Homework 1

Course: Algorithm Design and Analysis | **Semester**: Spring 2026

Instructor: Shi Li | **Due Date**: **Apr 5, 2026**

Student Name: _________ Student ID: _________

Problems	1	2	3	4	5	6	7	Total
Max. Score	15	16	12	12	15	15	20	100
Your Score								

The sum of maximum scores over all problems is 105, but your score will be capped at 100.

1 Asymptotic Notations

Problem 1. For each pair of asymptotically positive functions $f(n)$ and $g(n)$ in the table below, determine whether f is O, Ω, Θ, o, and ω of g. Fill in the table with "yes" or "no" for each cell.

$f(n)$	$g(n)$	O	Ω	Θ	o	ω
$n^3/10 - 5$	n^2					
$n^2 + 10n$	$3n^2$					
$\log(n!)$	$n \log n$					
2^n	3^n					
$\sqrt{n}$	$n^{\sin n}$					

Problem 2. Assume $f(n)$ and $g(n)$ are both asymptotically positive functions. Solve the following problems:

(2a) Prove that if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c > 0$, then $f(n) = \Theta(g(n))$.

(2b) Prove that if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f(n) = o(g(n))$.

(2c) Prove the converse of (2b): if $f(n) = o(g(n))$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

(2d) Provide an example of two functions $f(n)$ and $g(n)$ such that $f(n) = \Theta(g(n))$, but the limit $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist.

2 Basic Graph Algorithms

Problem 3. Below, all undirected and directed graphs are simple. Namely, there are no self-loops and parallel edges.

(3a) An (undirected) graph is said to be **acyclic** if it contains no (undirected) cycles. What is the maximum number of edges in an (undirected) acyclic graph with n vertices? Provide a brief justification.

(3b) A directed graph is said to be **acyclic** if it contains no directed cycles. What is the maximum number of edges in a directed acyclic graph (DAG) with n vertices? Provide a brief justification.

(3c) What is the maximum number of edges in a bipartite graph with n vertices? Provide a brief justification.

Problem 4. Let $G = (V, E)$ be a connected graph. Prove that an edge $e = (u, v)$ is a bridge if and only if e lies on no cycle in G.

Problem 5. Let $G = (V, E)$ be a directed graph, and assume there is vertex $s \in v$ which can reach all vertices. Let T be a Depth-First Search (DFS) tree of G with root s. Prove that G is acyclic if and only if there are no upward edges with respect to T. Recall that an edge $(u, v) \in E$ is an upward edge if v is an ancestor of u in the tree T.

3 Data Structures

Problem 6. Consider the following dynamic median problem. A stream of n integers arrives one by one. At any point, the algorithm must return the (lower) median of the elements seen so far.

Example: If the sequence is $[5, 15, 1, 3]$, the medians returned would be:

- After $[5]$: Median is 5.
- After $[5, 15]$: Lower Median is 5 .
- After $[5, 15, 1]$: Median is 5.
- After $[5, 15, 1, 3]$: Lower Median is 3.

Describe how to implement this using two heaps (a max-heap and a min-heap) and provide the complexity of the algorithm.

Problem 7: Largest Rectangle in a Histogram

Problem Description: Given a histogram represented by an array of heights, find the area of the largest rectangle that can be formed within the bounds of the histogram. For

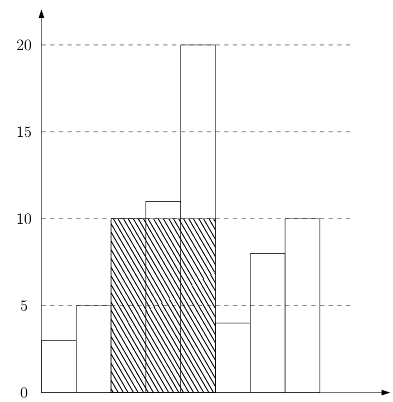

Figure 1: The instance for the largest rectangle in a histogram problem.

example, if the input is $(3, 5, 10, 11, 20, 4, 8, 10)$, then the largest rectangle has size 30 (with height 10 and width 3, covering column 3 to column 5). See Figure 1.

Assume you are also given a list of the columns sorted in non-decreasing order of their heights. For example, for the above instance, the sorted list is $(1, 6, 2, 7, 3, 8, 4, 5)$. Use the union-find data structure to design an $O(n\alpha(n))$-time algorithm for the problem, where $\alpha(n)$ is the inverse Ackermann function.