geq 0 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 4 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} \quad c = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \min \quad & c^T x \quad \text{s.t.} \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & b^T y \quad \text{s.t.} \\ & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

Primal LP

$$\min \quad c^T x \qquad \text{s.t.}$$
$$Ax \geq b$$
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- P = value of primal LP
- D = value of dual LP

Theorem (weak duality theorem) $D \leq P$.

Theorem (strong duality theorem) $D = P$.

- Can always prove the optimality of the primal solution, by adding up primal constraints.

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Dual LP

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- D = value of dual LP

Theorem (weak duality theorem) $D \leq P$.

Proof.

- x^*: optimal primal solution
- y^*: optimal dual solution

$$D = b^{\mathrm{T}} y^* \leq (Ax^*)^{\mathrm{T}} y^* = (x^*)^{\mathrm{T}} A^{\mathrm{T}} y^* \leq (x^*)^{\mathrm{T}} c = c^{\mathrm{T}} x^* = P. \qquad \square$$

Proof of Strong Duality Theorem

Lemma (Variant of Farkas Lemma) $Ax \leq b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}}b < 0, y \geq 0$ is feasible.

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- $\forall \epsilon > 0, D > P - \epsilon \quad \implies \quad D = P$ (since $D \le P$) □

Example

Primal LP

$$\min \quad 5x_1 + 6x_2 + x_3 \quad \text{s.t.}$$

$$\begin{aligned} 2x_1 + 5x_2 - 3x_3 &\geq 2 \\ 3x_1 - 2x_2 + x_3 &\geq 5 \\ x_1 + 2x_2 + 3x_3 &\geq 7 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Dual LP

$$\max \quad 2y_1 + 5y_2 + 7y_3 \quad \text{s.t.}$$

$$\begin{aligned} 2y_1 + 3y_2 + y_3 &\leq 5 \\ 5y_1 - 2y_2 + 2y_3 &\leq 6 \\ -3y_1 + y_2 + 3y_3 &\geq 1 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Primal Solution

$$x_1 = 1.6, x_2 = 0.6$$

$$x_3 = 1.4, \text{value} = 13$$

Dual Solution

$$y_1 = 1, y_2 = 5/8$$

$$y_3 = 9/8, \text{value} = 13$$

$$
\begin{aligned}
& 5x_1 + 6x_2 + x_3 \\
\geq\ & (2x_1 + 5x_2 - 3x_3) + \frac{5}{8}(3x_1 - 2x_2 + x_3) + \frac{9}{8}(x_1 + 2x_2 + 3x_3) \\
\geq\ & 2 + \frac{5}{8} \times 5 + \frac{9}{8} \times 7 \\
=\ & 13
\end{aligned}
$$

Outline

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- For some combinatorial optimization problems, a polynomial-sized LP $Ax \leq b$ already defines an integral polytope, whose vertices correspond to valid integral solutions.
- Such a problem can be solved directly using the LP:

$$\max / \min \quad c^{\mathrm{T}} x \quad Ax \leq b.$$

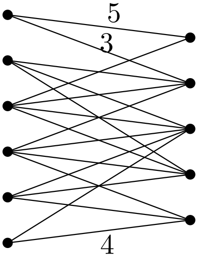
Outline

Maximum Weight Bipartite Matching

Input: bipartite graph $G = (L \uplus R, E)$

edge weights $w \in \mathbb{Z}_{>0}^E$

Output: a matching $M \subseteq E$ so as to maximize $\sum_{e \in M} w_e$



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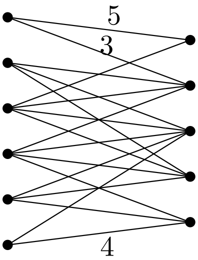

LP Relaxation

$$\max \quad \sum_{e \in E} w_e x_e$$

$$\sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in L \cup R$$

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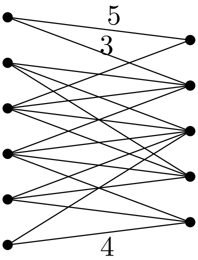

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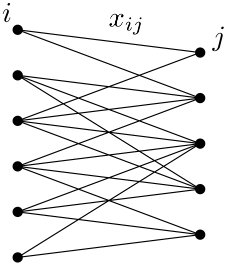

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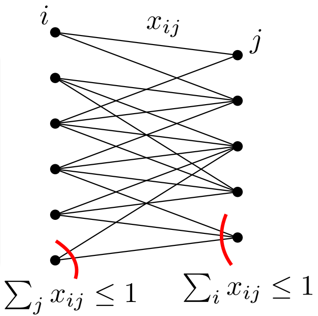

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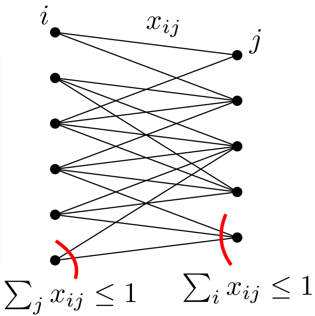

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- $\chi^M \in \{0, 1\}^E$: $\chi^M_e = 1$ iff $e \in M$

Theorem The LP polytope is integral: It is the convex hull of $\{\chi^M : M \text{ is a matching}\}$.

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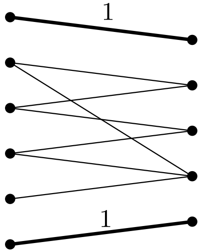

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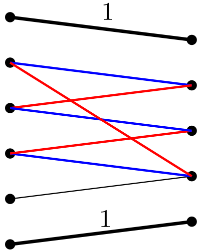

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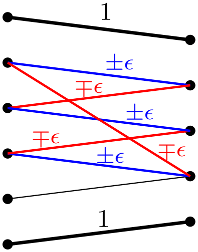

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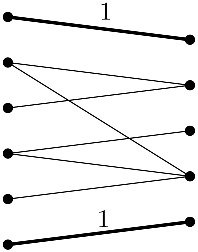

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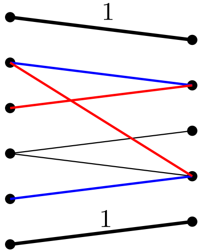

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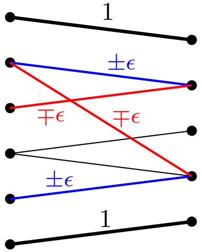
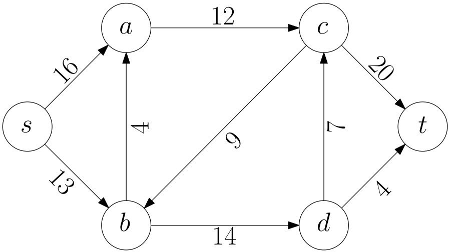


Outline

Example: s-t Flow Polytope

Flow Network

- directed graph $G = (V, E)$, source $s \in V$, sink $t \in V$, edge capacities $c_e \in \mathbb{Z}_{>0}, \forall e \in E$
  - s has no incoming edges, t has no outgoing edges



Def. A s-t flow is a vector $f \in \mathbb{R}^E_{\geq 0}$ satisfying the following conditions:

- $\forall e \in E, 0 \leq f(e) \leq c_e$ (capacity constraints)
- $\forall v \in V \setminus \{s, t\}$,

$$\sum_{e \in \delta^{\text{in}}(v)} f(e) = \sum_{e \in \delta^{\text{out}}(v)} f(e) \qquad \text{(flow conservation)}$$

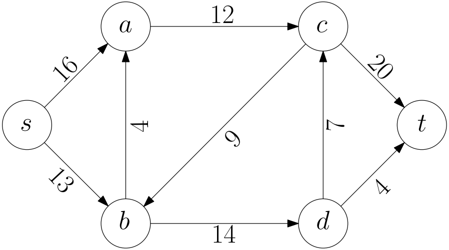
The value of flow f is defined as:

$$\text{val}(f) := \sum_{e \in \delta^{\text{out}}(s)} f(e) = \sum_{e \in \delta^{\text{in}}(t)} f(e)$$

Maximum Flow Problem

Input: flow network $(G = (V, E), c, s, t)$

Output: maximum value of a s-t flow f



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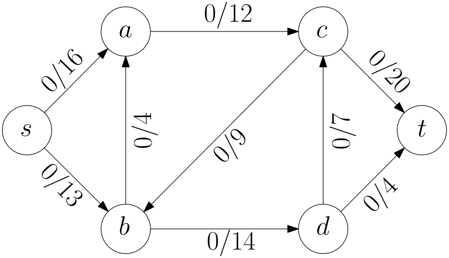

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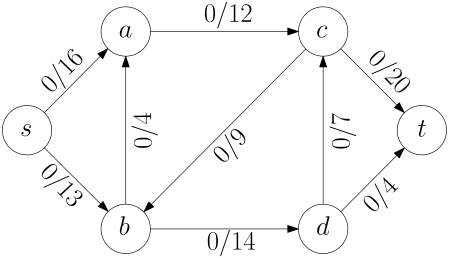

- Ford-Fulkerson method

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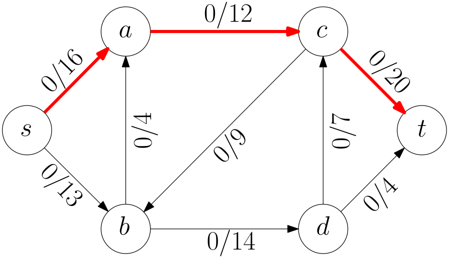

- Ford-Fulkerson method
- Maximum-Flow Min-Cut Theorem: value of the maximum flow is equal to the value of the minimum s-t cut

LP for Maximum Flow

$$\max \quad \sum_{e \in \delta_{\text{in}}(t)} x_e$$

$$x_e \le c_e \qquad \forall e \in E$$

$$\sum_{e \in \delta_{\text{out}}(v)} x_e - \sum_{e \in \delta_{\text{in}}(v)} x_e = 0 \qquad \forall v \in V \setminus \{s, t\}$$

$$x_e \ge 0 \qquad \forall e \in E$$

LP for Maximum Flow

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$$\begin{aligned} x_e &\le c_e && \forall e \in E \\ \sum_{e\in\delta_{\text{out}}(v)} x_e - \sum_{e\in\delta_{\text{in}}(v)} x_e &= 0 && \forall v \in V \setminus \{s,t\} \\ x_e &\ge 0 && \forall e \in E \end{aligned}$$

Theorem The LP polytope is integral.

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Theorem The LP polytope is integral.

Sketch of Proof.

- Take any s-t flow x; consider fractional edges E'
- Every $v \notin \{s, t\}$ must be incident to 0 or ≥ 2 edges in E'
- Ignoring the directions of E', it contains a cycle, or a s-t path
- We can increase/decrease flow values along cyle/path □

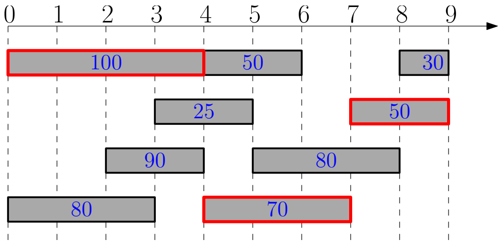
Outline

Weighted Interval Scheduling Problem

Input: n activities, activity i starts at time s_i, finishes at time f_i, and has weight $w_i > 0$

i and j can be scheduled together iff $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: maximum weight subset of jobs that can be scheduled



- optimum value$= 220$

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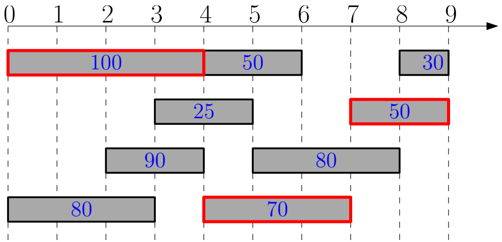

- optimum value$= 220$
- Classic Problem for Dynamic Programming

Weighted Interval Scheduling Problem

Linear Program

$$\max \quad \sum_{j \in [n]} x_j w_j$$

$$\sum_{j \in [n] : t \in [s_j, f_j)} x_j \leq 1 \qquad \forall t \in [T]$$

$$x_j \geq 0 \qquad \forall j \in [n]$$

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Weighted Interval Scheduling Problem

Linear Program

$$\max \quad \sum_{j\in[n]} x_j w_j$$

$$\sum_{j\in[n]:t\in[s_j,f_j)} x_j \leq 1 \qquad \forall t \in [T]$$

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- Every vertex $x \in P$ is the unique solution to the linear system (after permuting coordinates): $\begin{pmatrix} A' & 0 \\ 0 & I \end{pmatrix} x = \begin{pmatrix} b' \\ 0 \end{pmatrix}$, where
  - A' is a square submatrix of A with $\det(A') = \pm 1$, b' is a sub-vector of b,
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- Let $x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$, so that $A'x^1 = b'$ and $x^2 = 0$.
- Cramer's rule: $x_i^1 = \frac{\det(A_i'|b)}{\det(A')}$ for every $i \implies x_i^1$ is integer

  $A_i'|b$: the matrix of A' with the i-th column replaced by b □

Example for the Proof

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \geq \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

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The following equation system may give a vertex:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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Equivalently, the vertex satisfies

$$\begin{pmatrix} a_{1,2} & a_{1,3} & 0 & 0 & 0 \\ a_{3,2} & a_{3,3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_1 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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Coro. The matrix for s-t flow polytope is TUM; thus, the polytope is integral.

Example for the Proof

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

Example for the Proof

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

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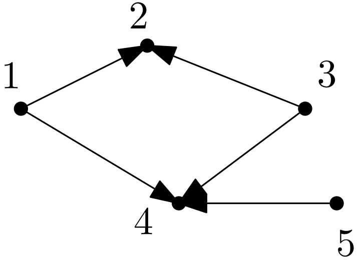

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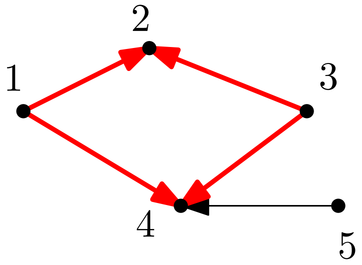

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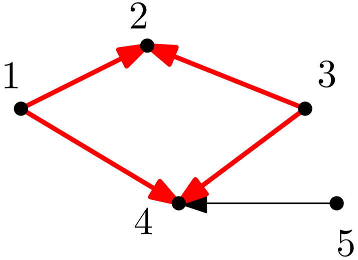

$$\begin{array}{cc} + & (1 \quad -1 \quad 0 \quad 0 \quad 0) \\ - & (0 \quad -1 \quad 1 \quad 0 \quad 0) \\ + & (0 \quad 0 \quad 1 \quad -1 \quad 0) \\ - & (1 \quad 0 \quad 0 \quad -1 \quad 0) \\ \\ = & (0 \quad 0 \quad 0 \quad 0 \quad 0) \end{array}$$

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- $A'M$ is a matrix satisfying condition of first lemma, where

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}. \ \det(M) = 1.$$

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- $\text{det}(A'M) \in \{0, \pm 1\} \implies \det(A') \in \{0, \pm 1\}$. □

Example for the Proof

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

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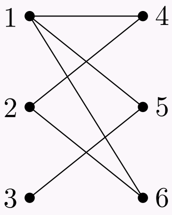

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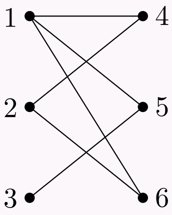

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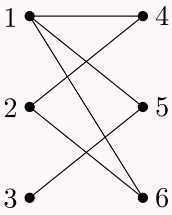

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

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Coro. Bipartite matching polytope is integral.

In summary, given a matrix $A \in \{-1, 0, 1\}^{m \times n}$, A is TUM if one of the conditions hold:

- every row of A has at most one 1 and at most one -1 (network flow polytope)
- $A \in \{0, 1\}^{m \times n}$, and the 1's in every row form an interval (interval scheduling polytope)
- A is edge-vertex incidence matrix of a bipartite graph (bipartite matching polytope)

- $G = (L \uplus R, E)$: bipartite graph
- $\mathsf{MM}(G)$: the size of the maximum matching of G
- $\mathsf{MVC}(G)$: the size of the minimum vertex cover of G
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- Using MFMC theorem, we know $\mathsf{MM}(G) = \mathsf{MVC}(G)$
- A new proof using LP duality:

LP for MM

$$
\begin{aligned}
\max \quad & \sum_{e \in E} x_e & \\
\sum_{e \in \delta(v)} x_e \leq 1 & \quad \forall v \in L \uplus R \\
x_e \geq 0 & \quad \forall e \in E
\end{aligned}
$$

LP for MVC

$$
\begin{aligned}
\min \quad & \sum_{v \in L \uplus R} y_v & \\
y_u + y_v \geq 1 & \quad \forall (u, v) \in E \\
\alpha_u \geq 0 & \quad \forall u \in L \uplus R
\end{aligned}
$$

- Both LP polytopes are integral
- $\mathsf{MM}(G) = \text{primal value} = \text{dual value} = \mathsf{MVC}(G)$