Homework 5

Course: Algorithm Design and Analysis **Semester**: Spring 2026

Instructor: Shi Li **Due Date**: **June 28, 2026**

Student Name: _____ Student ID: _____

Problems	1	2	3	4	5	Total
Max. Score	20	20	20	20	25	105
Your Score						

The sum of maximum scores over all problems is 105, but your score will be capped at 100.

NP-Completeness Reductions

Problem 1. Recall that a 3-CNF formula is a conjunction (which corresponds to the "and" operator "$\wedge$") of clauses where each clause is a disjunction (which corresponds to the "or" operator "$\vee$") of exactly three distinct literals. The **1-in-3-SAT** problem asks whether there exists a satisfying truth assignment to the variables such that each clause contains *exactly* one true literal. Prove that 3-SAT $\leq_P$ 1-in-3-SAT.

Hint: For a clause $(x \vee y \vee z)$ in a 3-SAT instance, where each of x, y and z is a literal, consider introducing four new dummy variables a, b, c, and d, and constructing the following 1-in-3-SAT formula:

$$(\neg x \vee a \vee b) \wedge (y \vee b \vee c) \wedge (\neg z \vee c \vee d).$$

Problem 2. Recall that the *(Directed) Hamiltonian Cycle* problem asks whether a given (Directed) graph contains a simple cycle that visits every vertex exactly once. In class, we proved that the Directed Hamiltonian Cycle problem (Directed-HC) is NP-Complete. In this problem, you need to prove that Directed-HC $\leq_P$ HC to establish that the Hamiltonian Cycle (HC) problem is also NP-Complete.

Hint: Consider splitting each vertex in the Directed-HC instance into three vertices.

Advanced Topics

Problem 3. Suppose we have a perfect binary tree where the paths from the root to any leaf contain exactly L levels of edges (hence, there are 2^L leaves). Every edge in the tree independently succeeds with probability $1/2$ and fails with probability $1/2$. The overall experiment is considered a success if there exists at least one root-to-leaf path where all L edges along the path successfully survived.

In class, we proved a lower bound showing that the probability of overall success is $\Omega(1/L)$. In this problem, you need to prove the bound is tight up to a constant. That is, prove that the probability of success is $O(1/L)$.

Problem 4. Linear Programming and Duality.
Consider the following linear program in standard minimization form:

$$\begin{aligned} \text{minimize} \quad & 7x_1 + 5x_2 + 6x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + 3x_3 \geq 8 \\ & x_1 + 2x_2 + x_3 \geq 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Based on this formulation, complete the following tasks:

(4a) Find the optimum solution (x_1^*, x_2^*, x_3^*) and the corresponding optimal objective value for this primal linear program. (Although you have not learned how to solve an LP manually, you can find the optimal value by testing all the extreme solutions.)

(4b) Write down the the dual linear program using dual variables y_1 and y_2.

(4c) Find the optimal solution (y_1^*, y_2^*) for the dual linear program and verify that the strong duality theorem holds.

Problem 5. Recall that in the unweighted *3-Dimensional Matching* (3DM) problem, we are given three disjoint sets X, Y, and Z, each of size n, and a set of tuples $T \subseteq X \times Y \times Z$. A 3-dimensional matching is a subset $M \subseteq T$ such that every element in $X \cup Y \cup Z$ appears in at most one tuple in M. A perfect matching is a matching M of size n.

In class, you learned that the problem of deciding whether a perfect 3-dimensional matching exists is NP-complete. In this problem, we consider the maximum 3-dimensional matching problem, where the goal is to find a matching M of maximum size. You need to design a polynomial-time approximation algorithm for this problem.

(5a) Design a polynomial-time 3-approximation algorithm for the maximum 3-Dimensional Matching problem and prove its approximation ratio.

(5b) Extend your algorithm for (5a) to the weighted version of the problem, where each tuple $t \in T$ has a non-negative weight $w(t) \geq 0$ and the goal is to maximize the total weight of the matching.