算法设计与分析(2025年春季学期)

Network Flow

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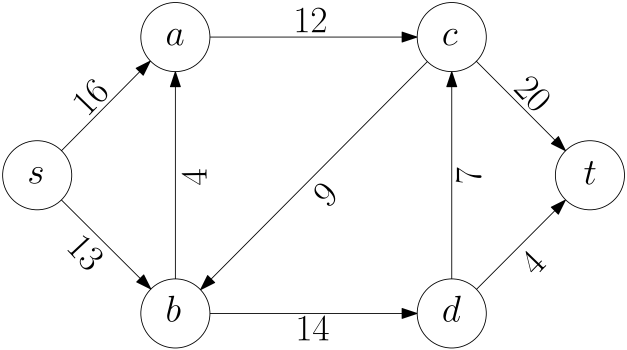
Outline

1. Network Flow
2. Ford-Fulkerson Method
3. Correctness of Ford-Fulkerson's Method and Maximum Flow Minimum Cut Theorem
4. Running Time of Ford-Fulkerson-Type Algorithm
   - Shortest Augmenting Path Algorithm
   - Capacity-Scaling Algorithm
5. Bipartite Matching Problem
6. s-t Edge-Disjoint Paths Problem
7. More Applications

Flow Network

- Abstraction of fluid flowing through edges
- Digraph $G = (V, E)$ with source $s \in V$ and sink $t \in V$
  - No edges enter s
  - No edges leave t
- Edge capacity $c_e \in \mathbb{R}_{>0}$ for every $e \in E$



Def. An s-t flow is a function $f : E \to \mathbb{R}$ such that

- for every $e \in E$: $0 \leq f(e) \leq c_e$ (capacity conditions)
- for every $v \in V \setminus \{s, t\}$:

$$\sum_{e \in \delta_{\text{in}}(v)} f(e) = \sum_{e \in \delta_{\text{out}}(v)} f(e). \qquad \text{(conservation conditions)}$$

The value of a flow f is

$$\text{val}(f) := \sum_{e \in \delta_{\text{out}}(s)} f(e).$$

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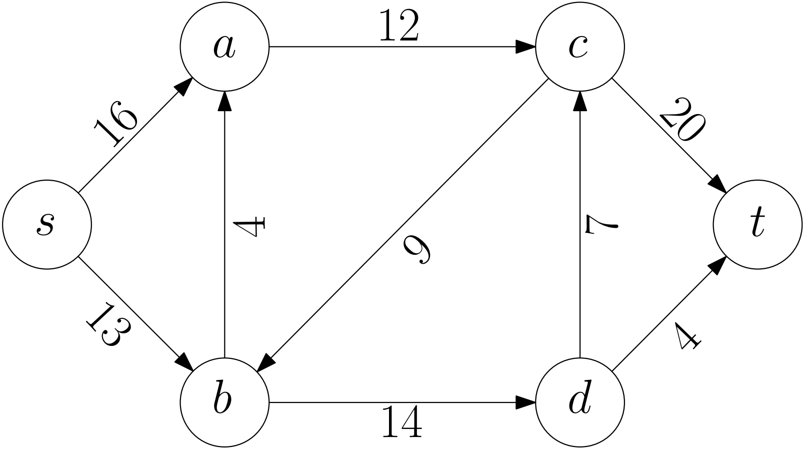
Maximum Flow Problem

Input: directed network $G = (V, E)$, capacity function $c : E \to \mathbb{R}_{>0}$, source $s \in V$ and sink $t \in V$

Output: an s-t flow f in G with the maximum $\text{val}(f)$

Maximum Flow Problem: Example



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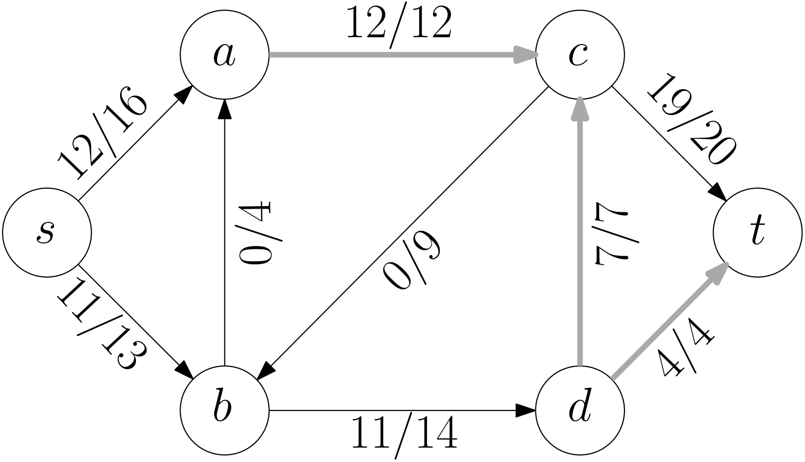

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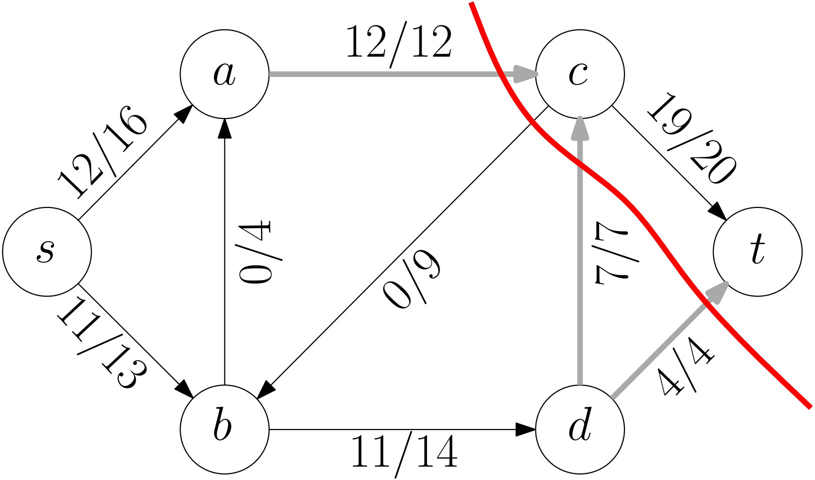

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- Repeat until we got stuck

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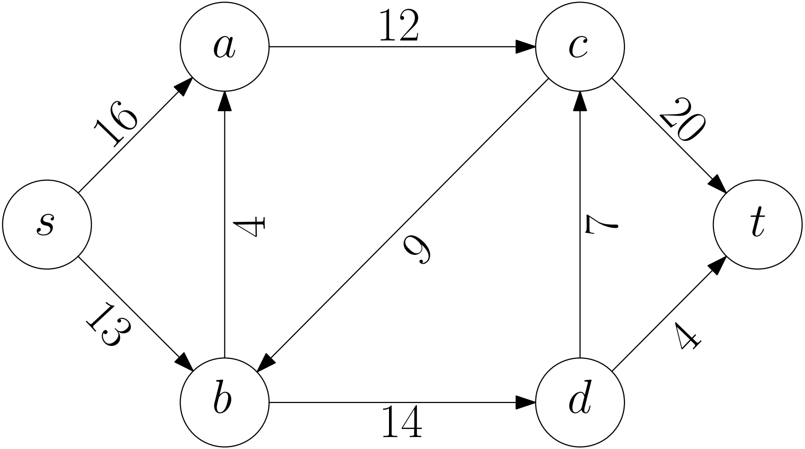

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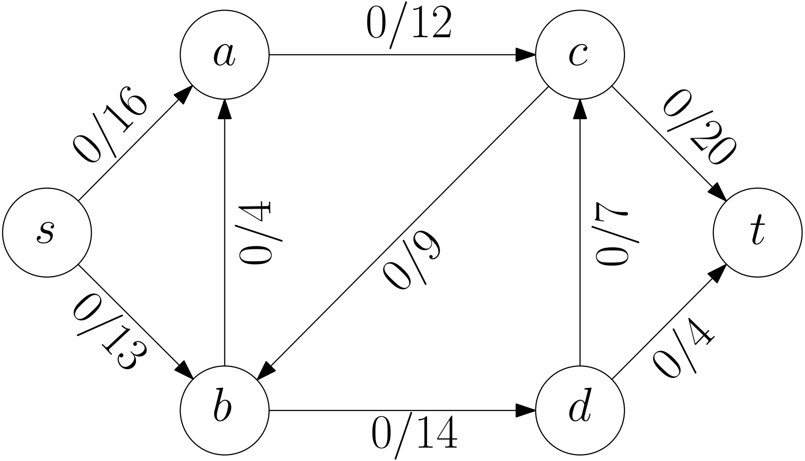

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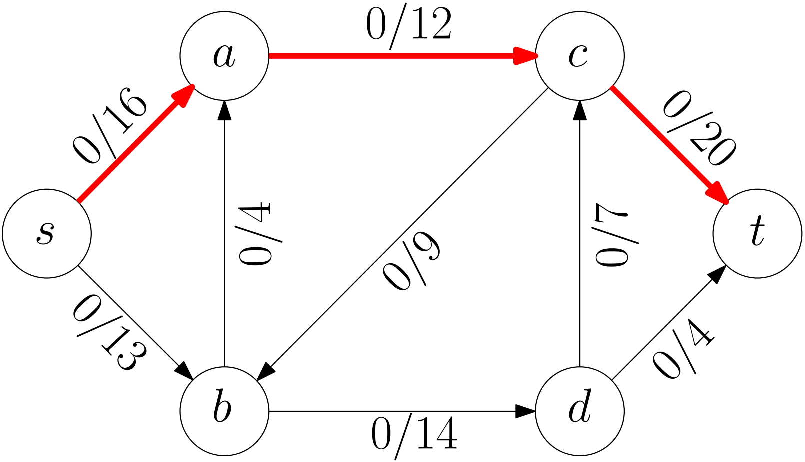

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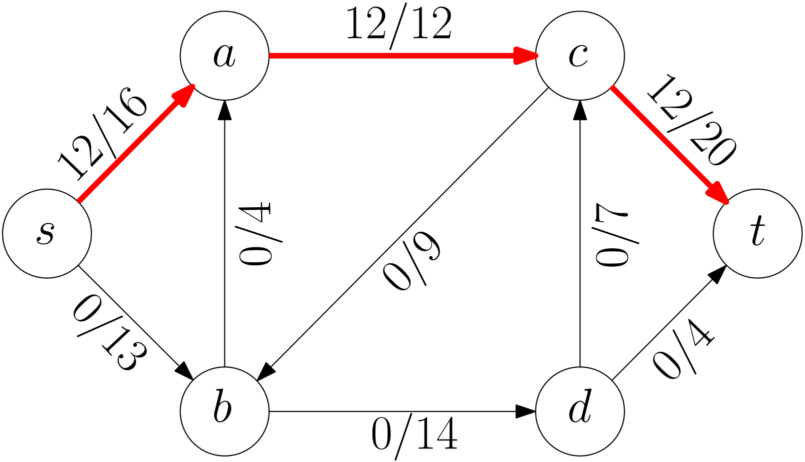

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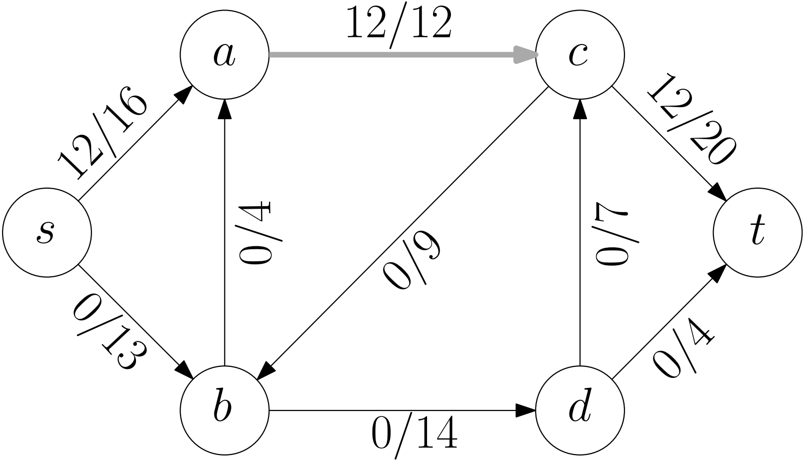

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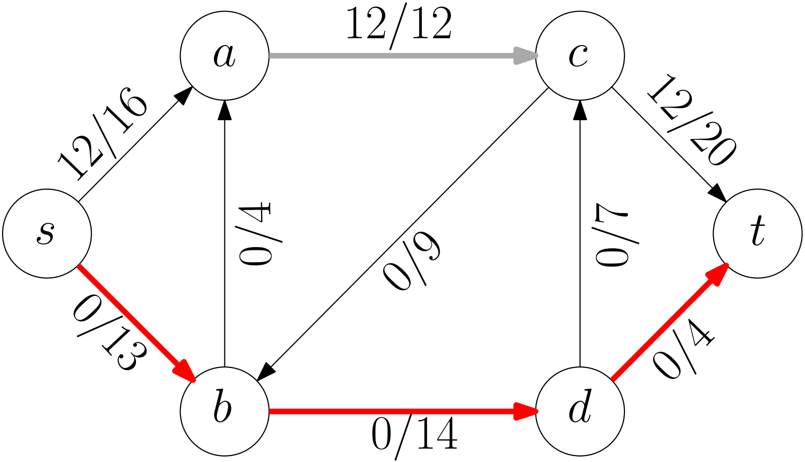

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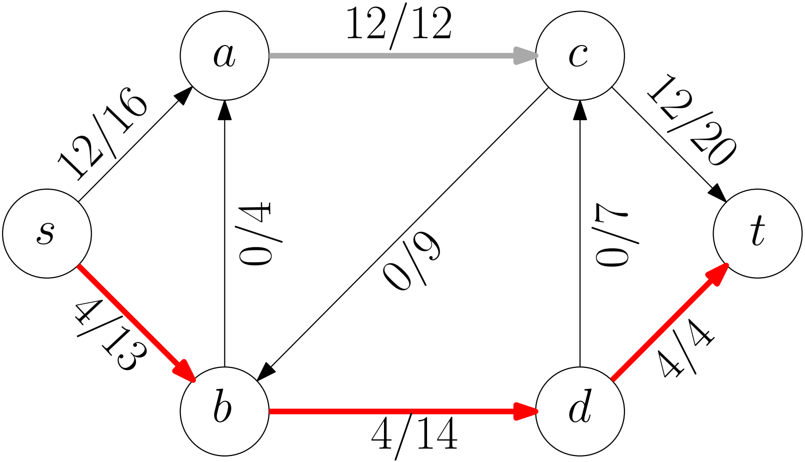

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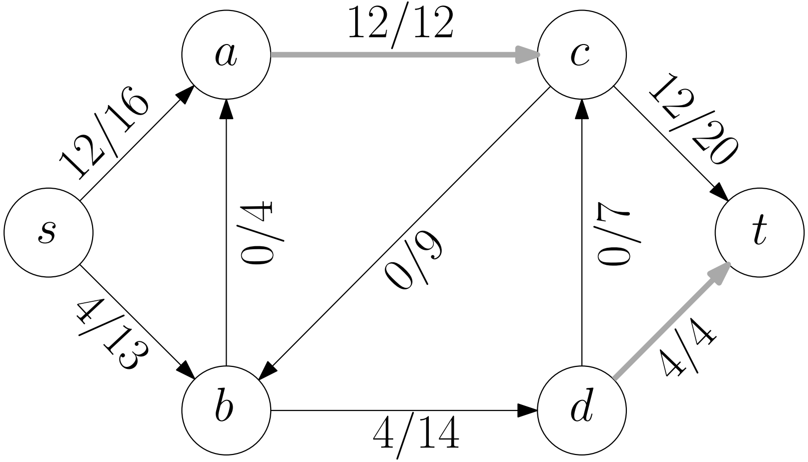

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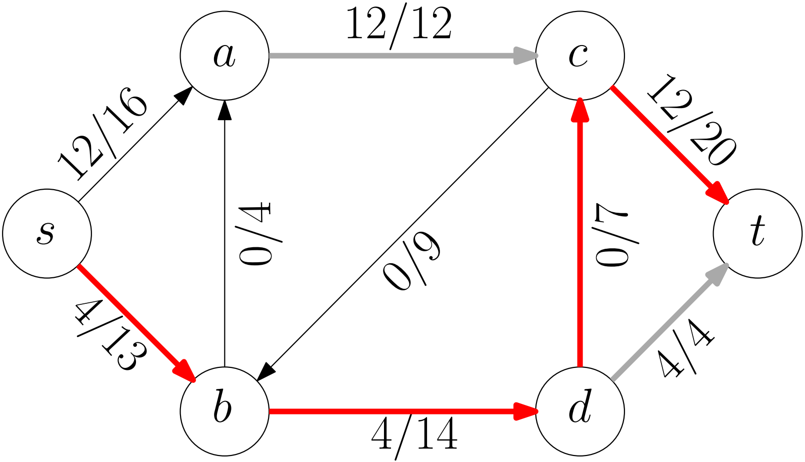

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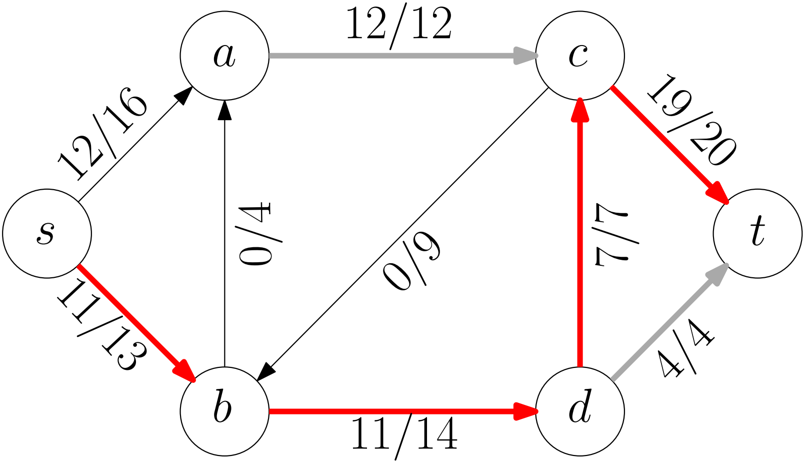

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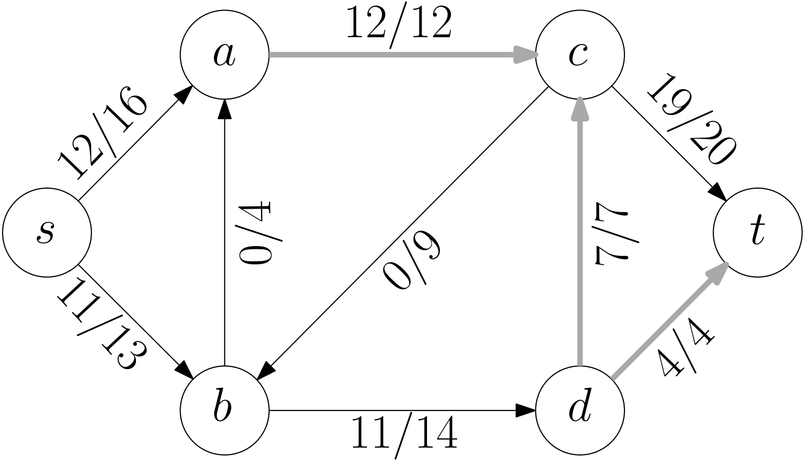

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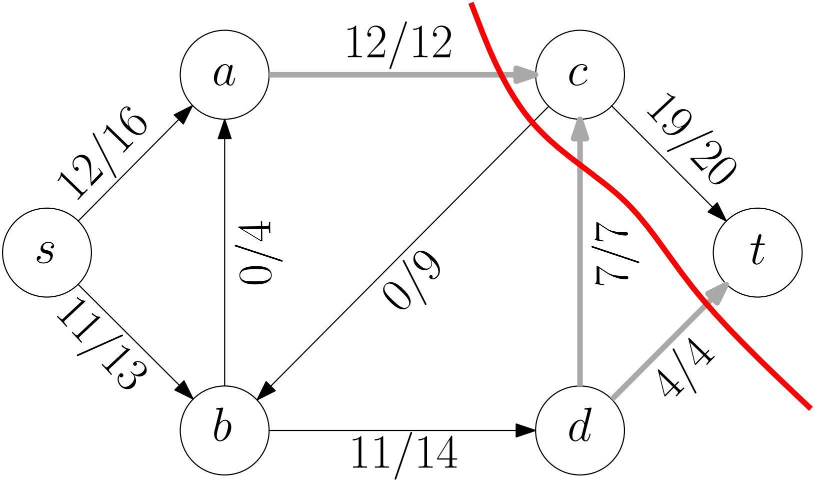

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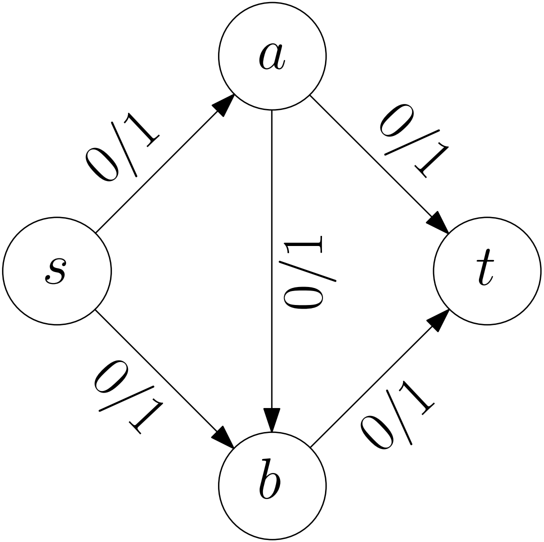

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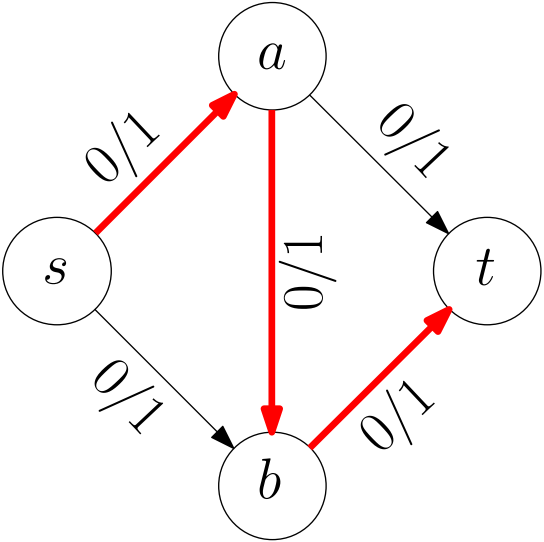

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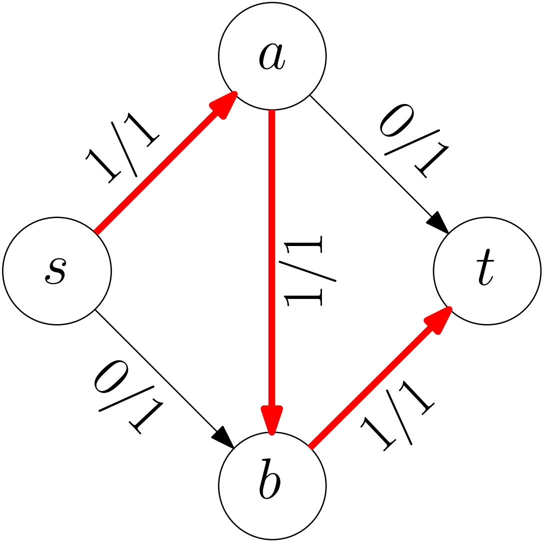

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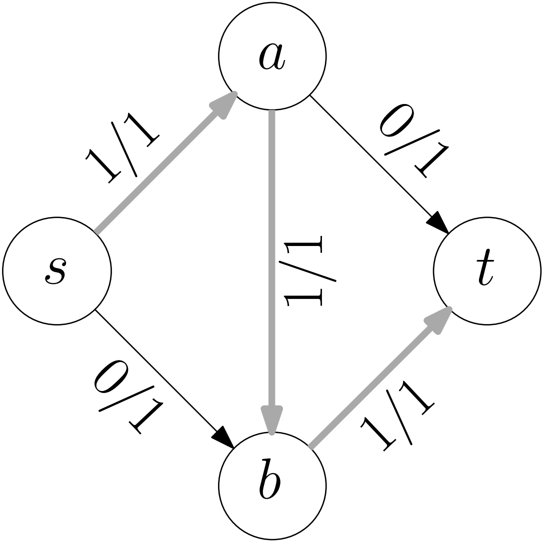

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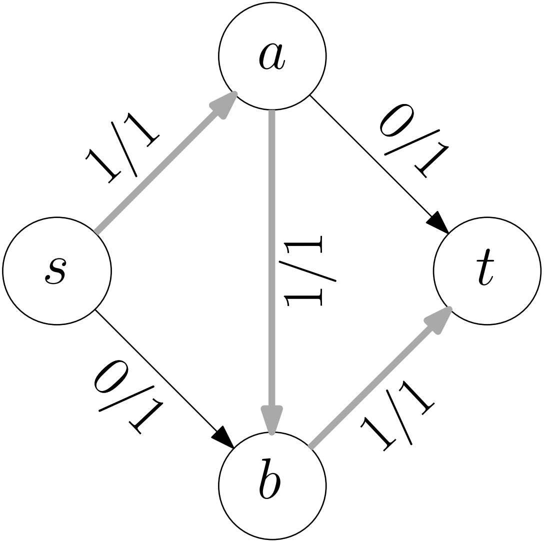

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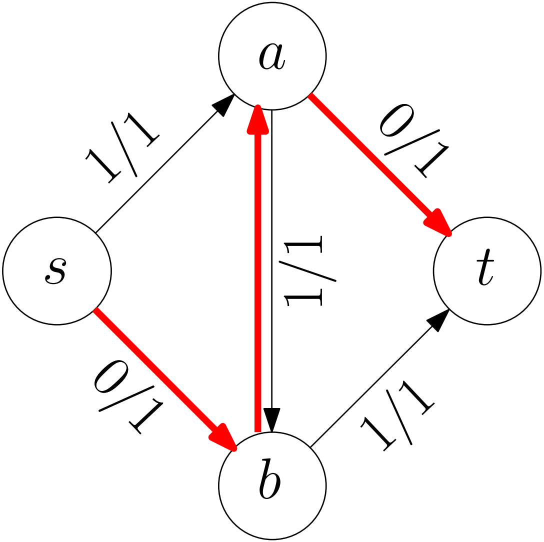

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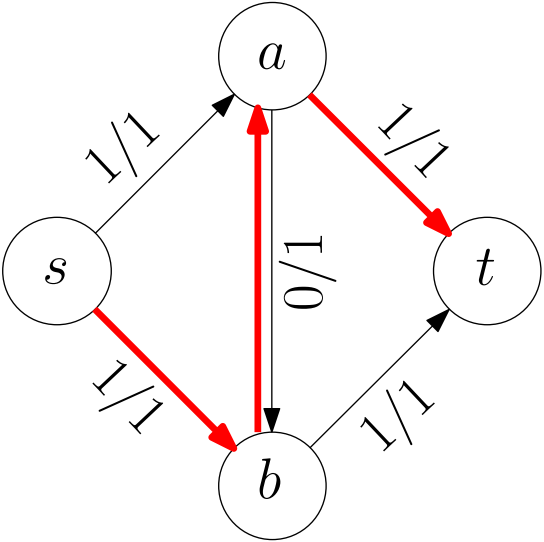

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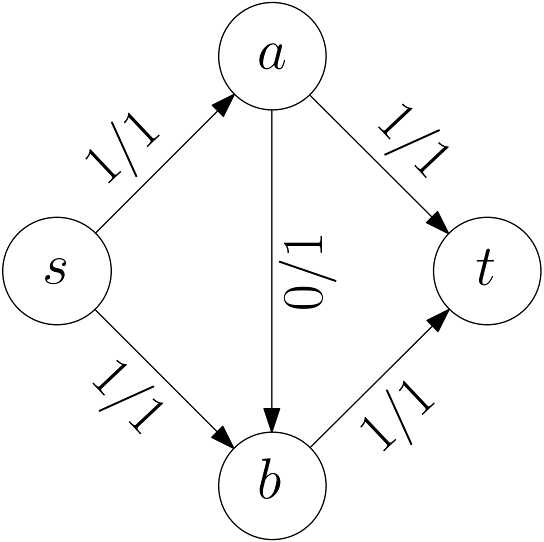

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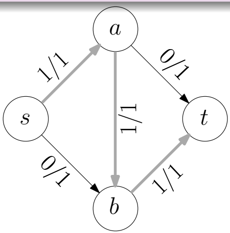
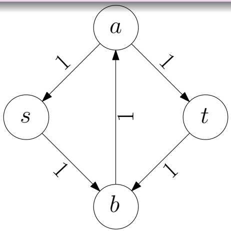
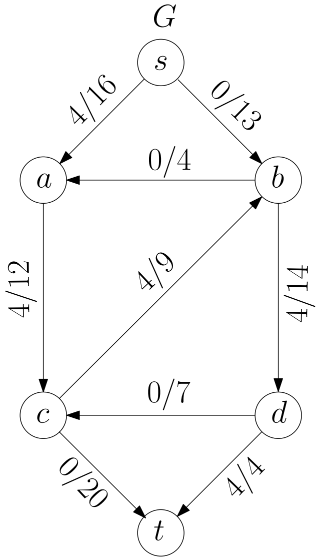


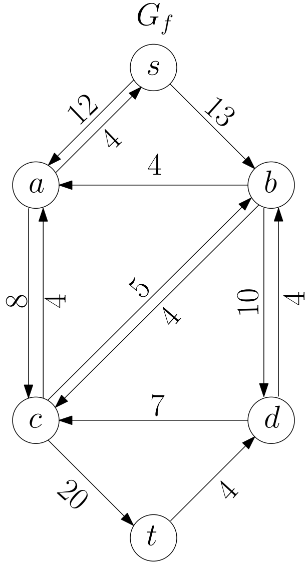
Original graph G and f



Residual Graph G_f

Residual Graph: One More Example





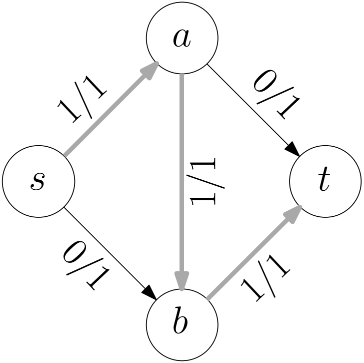
Agumenting Path

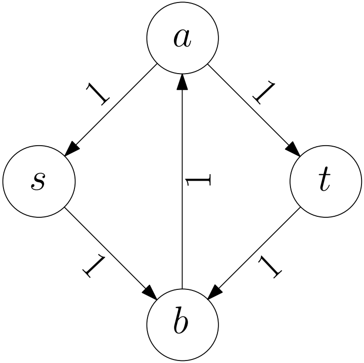
Augmenting the flow along a path P from s to t in G_f

Augment(P)

```
b ← $\min_{e \in P} c_f(e)$
for every $(u,v) \in P$ do
    if $(u,v)$ is a forward edge then
        $f(u,v) \leftarrow f(u,v) + b$
    else                                        ▷ $(u,v)$ is a backward edge
        $f(v,u) \leftarrow f(v,u) - b$
return $f$
```

Example for Augmenting Along a Path





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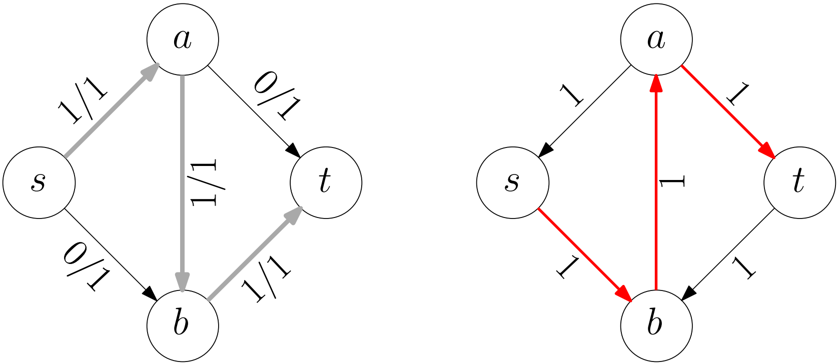

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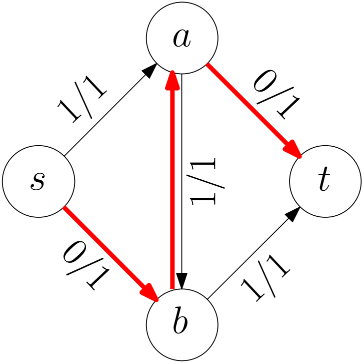

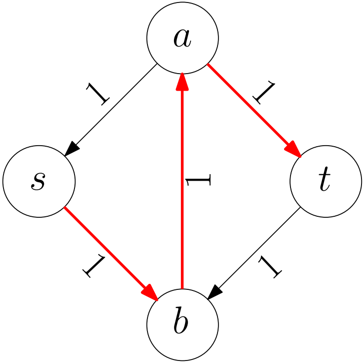

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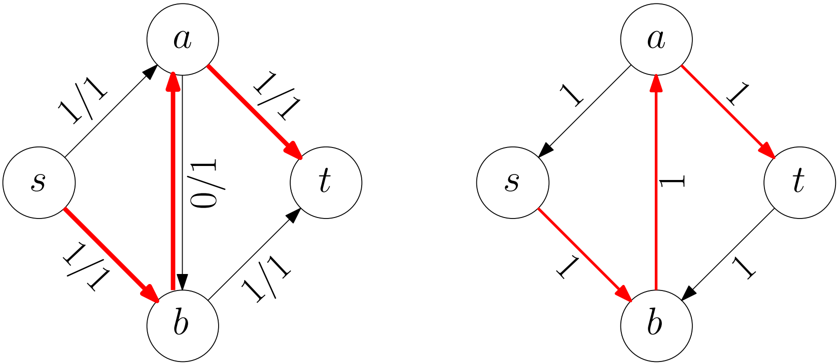

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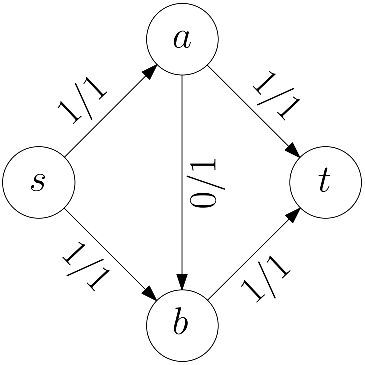
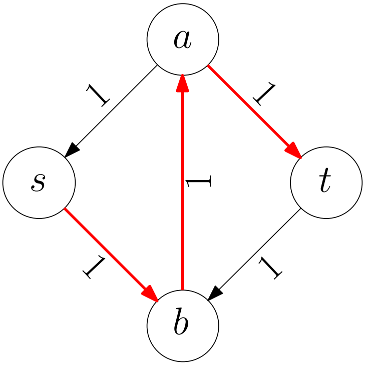


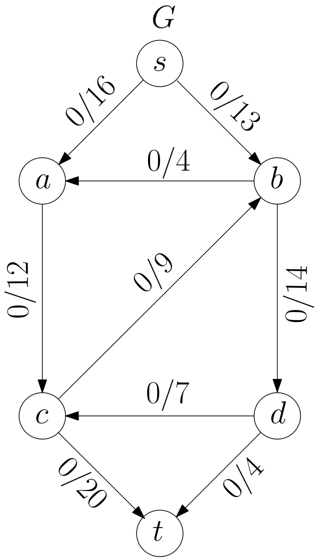


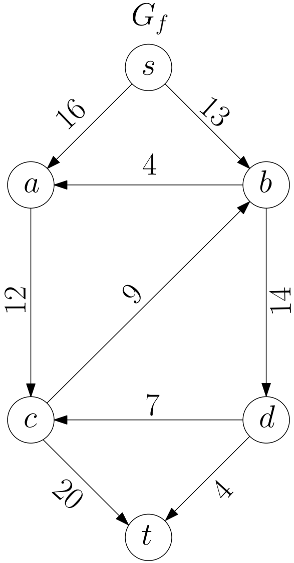
Ford-Fulkerson's Method

Ford-Fulkerson(G, s, t, c)

```
let $f(e) \leftarrow 0$ for every $e$ in $G$
while there is a path from $s$ to $t$ in $G_f$ do
    let $P$ be any simple path from $s$ to $t$ in $G_f$
    $f \leftarrow$augment$(f, P)$
return $f$
```

Ford-Fulkerson: Example





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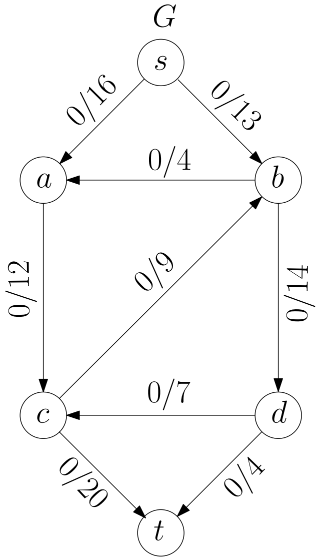

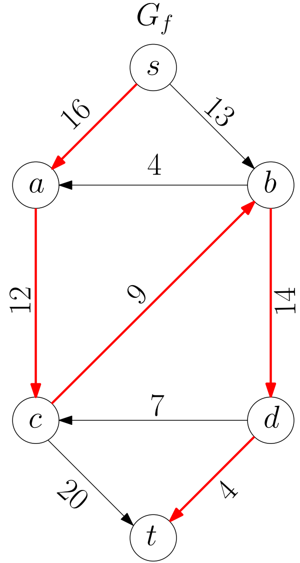

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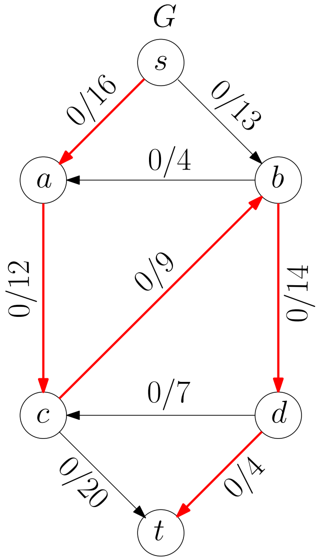

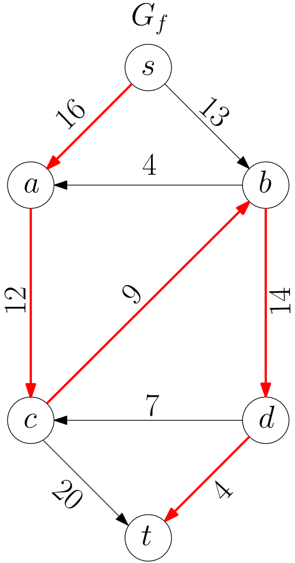

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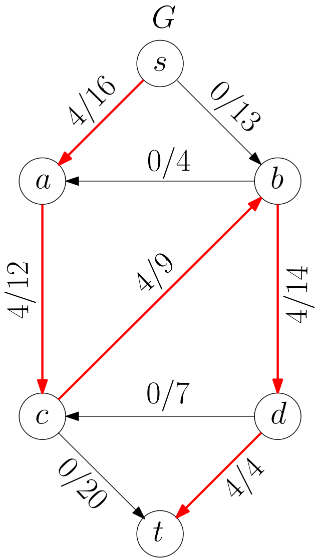

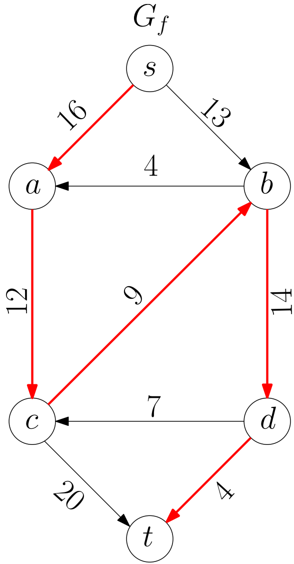

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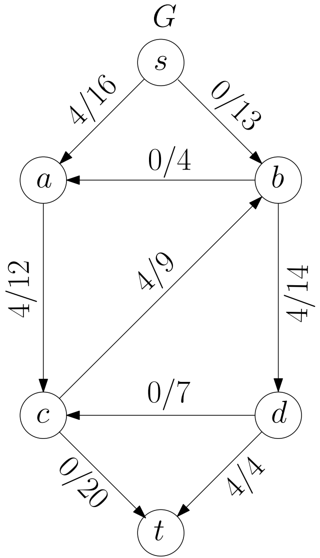

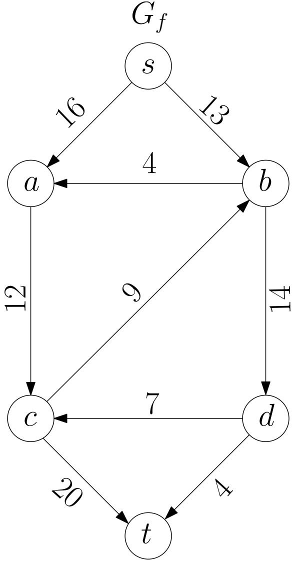

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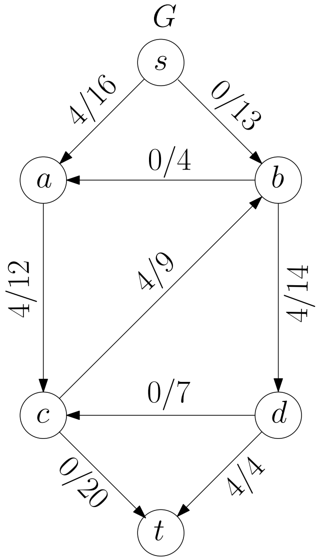

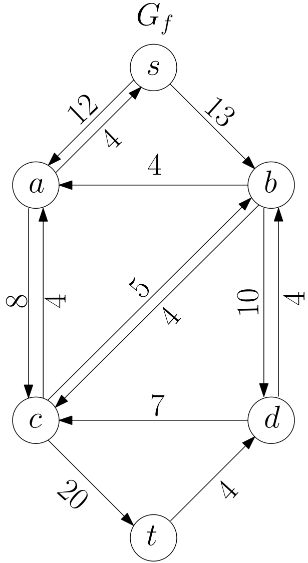

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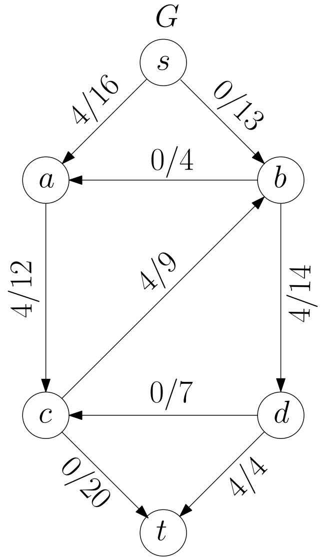

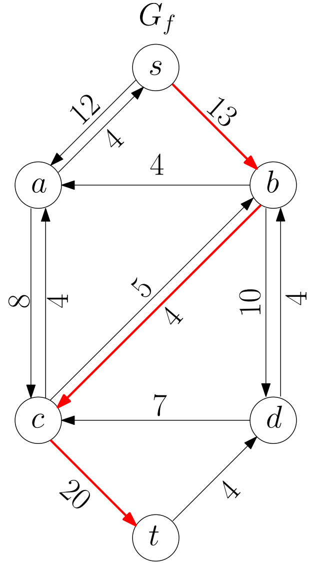

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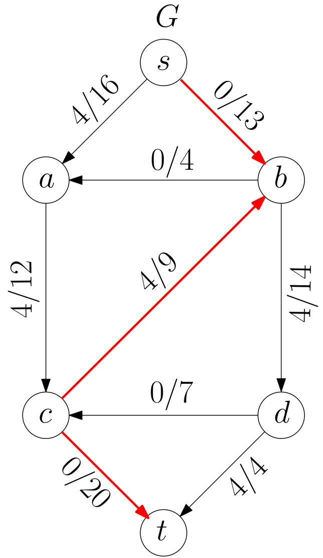

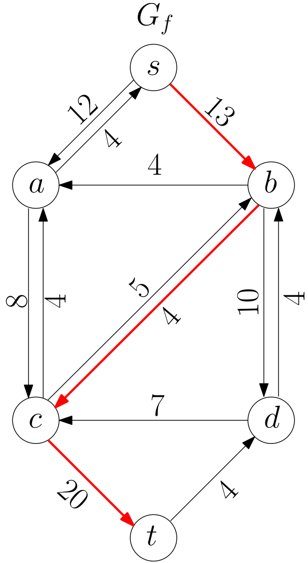

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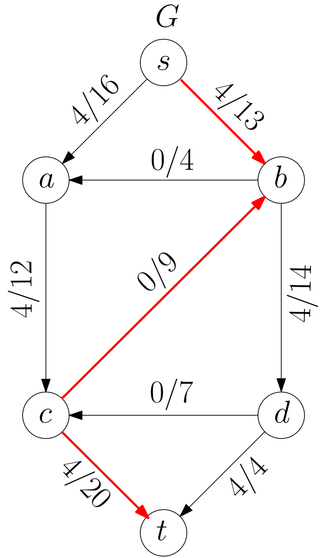

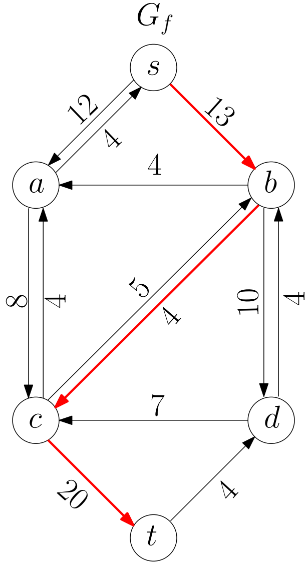

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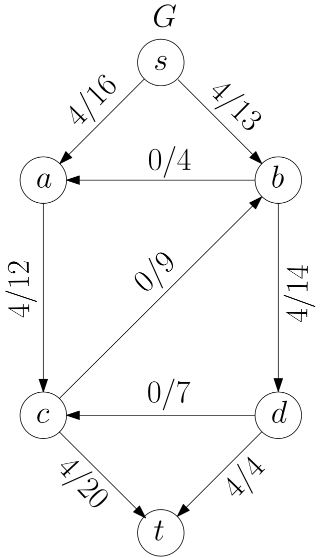

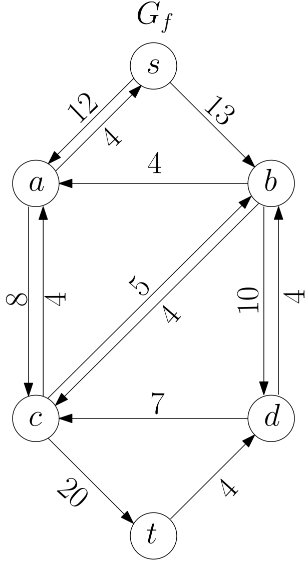

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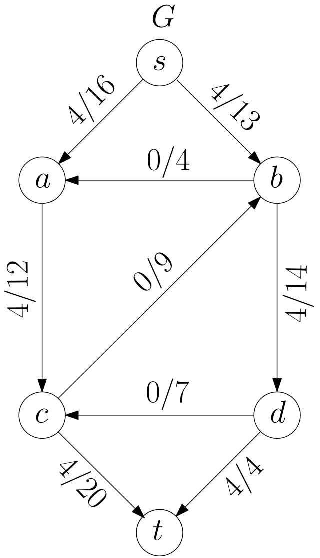

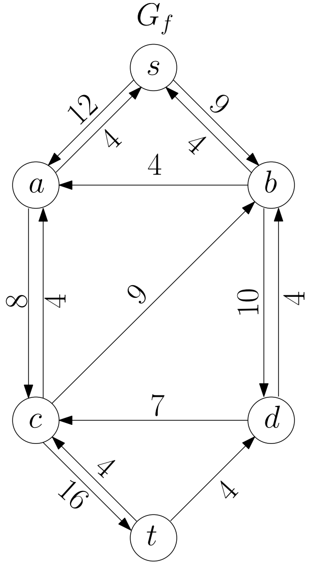

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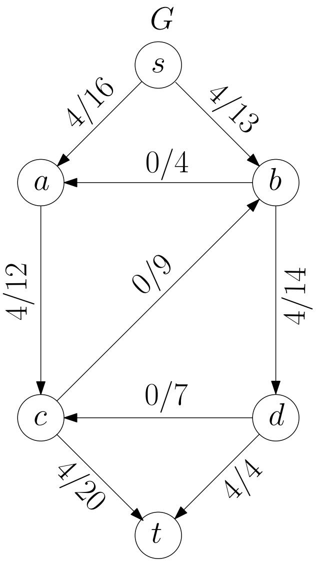

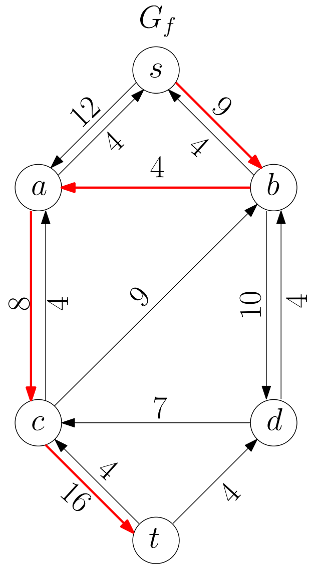

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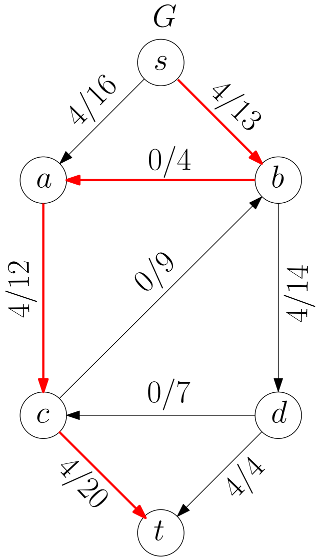

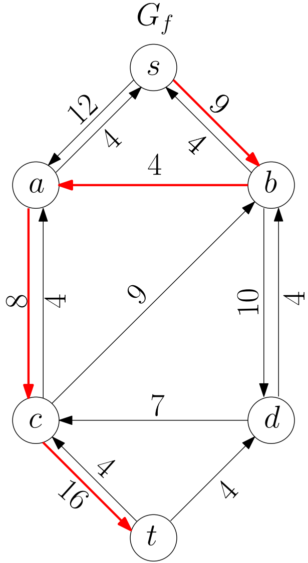

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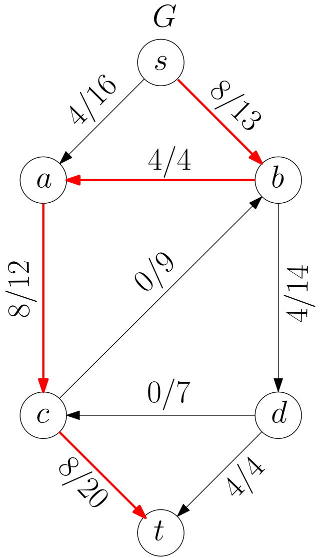

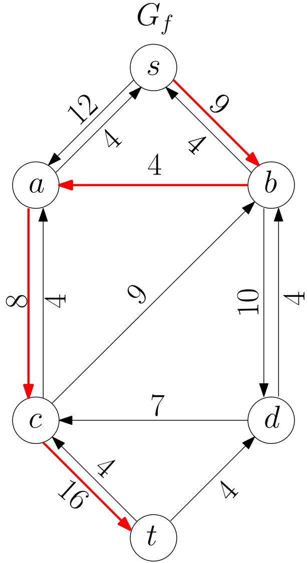

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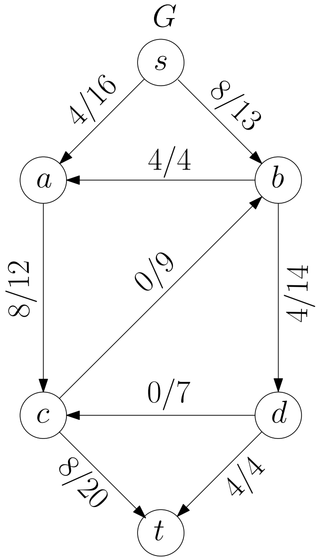

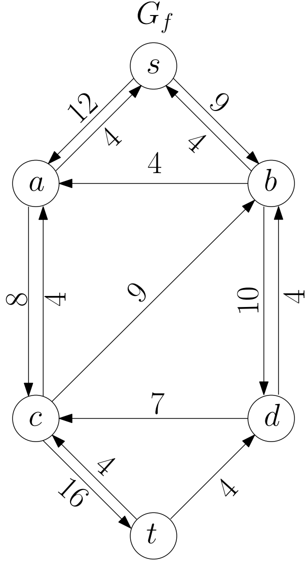

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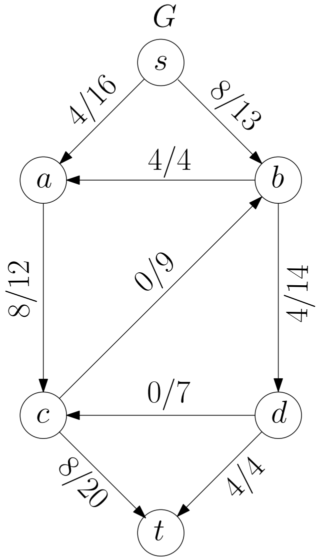

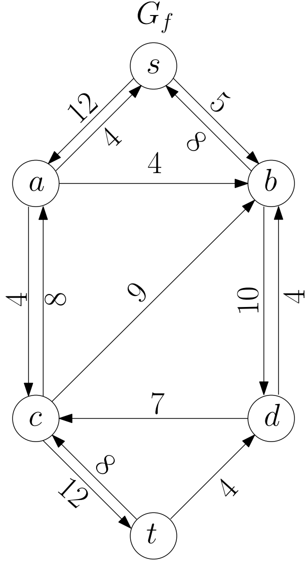

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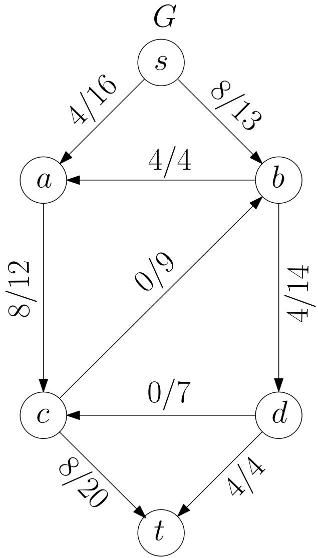

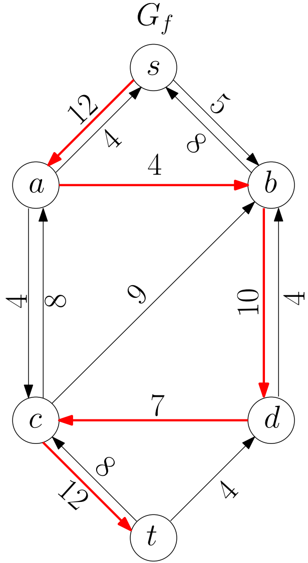

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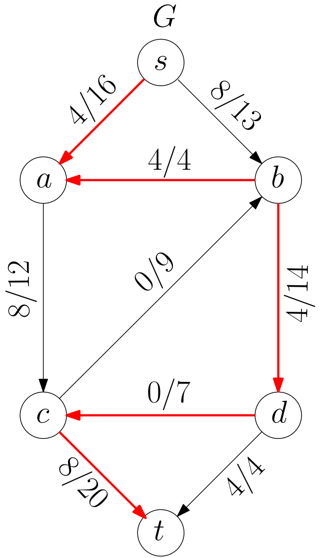

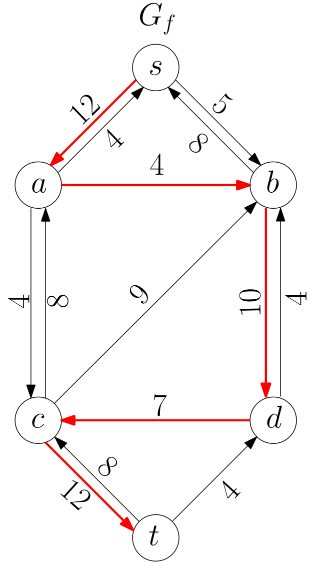

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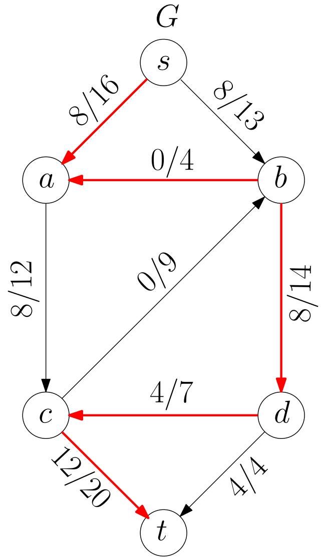

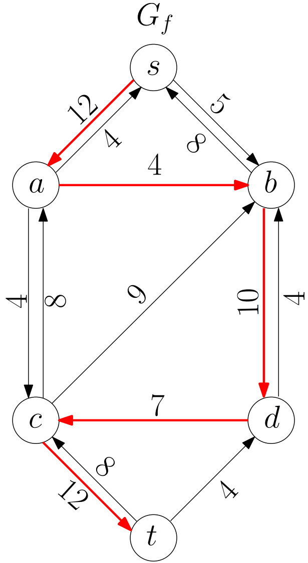

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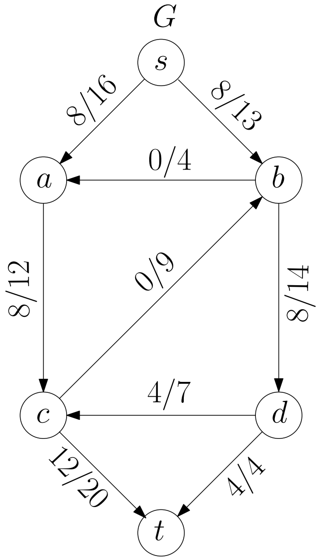

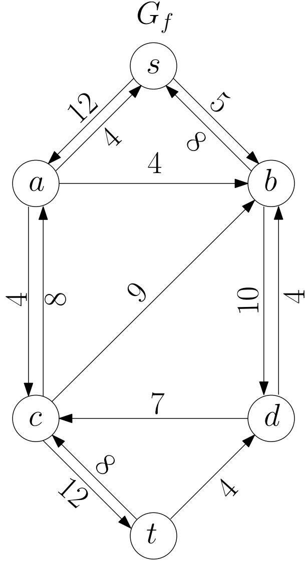

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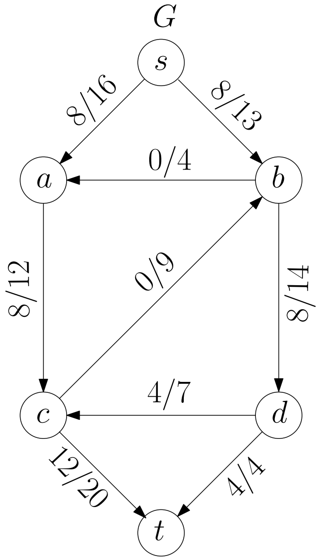

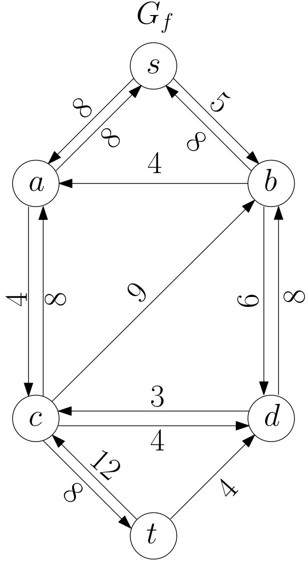

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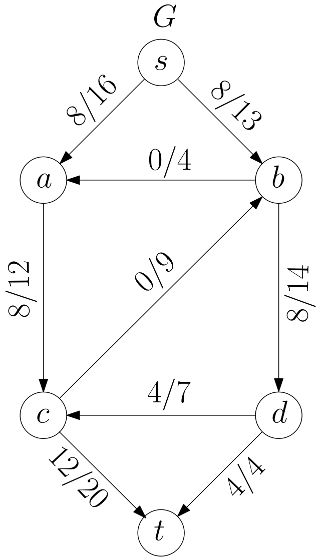

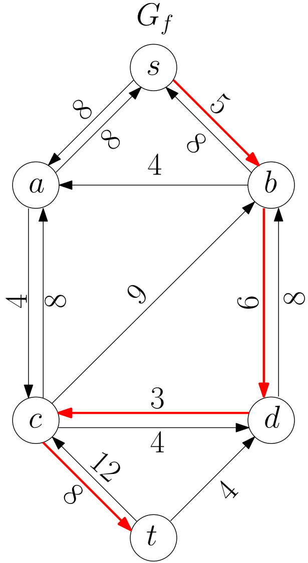

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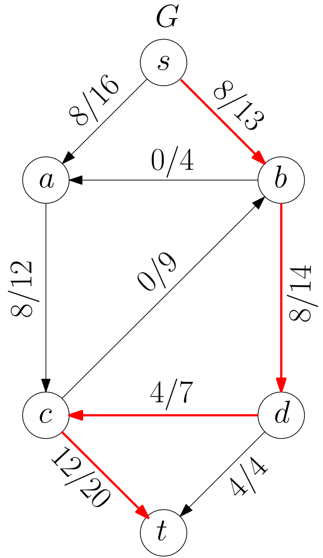

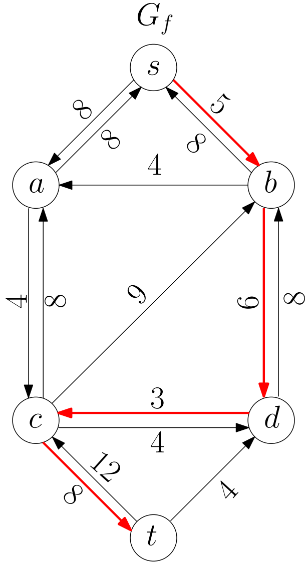

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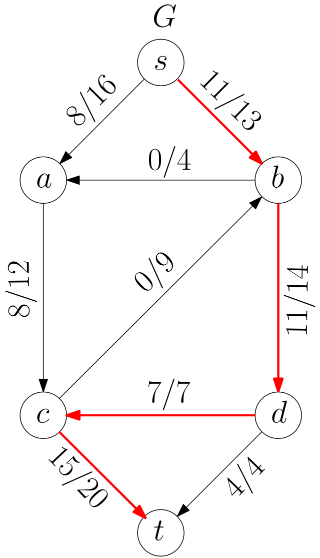

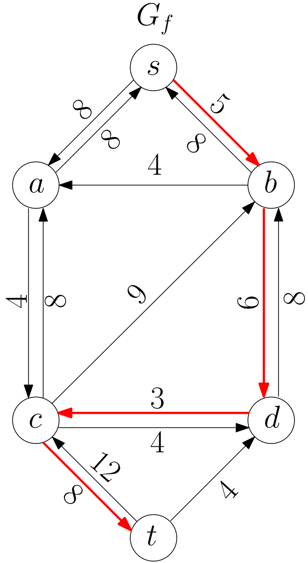

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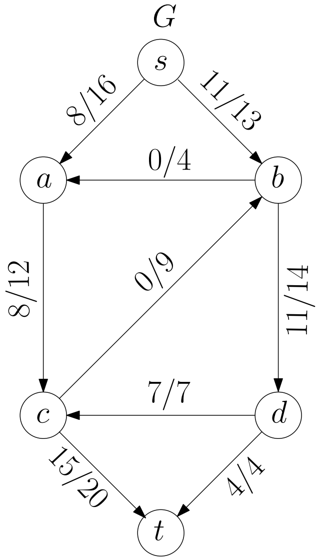

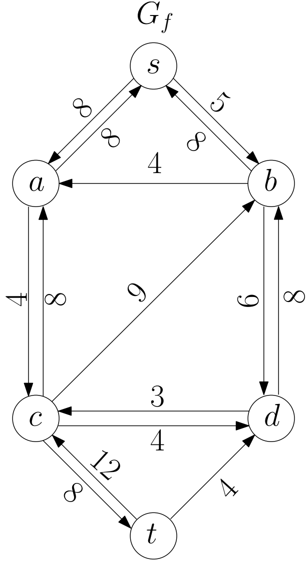

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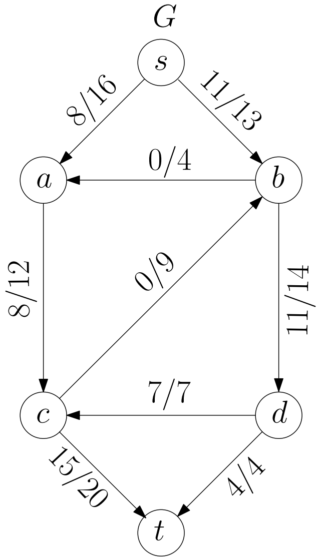

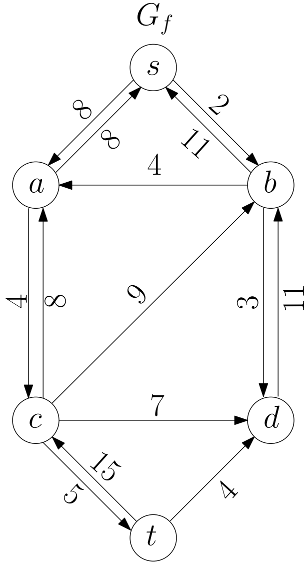

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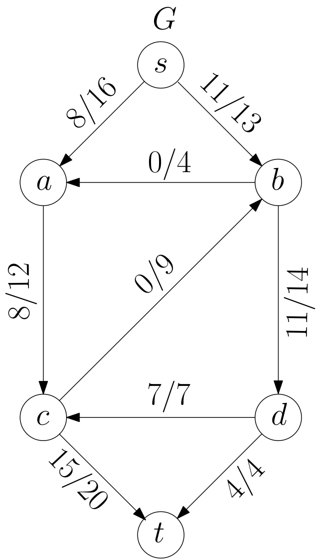

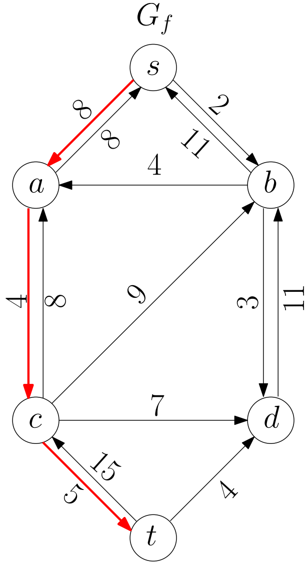

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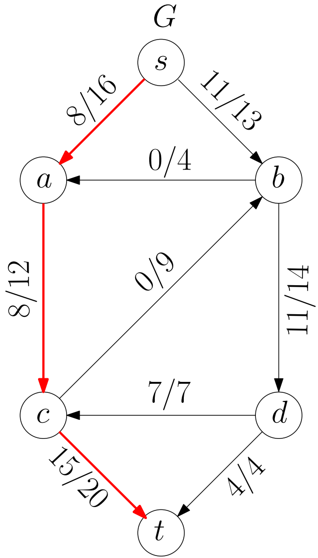

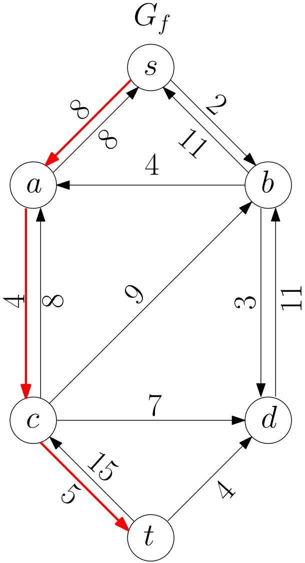

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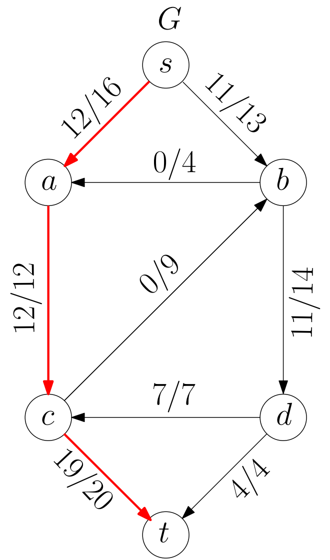

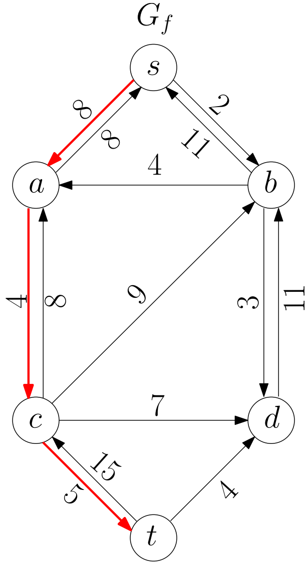

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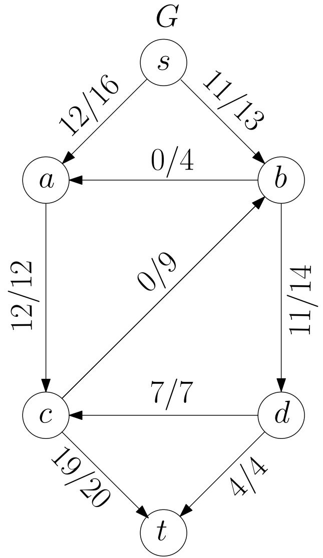

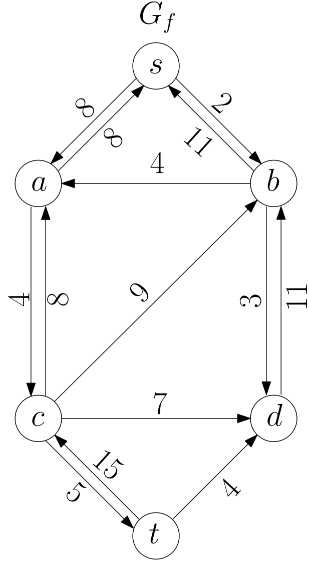

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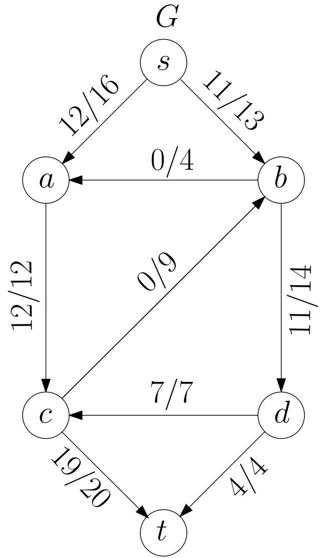

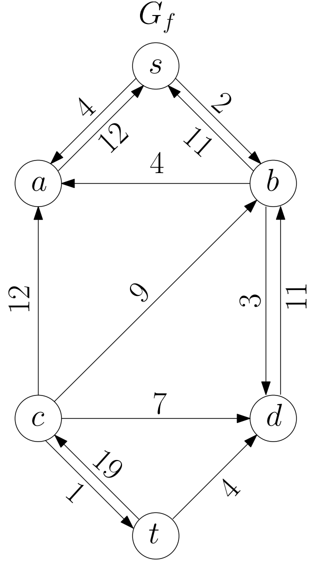

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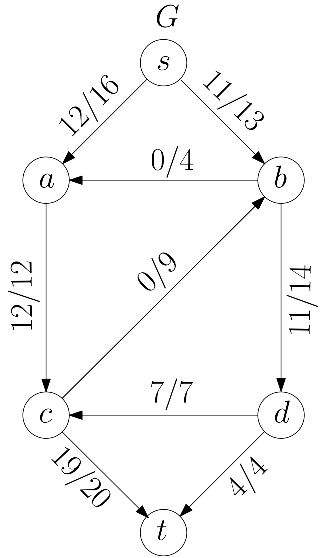

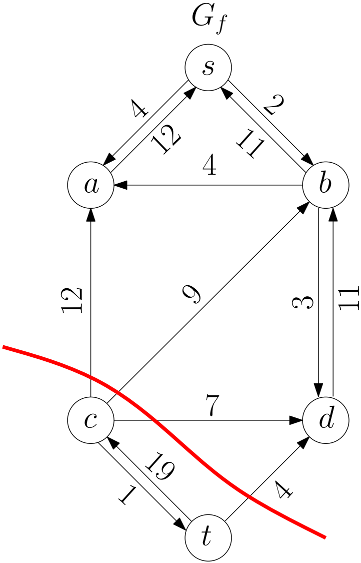

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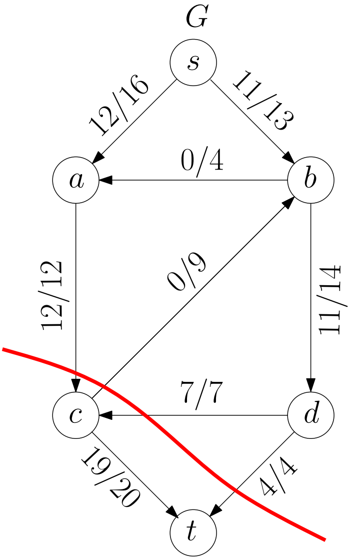

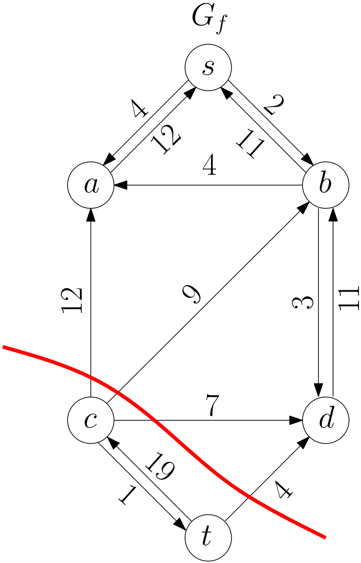

Outline

1. Network Flow
2. Ford-Fulkerson Method
3. Correctness of Ford-Fulkerson's Method and Maximum Flow Minimum Cut Theorem
4. Running Time of Ford-Fulkerson-Type Algorithm
   - Shortest Augmenting Path Algorithm
   - Capacity-Scaling Algorithm
5. Bipartite Matching Problem
6. s-t Edge-Disjoint Paths Problem
7. More Applications

Correctness of Ford-Fulkerson's Method

1. The procedure augment(f, P) maintains the two conditions:
   - for every $e \in E$: $0 \leq f(e) \leq c_e$ (capacity conditions)
   - for every $v \in V \setminus \{s, t\}$:

$$\sum_{e \in \delta_{\text{in}}(v)} f(e) = \sum_{e \in \delta_{\text{out}}(v)} f(e). \qquad \text{(conservation conditions)}$$

2. When Ford-Fulkerson's Method terminates, val(f) is maximized
3. Ford-Fulkerson's Method will terminate

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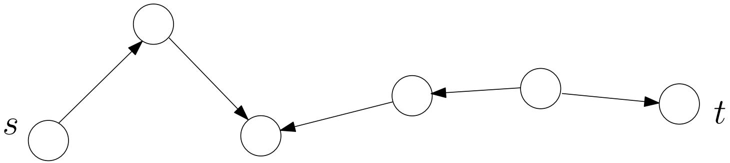

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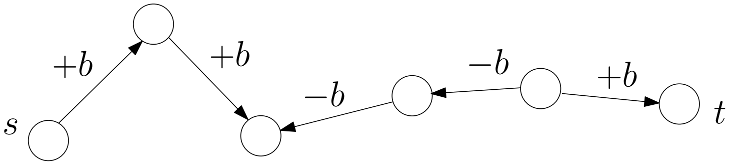

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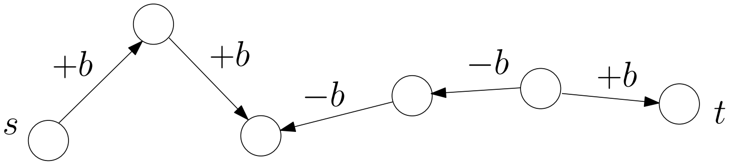

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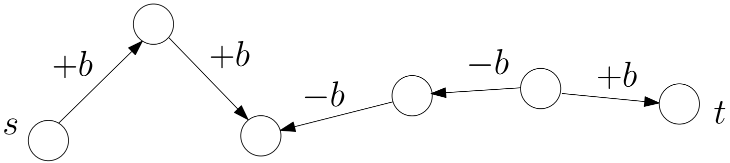

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- for an edge e correspondent to a backward edge :
  $b \leq f(e) \implies f(e) - b \geq 0$

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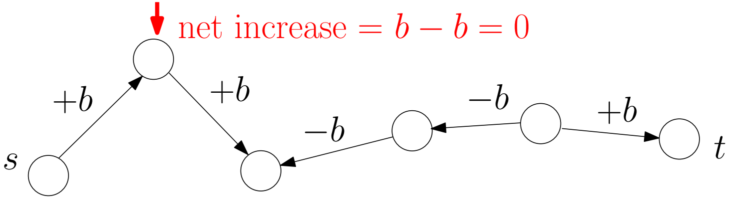

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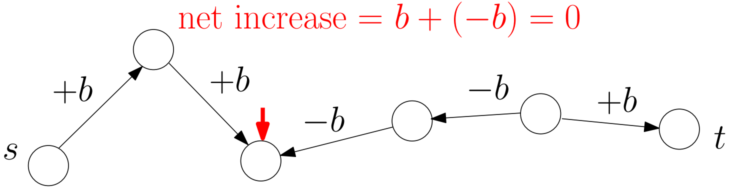

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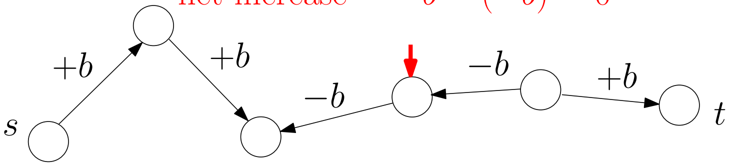
$\text{net increase} = -b - (-b) = 0$



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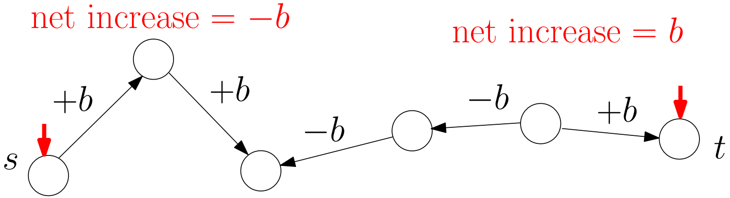

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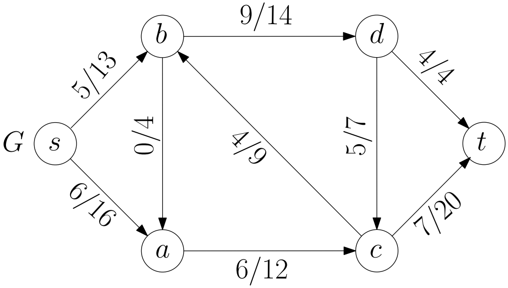
Def. Given an s-t flow f and an s-t cut (S, T), the net flow sent from S to T is

$$f(S, T) := \sum_{e=(u,v) \in E : u \in S, v \in T} f(e) - \sum_{e=(u,v) \in E : u \in T, v \in S} f(e).$$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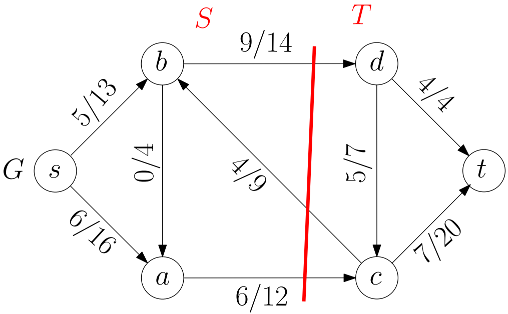S
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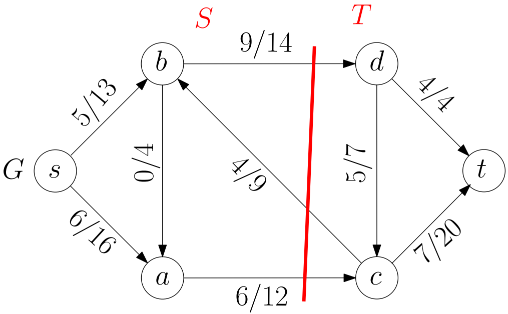

$$c(S, T) = 14 + 12 = 26$$

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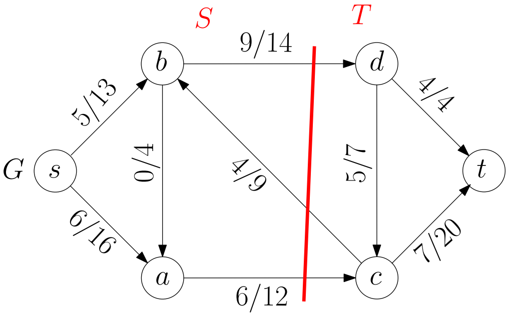

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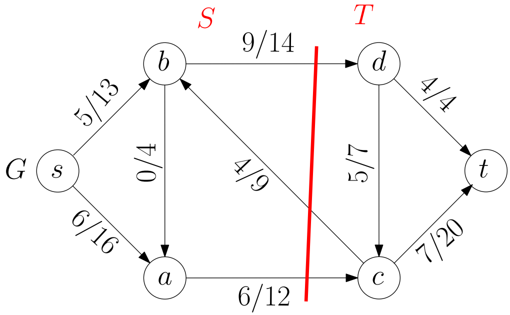
Obs. $f(S,T) \le c(S,T)$ s-t cut (S,T).



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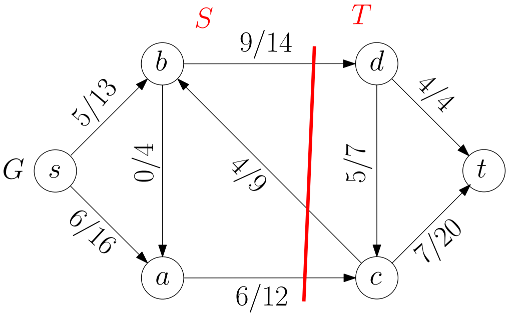

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Corollary and Main Lemma implies

Maximum Flow Minimum Cut Theorem
$$\sup_{s\text{-}t \text{ flow } f} \text{val}(f) = \min_{s\text{-}t \text{ cut } (S,T)} c(S,T).$$

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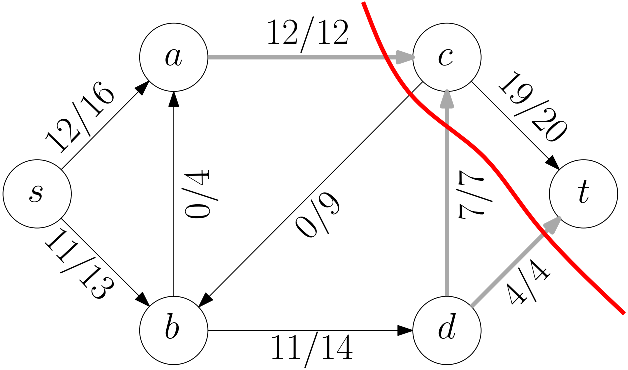

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- Thus,
  $$\text{val}(f) = f(S,T) = \sum_{e=(u,v)\in E, u\in S, v\in T} f(e) - \sum_{e=(u,v)\in E, u\in T, v\in S} f(e) = \sum_{e=(u,v)\in E, u\in S, v\in T} c_e = c(S,T).$$

□

Correctness of Ford-Fulkerson's Method

1. The procedure augment(f, P) maintains the two conditions:
   - for every $e \in E$: $0 \le f(e) \le c_e$ (capacity conditions)
   - for every $v \in V \setminus \{s, t\}$:

$$\sum_{e \in \delta_{\text{in}}(v)} f(e) = \sum_{e \in \delta_{\text{out}}(v)} f(e). \qquad \text{(conservation conditions)}$$

2. When Ford-Fulkerson's Method terminates, val(f) is maximized
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However, the algorithm may not terminate if some capacities are irrational numbers. ("Pathological cases")

Lemma Ford-Fulkerson's Method will terminate if all capacities are integers.

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- Integers can be replaced by rational numbers.

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Running time of the Generic Ford-Fulkerson's Algorithm

Ford-Fulkerson(G, s, t, c)

```
1: let $f(e) \leftarrow 0$ for every $e$ in $G$
2: **while** there is a path from $s$ to $t$ in $G_f$ **do**
3:     let $P$ be any simple path from $s$ to $t$ in $G_f$
4:     $f \leftarrow$augment$(f, P)$
5: **return** $f$
```

- $O(m)$-time for Steps 3 and 4 in each iteration
- Total time $= O(m) \times$ number of iterations

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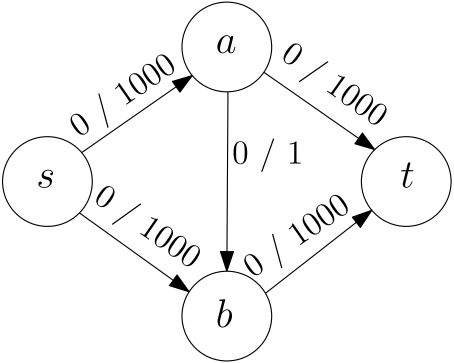
- $O(m)$-time for Steps 3 and 4 in each iteration
- Total time $= O(m) \times$ number of iterations
- Assume all capacities are integers, then algorithm may run up to val(f^*) iterations, where f^* is the optimum flow
- Total time $= O(m \cdot \text{val}(f^*))$
- Running time is "Pseudo-polynomial"

The Upper Bound on Running Time Is Tight!



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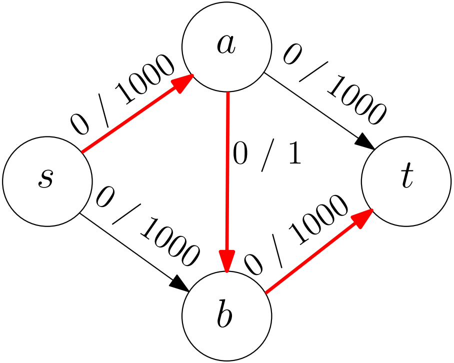

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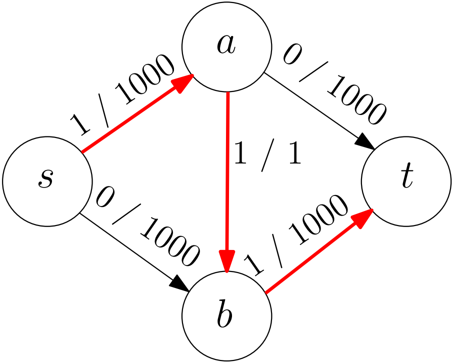

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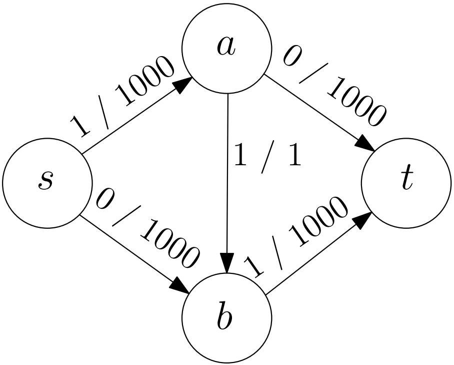

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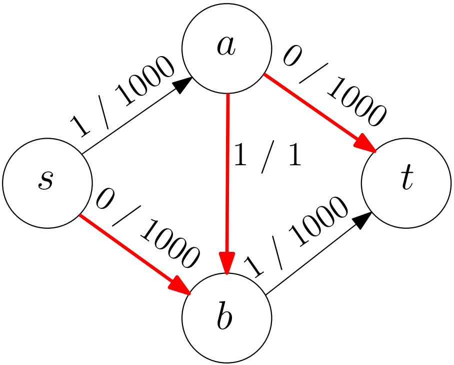

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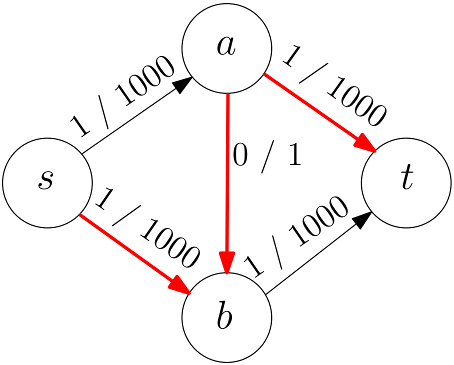

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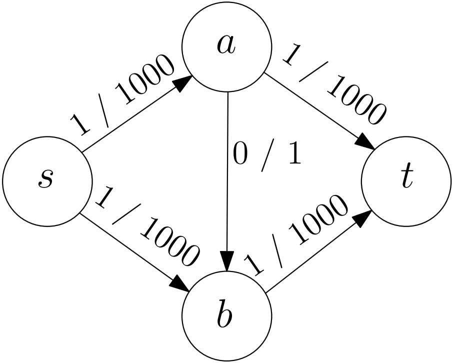

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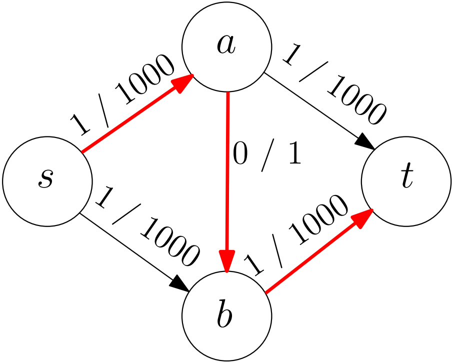

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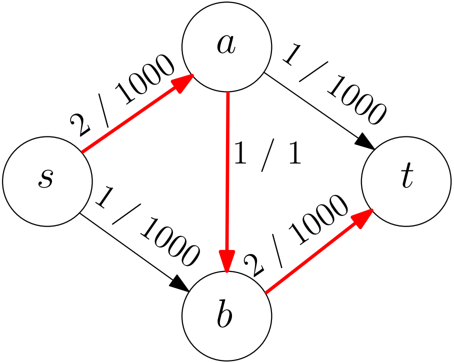

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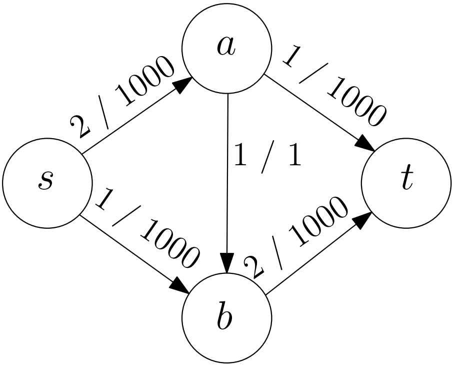

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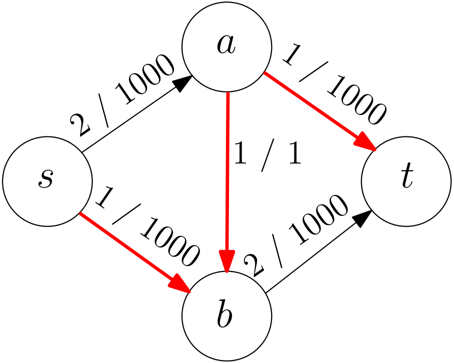

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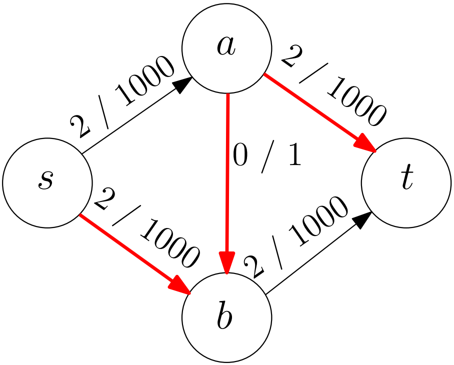

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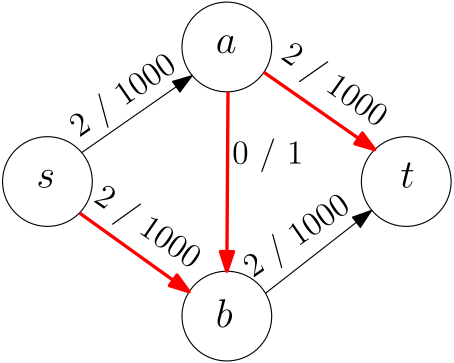

Better choices for choosing augmentation paths:

- Choose the shortest augmentation path
- Choose the augmentation path with the largest bottleneck capacity

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Shortest Augmenting Path

```
shortest-augmenting-path$(G, s, t, c)$
let $f(e) \leftarrow 0$ for every $e$ in $G$
while there is a path from $s$ to $t$ in $G_f$ do
    $P \leftarrow$ breadth-first-search$(G_f, s, t)$
    $f \leftarrow$augment$(f, P)$
return $f$
```

Due to [Dinitz 1970] and [Edmonds-Karp, 1970]

Running Time of Shortest Augmenting Path Algorithm

Lemma 1. Throughout the algorithm, length of shortest path from s to t in G_f never decreases.
2. After at most m shortest path augmentations, the length of the shortest path from s to t in G_f strictly increases.

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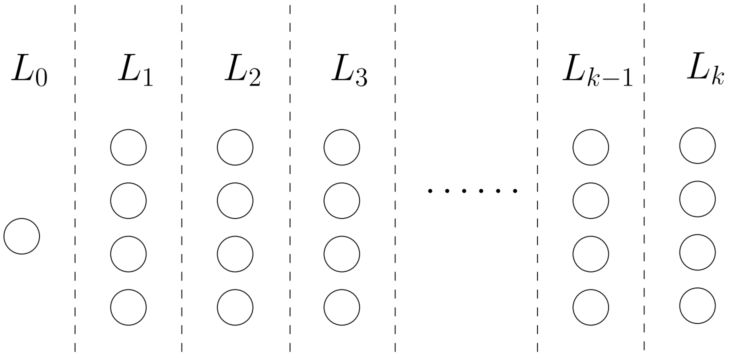
Theorem The shortest-augmenting-path algorithm runs in time $O(m^2n)$.

Proof of Lemma: Focus on G_f



- Divide V into levels: L_i contains the set of vertices v such that the length of shortest path from s to v in G_f is i

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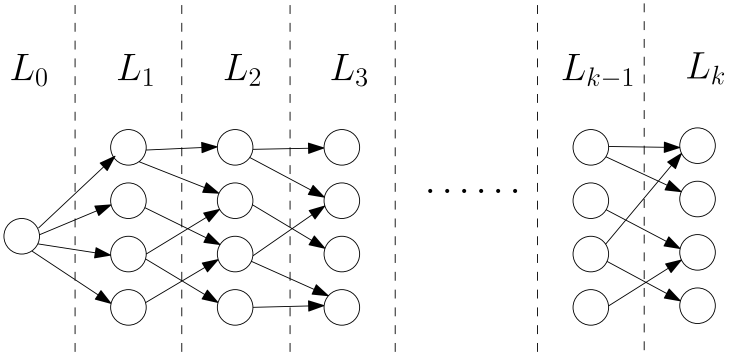

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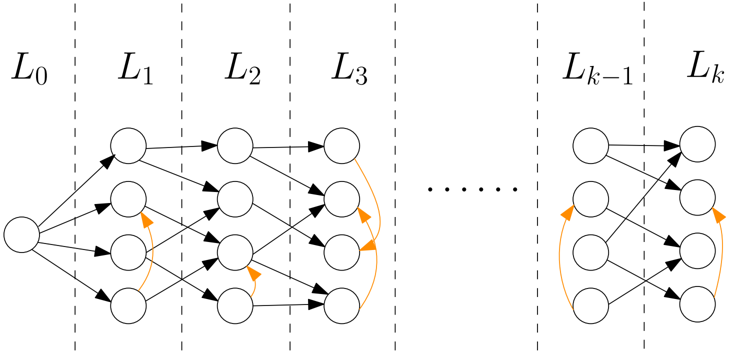

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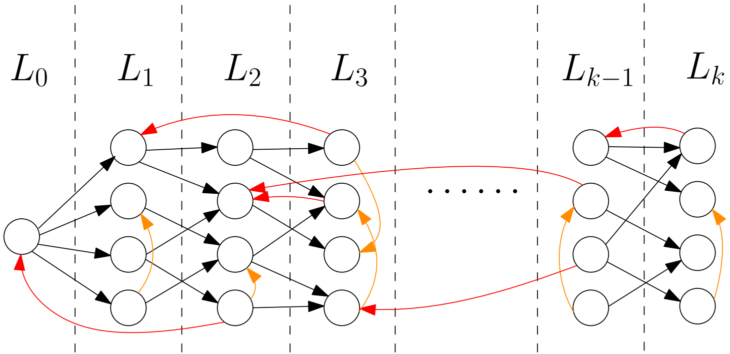

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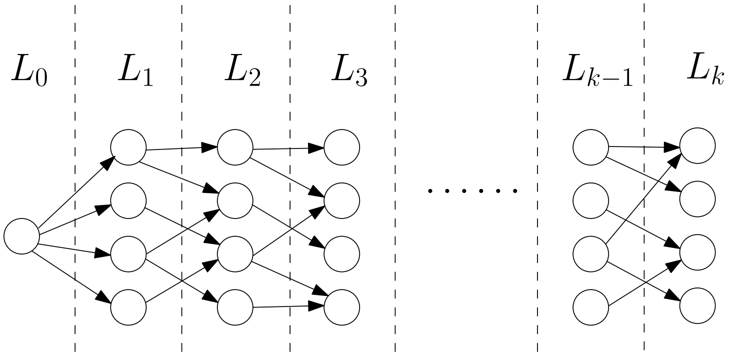

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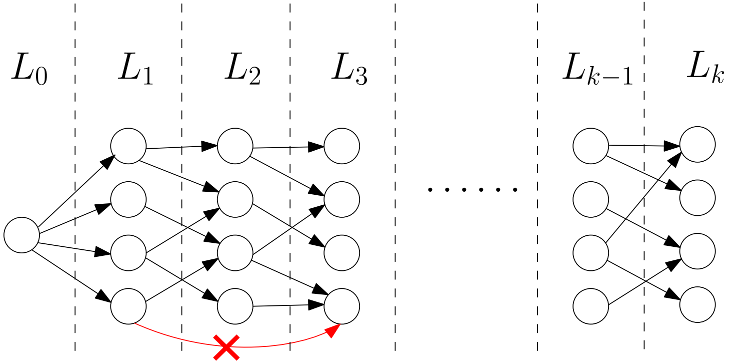

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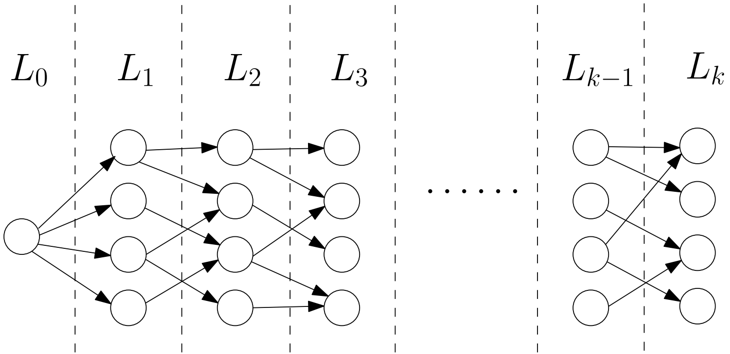

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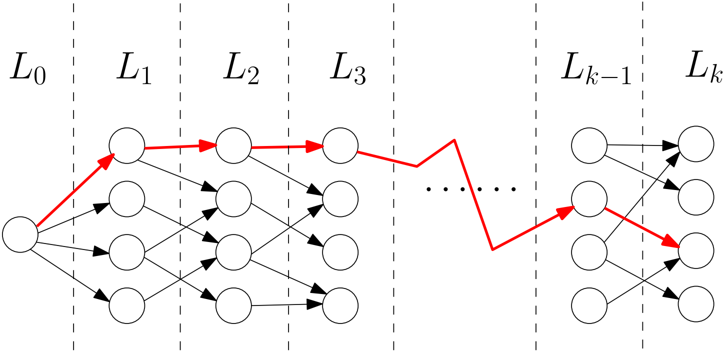

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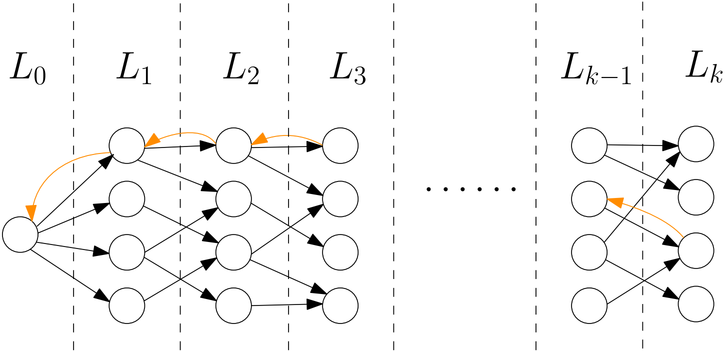

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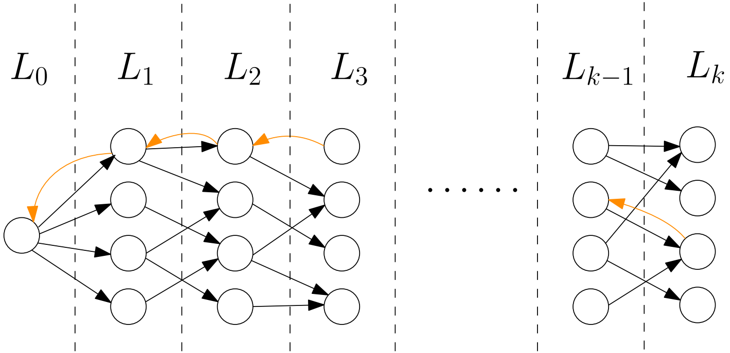

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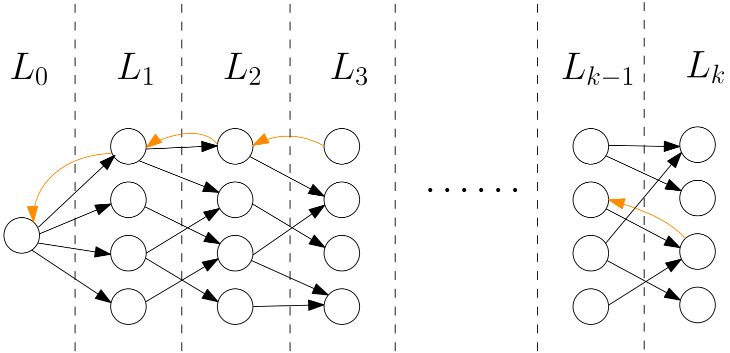

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- One forth edge will be removed from G_f
- In $O(m)$ iterations, there will be no paths from s to t of length k in G_f.

Improving the $O(m^2n)$ Running Time for Shortest Path Augmentation Algorithm

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- Dynamic Trees $\Rightarrow O(mn\log n)$ [Sleator-Tarjan 1983]

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2. Ford-Fulkerson Method
3. Correctness of Ford-Fulkerson's Method and Maximum Flow Minimum Cut Theorem
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capacity-scaling(G, s, t, c)

```
let $f(e) \leftarrow 0$ for every $e$ in $G$
$\Delta \leftarrow$ largest power of $2$ which is at most $C$
while $\Delta \geq 1$ do do
    while there exists an augmenting path $P$ with bottleneck capacity at least $\Delta$ do
        $f \leftarrow$augment$(f, P)$
    $\Delta \leftarrow \Delta/2$
return $f$
```

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Theorem The number of augmentations in the scaling max-flow algorithm is at most $O(m \log C)$. The running time of the algorithm is $O(m^2 \log C)$.

Polynomial Time

Assume all capacities are integers between 1 and C.

Ford-Fulkerson	$O(m^2C)$	pseudo-polynomial
Capacity-scaling:	$O(m^2 \log C)$	weakly-polynomial
Shortest-Path-Augmenting:	$O(m^2n)$	strongly-polynomial

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Brief History

Algorithm	**Year**	**Time**	**Description**
Ford-Fulkerson	1956	$O(mf)$	Ford-Fulkerson Method.
Edmonds-Karp	1972	$O(nm^2)$	Shortest Augmenting Paths
Dinic	1970	$O(n^2m)$	SAP with blocking Flows
Goldberg-Tarjan	1988	$O(n^3)$	Generic Push-Relabel
Goldberg-Tarjan	1988	$O(n^2\sqrt{m})$	PR using highest-label nodes
Chen et al.	2022	$O(m^{1+o(1)})$	LP-solver, dynamic algorithms

- Chen et al. [Chen-Kyng-Liu-Peng-Gutenberg-Sachdeva, 2022].

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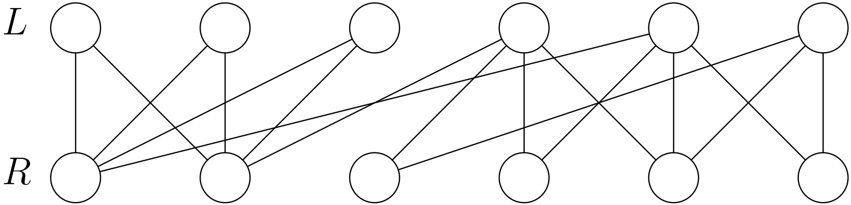
Bipartite Graphs

Def. A graph $G = (V, E)$ is bipartite if the vertices V can be partitioned into two subsets L and R such that every edge in E is between a vertex in L and a vertex in R.



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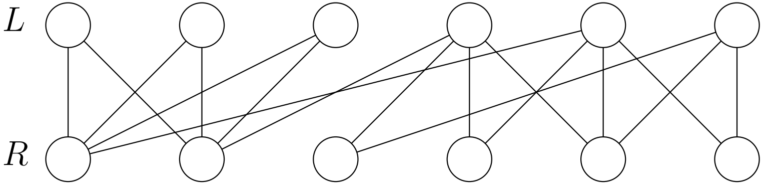

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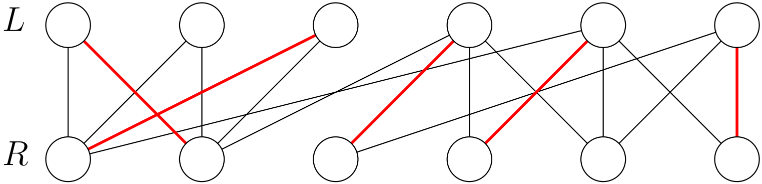

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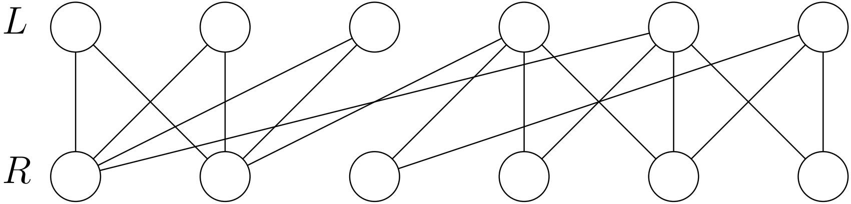

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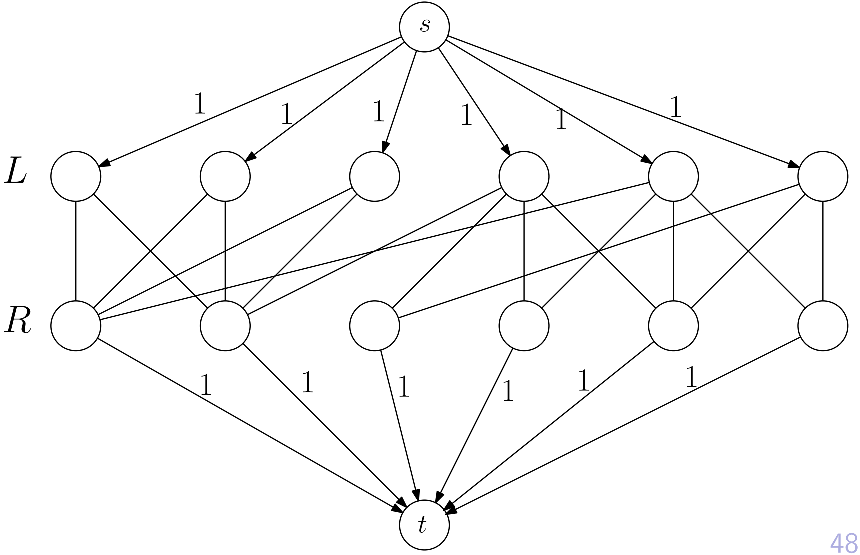

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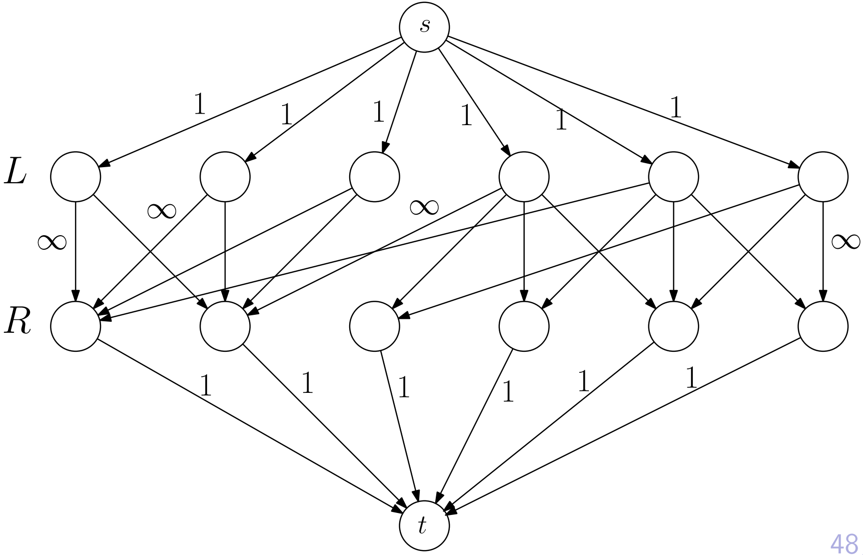

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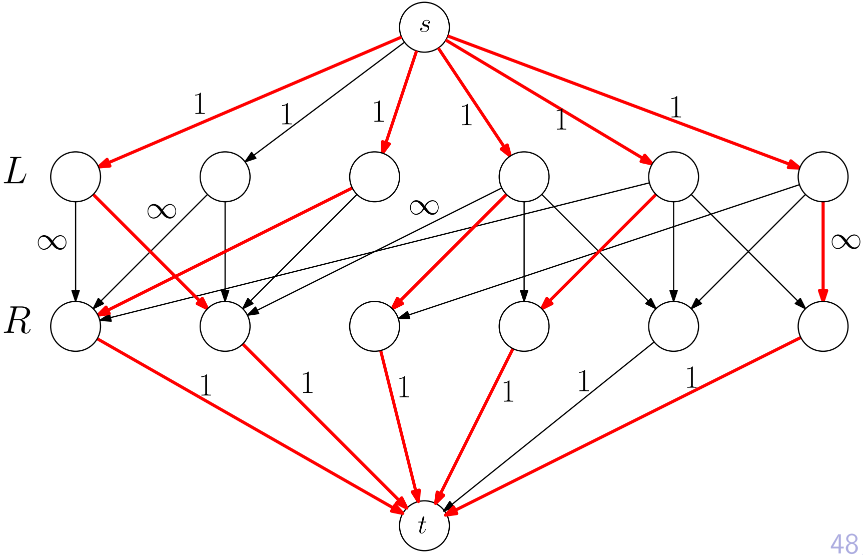

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- Create a digraph $G' = (L \cup R \cup \{s, t\}, E')$ with capacity $c : E' \to \mathbb{R}_{\geq 0}$:
  - Add a source s and a sink t
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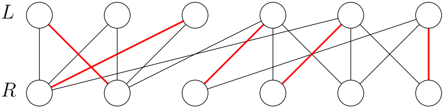

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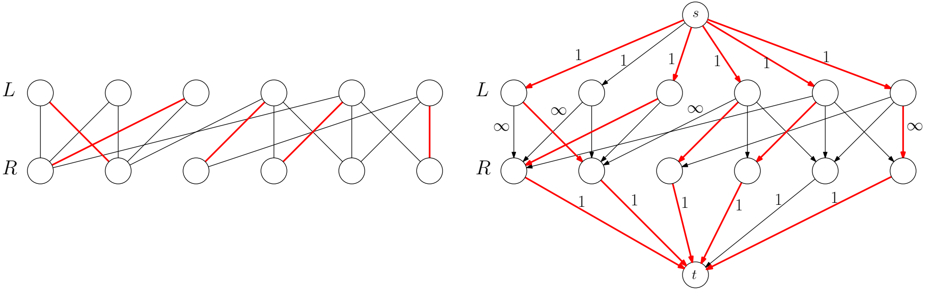

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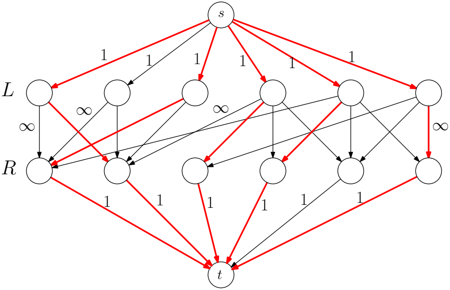

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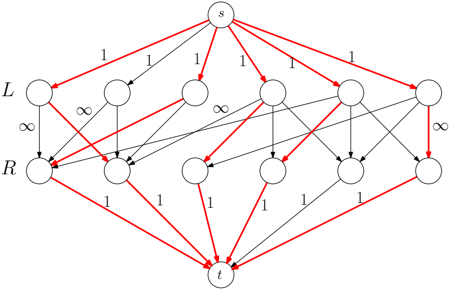

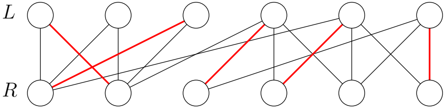

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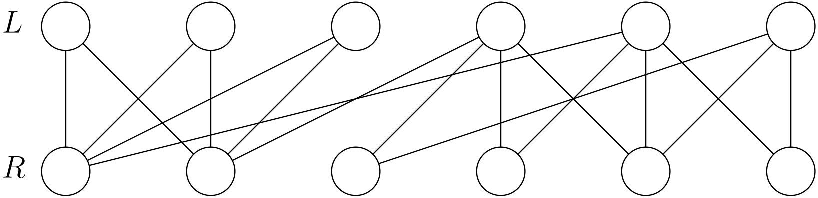

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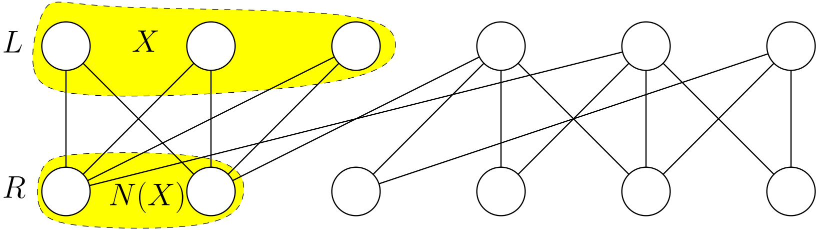

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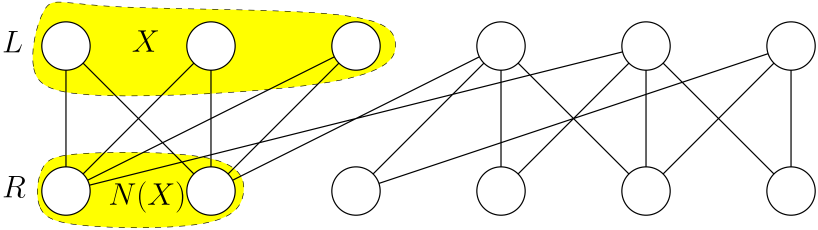

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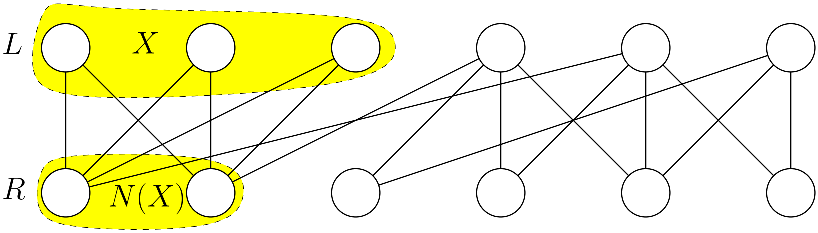

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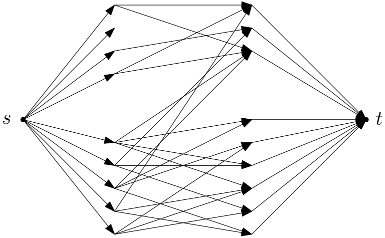

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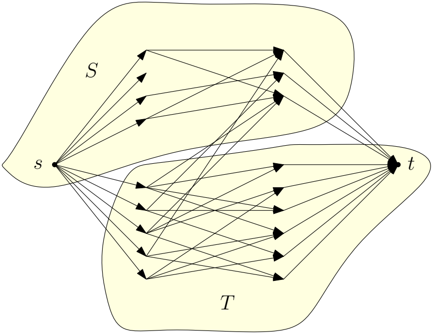

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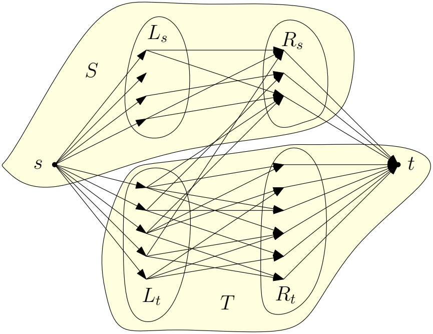

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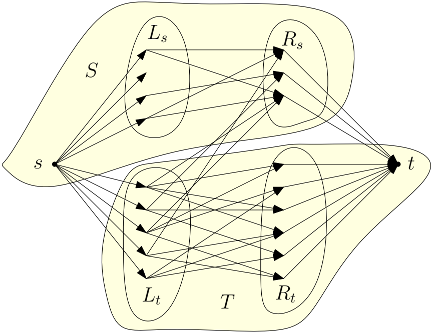

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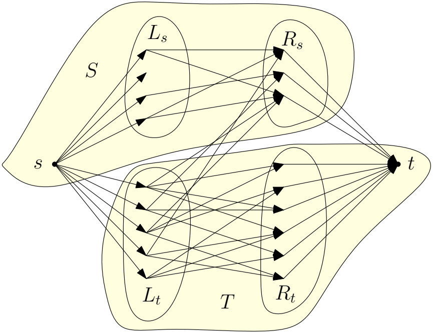

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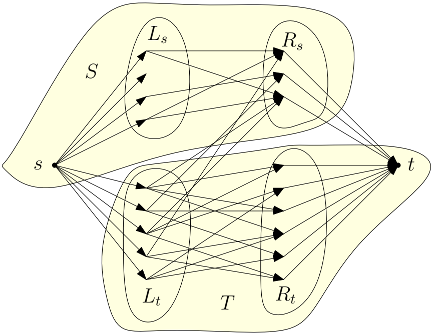

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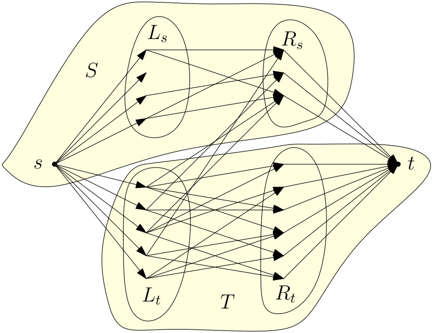

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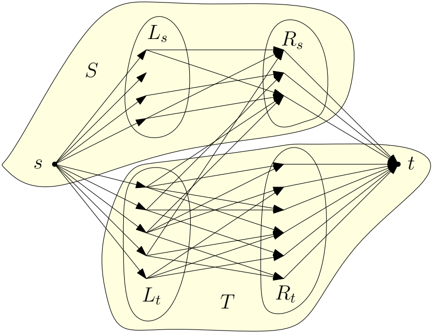
- $|N(L_s)| \leq |R_s| < n - |L_t| = |L_s|$. □



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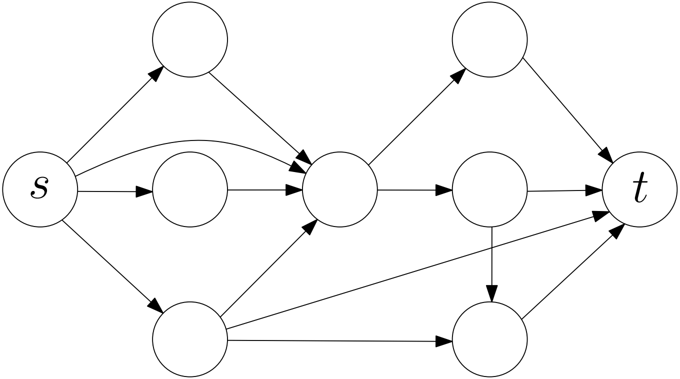
s-t Edge Disjoint Paths

Input: a directed (or undirected) graph $G = (V, E)$ and $s, t \in V$

Output: the maximum number of edge-disjoint paths from s to t in G



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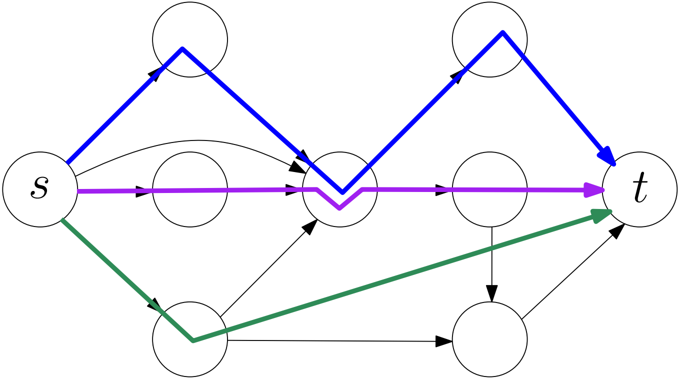
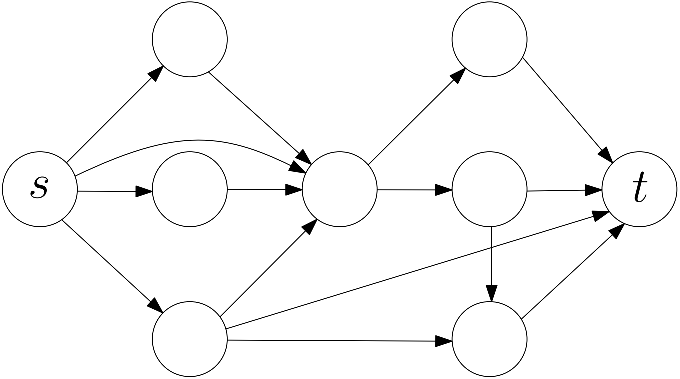


- Solving the maximum flow problem, where all capacities are 1
- All flow values are integral (i.e, either 0 or 1)



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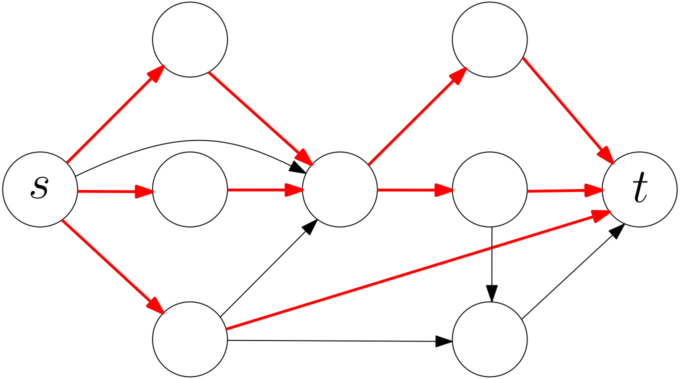

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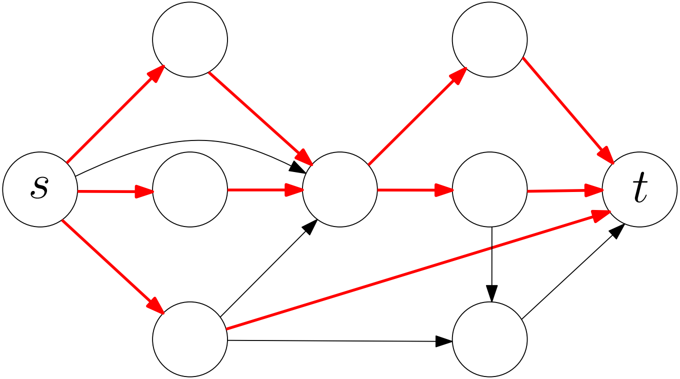
From flow to disjoint paths



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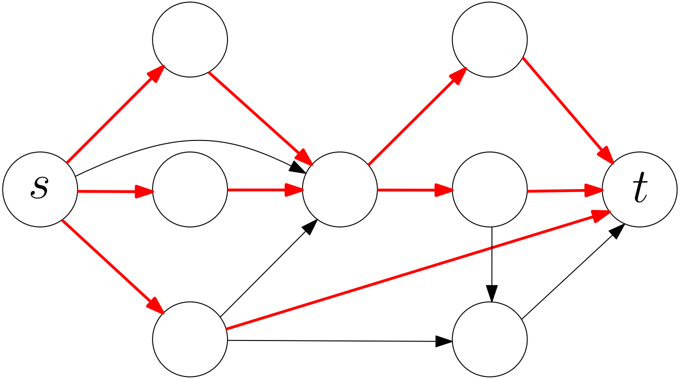

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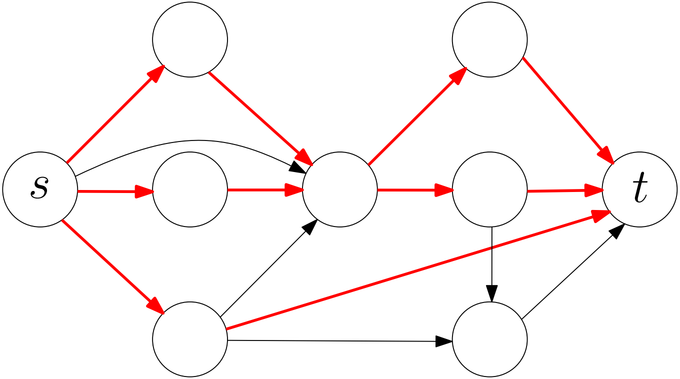
From flow to disjoint paths

- find an arbitrary $s \to t$ path where all edges have flow value 1
- change the flow values of the path to 0 and repeat



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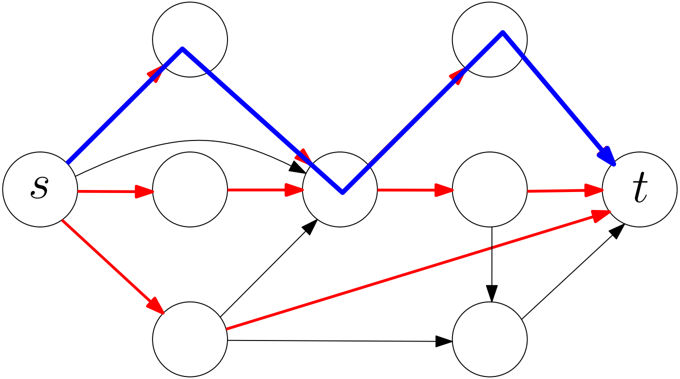

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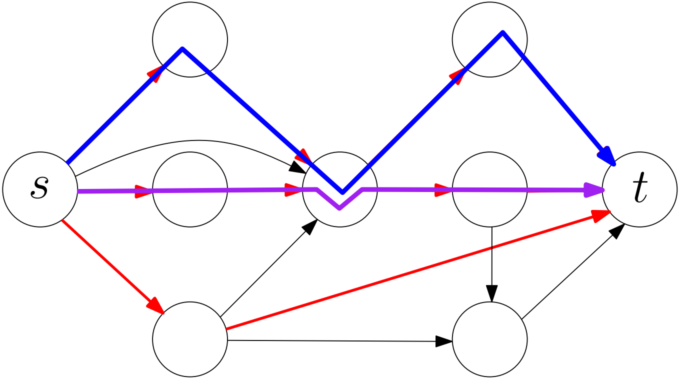

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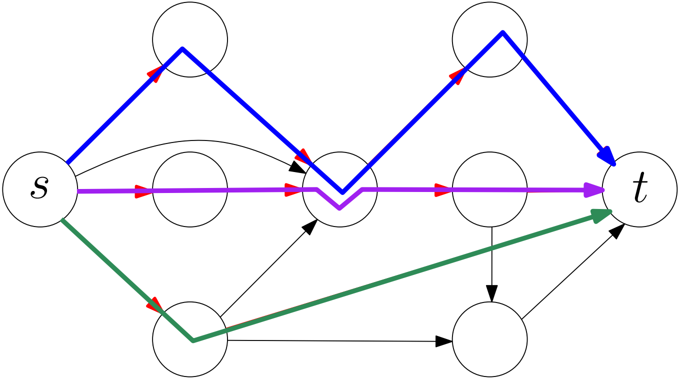
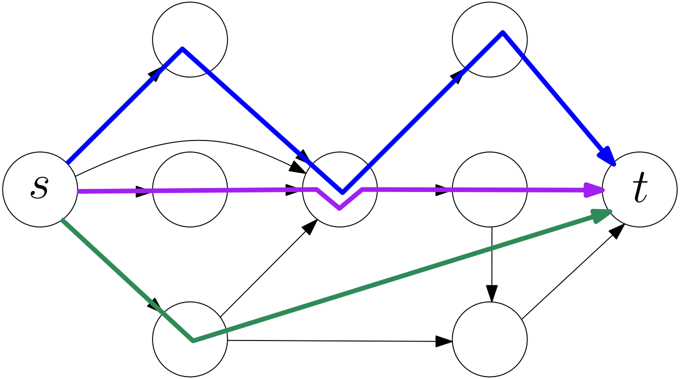


Theorem The maximum number of edge disjoint paths from s to t equals the minimum value of an s-t cut (S, T).



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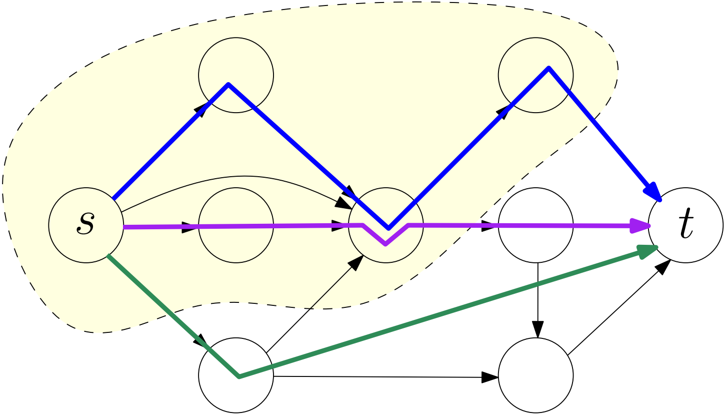
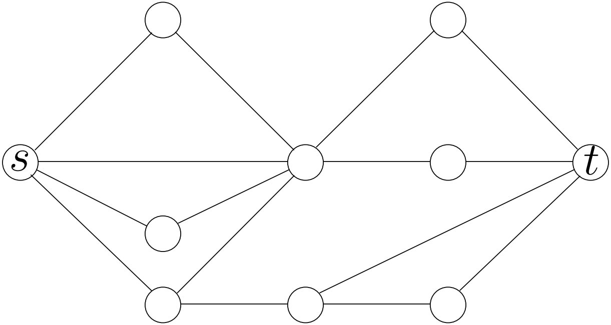


s-t Edge Disjoint Paths in Undirected Graphs



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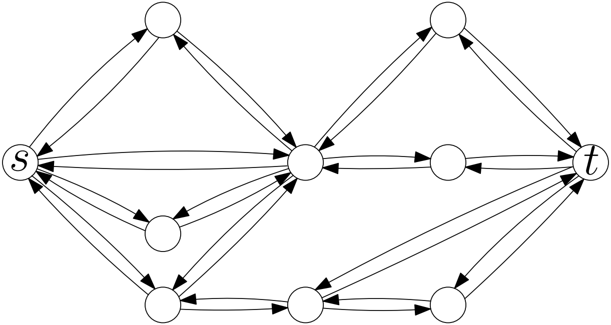

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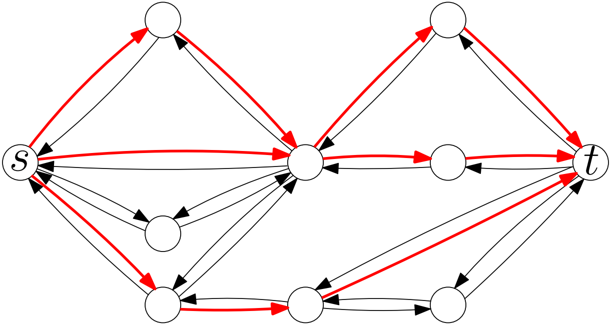

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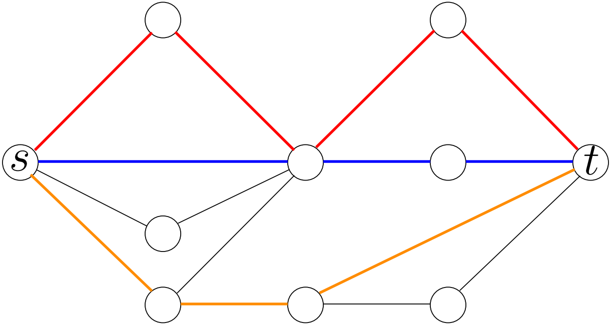

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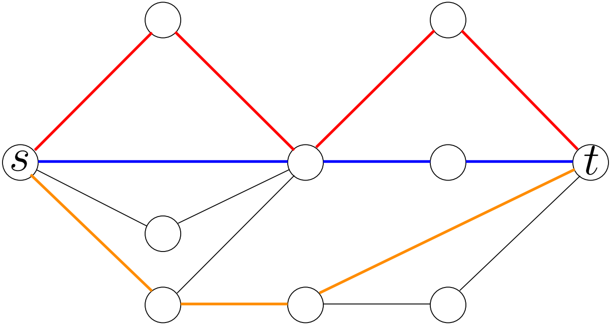

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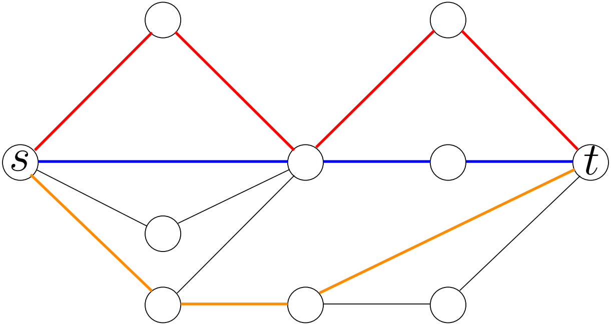

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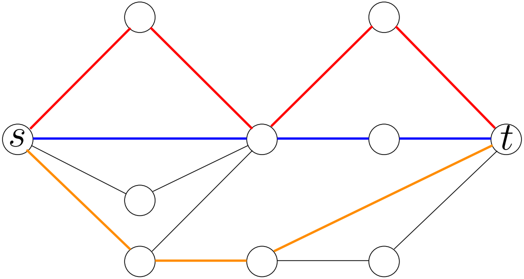
Menger's Theorem

Menger's Theorem In an undirected graph, the maximum number of edge-disjoint paths between s to t is equal to the minimum number of edges whose removal disconnects s and t.



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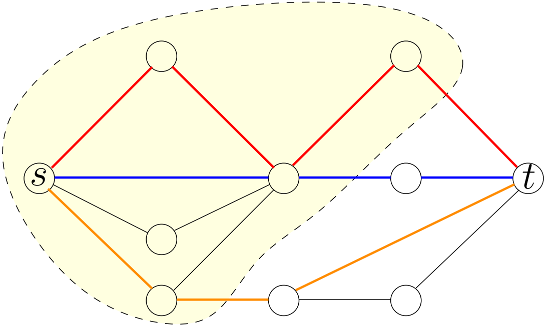

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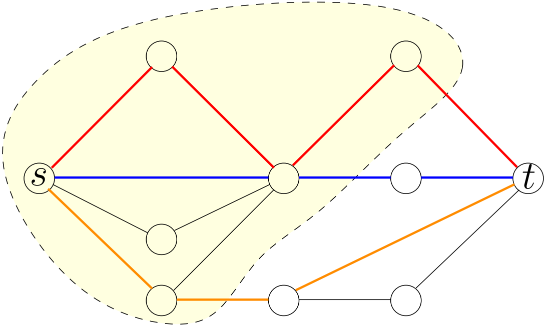
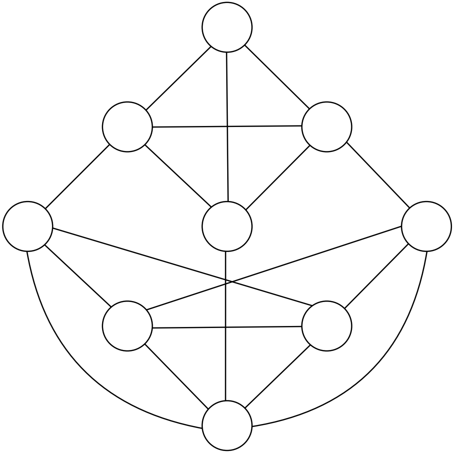


s-t connectivity measures how well s and t are connected.

Global Min-Cut Problem

Input: a connected graph $G = (V, E)$

Output: the minimum number of edges whose removal will disconnect G



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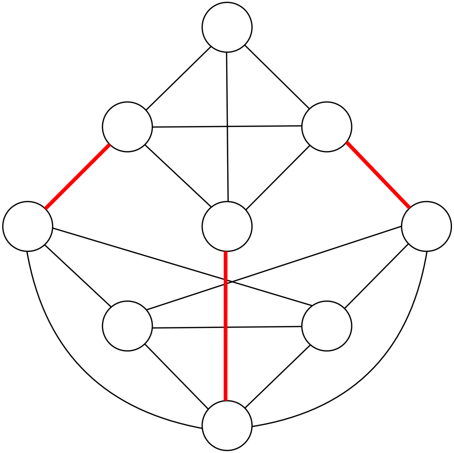

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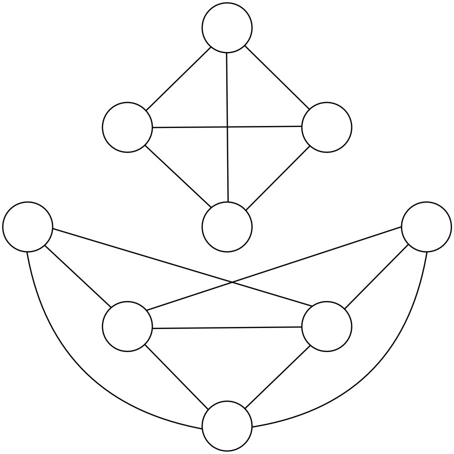

Solving Global Min-Cut Using Maximum Flow

```
let $G'$ be the directed graph obtained from $G$ by replacing every edge with two anti-parallel edges
for every pair $s \neq t$ of vertices do
    obtain the minimum cut separating $s$ and $t$ in $G$, by solving the maximum flow instance with graph $G'$,source $s$ and sink $t$
output the smallest minimum cut we found
```

- Need to solve $\Theta(n^2)$ maximum flow instances

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- Need to solve $\Theta(n^2)$ maximum flow instances
- Can we do better?
- Yes. We can fix s. We only need to enumerate t

Outline

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Extension of Network Flow: Circulation Problem

Input: A digraph $G = (V, E)$

capacities $c \in \mathbb{Z}_{\geq 0}^E$

supply vector $d \in \mathbb{Z}^V$ with $\sum_{v \in V} d_v = 0$

Output: whether there exists $f : E \to \mathbb{Z}_{\geq 0}$ s.t.

$$\sum_{e \in \delta^{\mathsf{out}}(v)} f(e) - \sum_{e \in \delta^{\mathsf{in}}(v)} f(e) = d_v \qquad \forall v \in V$$

$$0 \leq f(e) \leq c_e \qquad \forall e \in E$$

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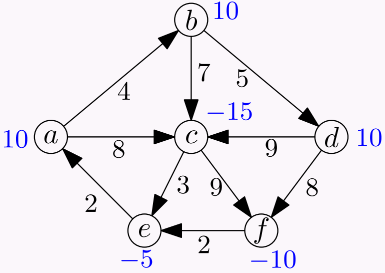
- d_v denotes the net supply of a good
- $d_v > 0$: there is a supply of d_v at v
- $d_v < 0$: there is a demand of $-d_v$ at v
- problem: whether we can match the supplies and demands without violating capacity constraints

Example



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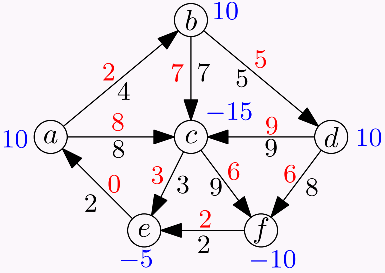

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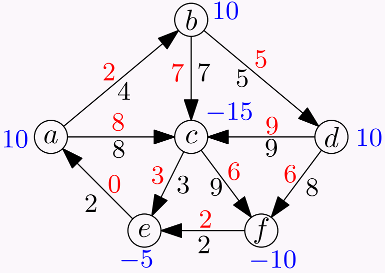
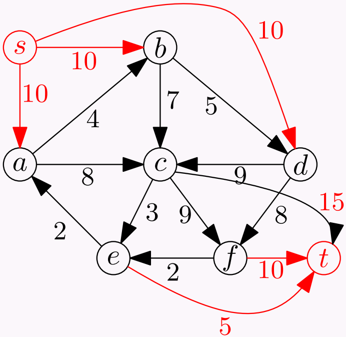


Reduction



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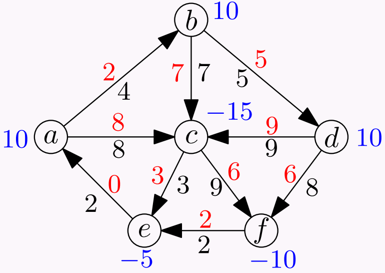

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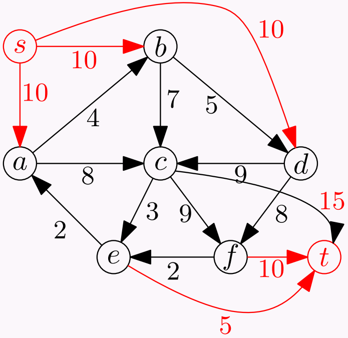

Reduction to maximum flow

- add a super-source s and a super-sink t to network
- for every $v \in V$ with $d_v > 0$: add edge (s, v) of capacity d_v
- for every $v \in V$ with $d_v < 0$: add edge (v, t) of capacity $-d_v$
- check if maximum flow has value $\sum_{v:d_v>0} d_v$

- $d(S) := \sum_{v \in S} d_v, \forall S \subseteq V$.
- $c(S, V \setminus S) := \sum_{(u,v) \in E: u \in S, v \notin S} c_{(u,v)}$.

Lemma The instance is feasible if and only if for every $S \subseteq V$, $d(S) \le c(S, V \setminus S)$.

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- if for some $S \subseteq V$, $c(S, V \setminus S) < d(S)$, then the demand in S can not be sent out of S. □

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- It remains to consider the "if" direction

Proof of “if” Direction

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- assume instance is infeasible: max-flow $< d(A)$
- $A := \{v \in V : d_v > 0\}$
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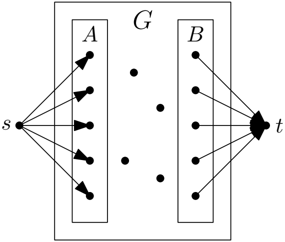

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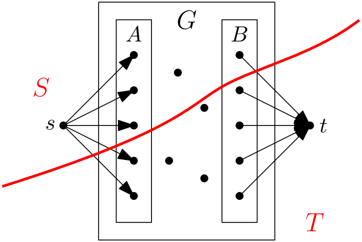

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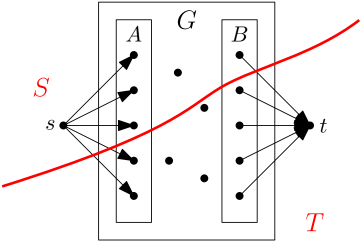

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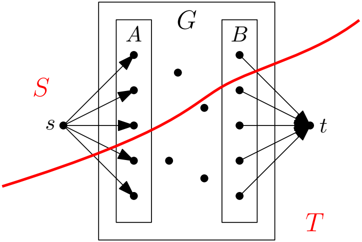

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- Define $S' = S \setminus \{s\}$: $d(S') > c(S', V \setminus S')$.

Circulation Problem with Capacity Lower Bounds

Input: A digraph $G = (V, E)$

capacities $c \in \mathbb{Z}_{\geq 0}^{E}$

capacity lower bounds $l \in \mathbb{Z}_{\geq 0}^{E}$, $0 \leq l_e \leq c_e$

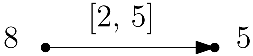
supply vector $d \in \mathbb{Z}^{V}$ with $\sum_{v \in V} d_v = 0$

Output: whether there exists $f : E \to \mathbb{Z}_{\geq 0}$ s.t.

$$\sum_{e \in \delta^{\mathrm{out}}(v)} f(e) - \sum_{e \in \delta^{\mathrm{in}}(v)} f(e) = d_v \qquad \forall v \in V$$

$$l_e \leq f(e) \leq c_e \qquad \forall e \in E$$

Removing Capacity Lower Bounds



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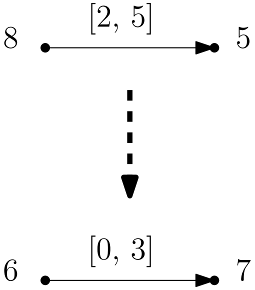

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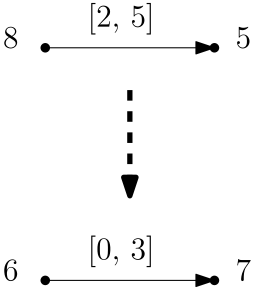

handling $e = (u, v)$ with $l_e > 0$

- $d'_u \leftarrow d_u - l_e$
- $d'_v \leftarrow d_v + l_e$
- $c'_e \leftarrow c_e - l_e$
- $l'_e \leftarrow 0$

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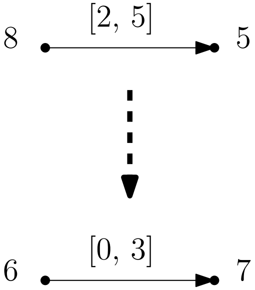

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- in old instance: flow is $f(e) \in [l_e, c_e] \implies f(e) - l_e \in [0, c_e - l_e]$
- in new instance: flow is $f(e) - l_e \in [0, c'_e]$

Survey Design

Input: integers $n, k \geq 1$ and $E \subseteq [n] \times [k]$

integers $0 \leq c_i \leq c'_i, \forall i \in [n]$

integers $0 \leq p_j \leq p'_j, \forall j \in [k]$

Survey Design

Input: integers $n, k \geq 1$ and $E \subseteq [n] \times [k]$

integers $0 \leq c_i \leq c_i', \forall i \in [n]$

integers $0 \leq p_j \leq p_j', \forall j \in [k]$

Output: $E' \subseteq E$ s.t.

$$\begin{aligned} c_i \leq |\{j \in [k] : (i,j) \in E'\}| \leq c_i', & \qquad \forall i \in [n] \\ p_j \leq |\{i \in [m] : (i,j) \in E'\}| \leq p_j', & \qquad \forall j \in [k] \end{aligned}$$

Survey Design

Input: integers $n, k \geq 1$ and $E \subseteq [n] \times [k]$

integers $0 \leq c_i \leq c'_i, \forall i \in [n]$

integers $0 \leq p_j \leq p'_j, \forall j \in [k]$

Output: $E' \subseteq E$ s.t.

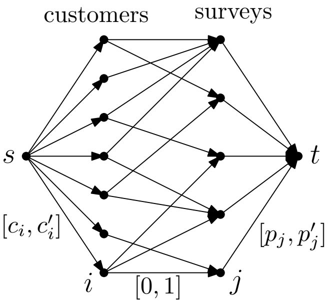
$$\begin{aligned} c_i \leq |\{j \in [k] : (i,j) \in E'\}| &\leq c'_i, & \forall i \in [n] \\ p_j \leq |\{i \in [m] : (i,j) \in E'\}| &\leq p'_j, & \forall j \in [k] \end{aligned}$$

Background

- $[n]$: customers, $[k]$:products
- $ij \in E$: customer i purchased product j and can do a survey
- every customer i needs to do between c_i and c'_i surveys
- every product j needs to collect between p_j and p'_j surveys

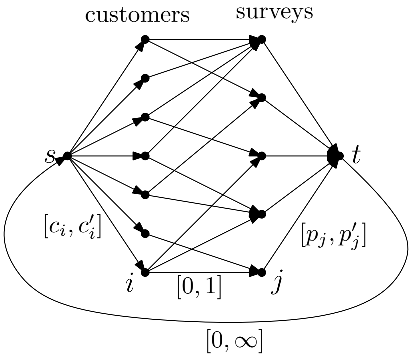
Reduction to Circulation

- vertices $\{s,t\} \uplus [n] \uplus [k]$,
- $(i,j) \in E$: (i,j) with bounds $[0,1]$
- $\forall i$: (s,i) with bounds $[c_i, c'_i]$
- $\forall j$: (j,t) with bounds $[p_j, p'_i]$



Reduction to Circulation

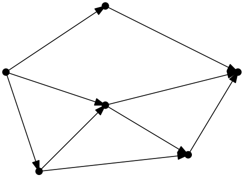
- vertices $\{s, t\} \uplus [n] \uplus [k]$,
- $(i, j) \in E$: (i, j) with bounds $[0, 1]$
- $\forall i$: (s, i) with bounds $[c_i, c_i']$
- $\forall j$: (j, t) with bounds $[p_j, p_i']$
- (t, s) with bounds $[0, \infty]$



Airline Scheduling

Input: a DAG $G = (V, E)$

Output: the minimum number of disjoint paths in G to cover all vertices



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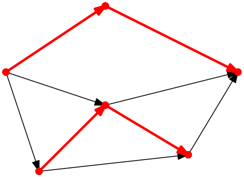

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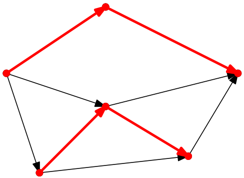
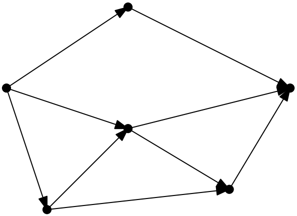


Background

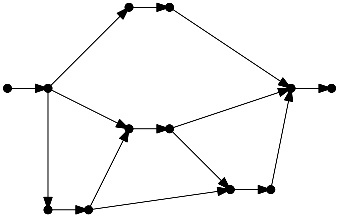
- vertex : a flight
- edge (u, v): an aircraft that serves u can serve v immediately
- goal: minimize the number of aircrafts

Reduction to the Circulation Problem



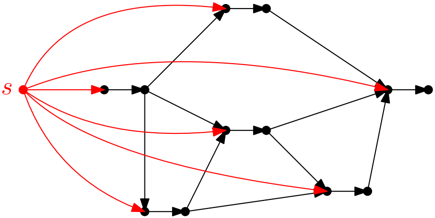
Reduction to the Circulation Problem

- split v into $(v_{\text{in}}, v_{\text{out}})$



Reduction to the Circulation Problem

- split v into $(v_{\textsf{in}}, v_{\textsf{out}})$
- add s, and $(s, v_{\textsf{in}}), \forall v$



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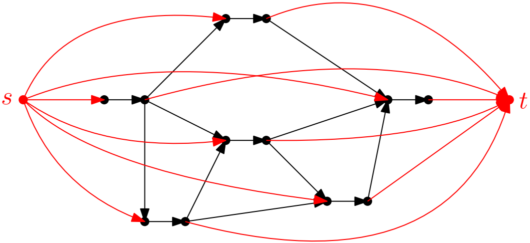

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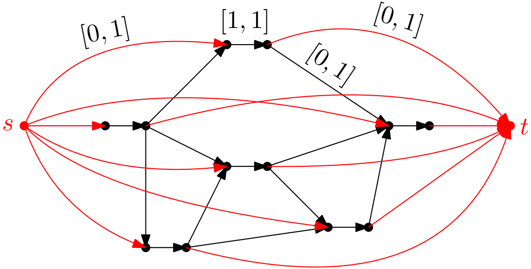

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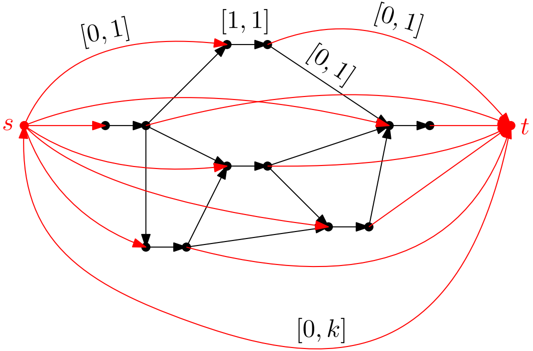

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- set lower and upper bounds
- add $t \to s$ of capacity k
- find minimum k s.t. instance is feasible

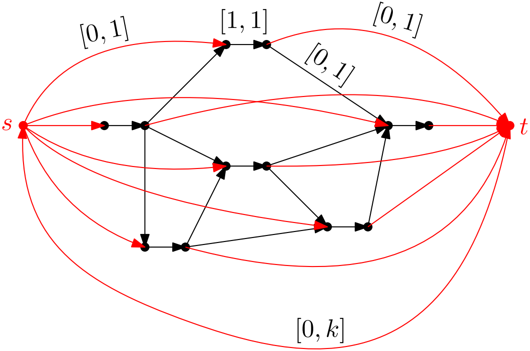

Image Segmentation

Input: A graph $G = (V, E)$, with edge costs $c \in \mathbb{Z}^E_{\geq 0}$
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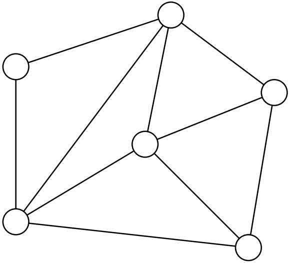
Background

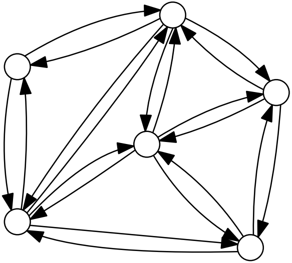
- a_v: the likelihood of v being a foreground pixel
- b_v: the likelihood of v being a background pixel
- $c_{(u,v)}$: the penalty for separating u and v
- need to maximize total reward - total penalty

Reduction to Network Flow



Reduction to Network Flow

- replace (u, v) with two anti-parallel arcs



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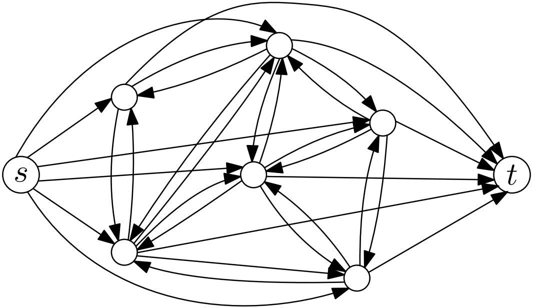

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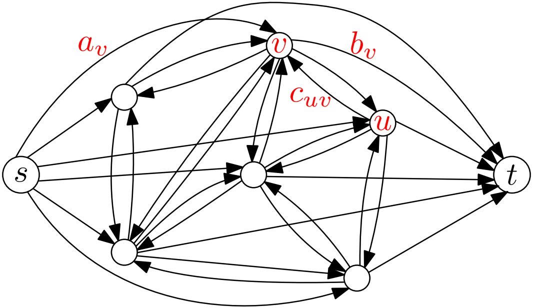

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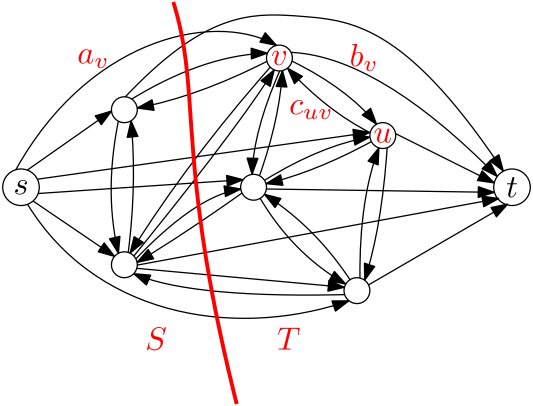

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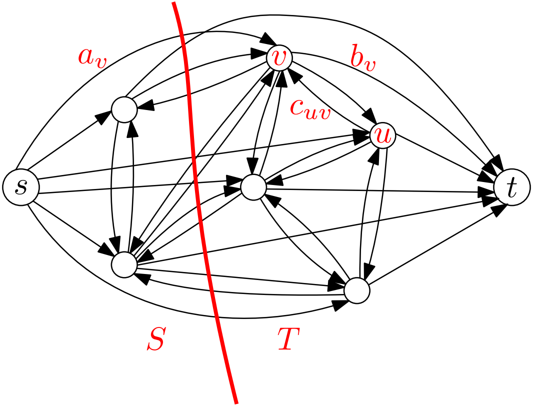

- The cut value of $(S = \{s\} \cup A, \{t\} \cup B)$ is

$$\sum_{v\in B} a_v + \sum_{v\in A} b_v + \sum_{(u,v)\in E:|\{u,v\}\cap A|=1} c_{(u,v)}$$

$$= \sum_{v\in V}(a_v + b_v) - \Bigg(\sum_{v\in A} a_v + \sum_{v\in B} b_v - \sum_{(u,v)\in E:|\{u,v\}\cap A|=1} c_{(u,v)}\Bigg)$$

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\end{aligned}$$

- So, maximizing the objective of (A, B) is equivalent to minimizing the cut value.

Project Selection

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revenue on vertices: $p \in \mathbb{Z}^V$; p_v's could be negative.

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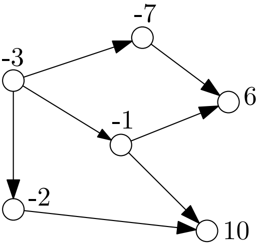

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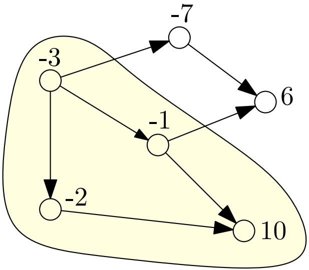
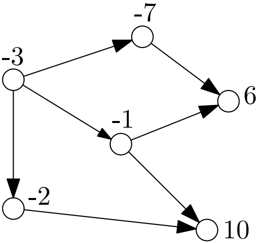


Reduction



Reduction

- add source s and sink t
- $p_v < 0$: (s, v) of capacity $-p_v$
- $p_v > 0$: (v, t) of capacity p_v
- $L = \{v : p_v < 0\}$
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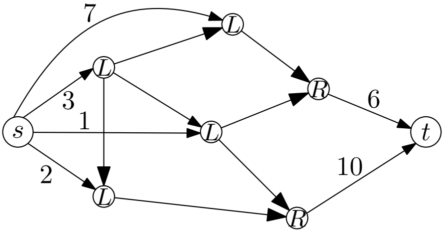

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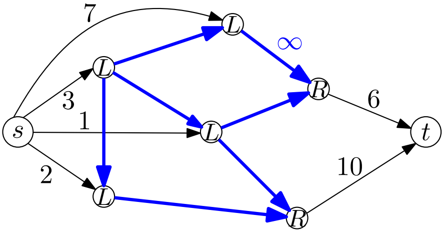

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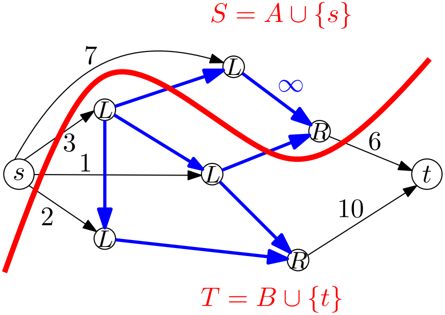

- min-cut $(S = \{s\} \cup A, T = \{t\} \cup B)$

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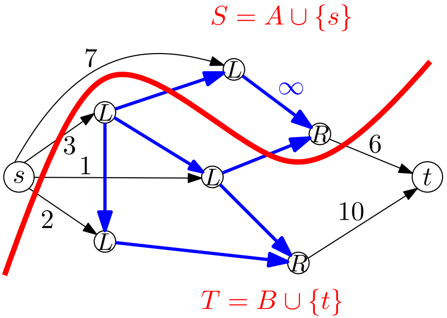

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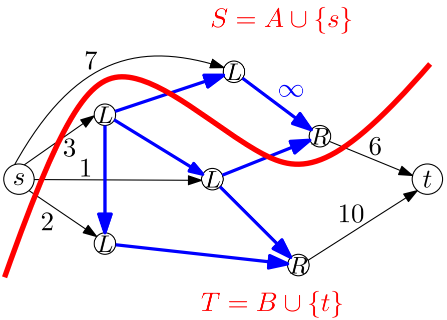
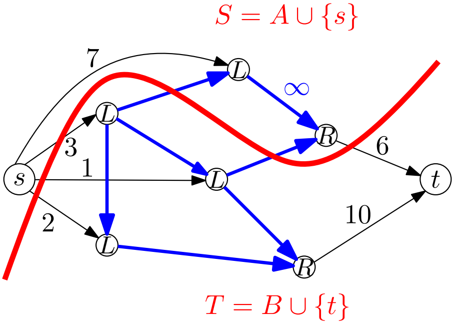


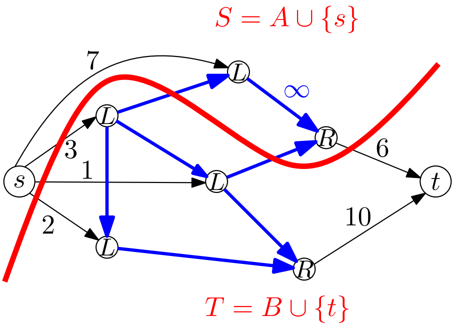
- min-cut $(S = \{s\} \cup A, T = \{t\} \cup B)$
- no ∞-capacity edges from A to B
- cut value is

$$\sum_{v \in B \cap L} (-p_v) + \sum_{v \in A \cap R} p_v = -\sum_{v \in B \cap L} p_v - \sum_{v \in B \cap R} p_v + \sum_{v \in R} p_v$$

$$= \sum_{v \in R} p_v - \sum_{v \in B} p_v$$



- B is a valid solution $\iff c(S,T) \neq \infty$



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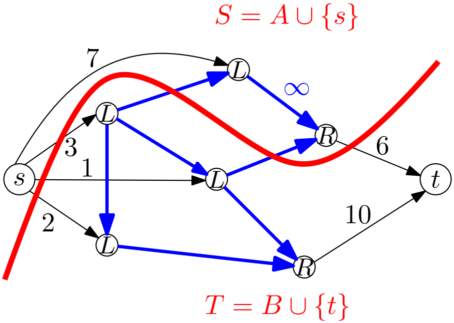
- when B is valid, $c(S,T) = \sum_{v \in R} p_v - \sum_{v \in B} p_v$



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- when B is valid, $c(S,T) = \sum_{v \in R} p_v - \sum_{v \in B} p_v$
- so, to maximize $\sum_{v \in B} p_v$, we need to minimize $c(S,T)$.

More Applications

- Graph orientation
- maximum independent set (and minimum vertex cover) in a bipartite graph
- $\cdots$