算法设计与分析(2026年春季学期)
Advanced Topics

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Outline

1. Randomized Algorithms
   - Freivald's matrix multiplication verification algorithm
   - Randomized Algorithm for Global Min-Cut
2. Finding Small Vertex Covers: Fixed Parameterized Tractability
3. Approximation Algorithms using Greedy
4. Approximation Using LP Rounding and Primal Dual
5. More Advanced Algorithms *

Why do we use randomized algorithms?

- simpler algorithms: quick-sort, minimum-cut, and Max 3-SAT.
- faster algorithms: polynomial identity testing, Freivald's matrix multiplication verification algorithm, sampling and fingerprinting.
- mathematical beauty: Nash equilibrium for 0-sum game
- proof of existence of objects: union bound, Lovasz local lemma.

Price of using randomness

- The algorithm may be incorrect with some probability (Monto Carlo Algorithm)
- The algorithm may take a long time to terminate (Las Vegas Algorithm)

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Matrix Multiplication Verification

Input: 3 matrices $A, B, C \in \mathbb{Z}^{n\times n}$

Output: whether if $C = AB$

- trivial: compute $C' = AB$ and check if $C' = C$.
- time $=$ matrix multiplication time
  - naive algorithm: $O(n^3)$
  - Strassen's algorithm: $O(n^{2.81})$
  - Best known algorithm for matrix multiplication: $O(n^{2.3719})$.
- Freivald's algorithm: randomized algorithm with $O(n^2)$ time.

Freivald's Algorithm: one experiment

1: randomly choose a vector $r \in \{0, 1\}^n$
2: **return** $ABr = Cr$

Q: What is the running time of the algorithm?

- $(AB)r$: matrix-multiplication time
- $A(Br)$: $O(n^2)$ time

Analysis of correctness

- $AB = C$: algorithm outputs true with probability 1.
- $AB \neq C$: algorithm may incorrectly output true.

Lemma If $AB \neq C$, then the algorithm outputs false with probability at least $1/2$.

Lemma If $AB \neq C$, then the algorithm outputs false with probability at least $1/2$.

Proof.

- $D := C - AB \neq 0$ $\qquad Cr = ABr \iff Dr = 0$
- $\exists i, j \in [n], D_{i,j} \neq 0$

$$D_i r = \sum_{j'=1}^{n} D_{i,j'} r_{j'} = X + Y, \quad X = \sum_{j' \in [n], j' \neq j} D_{i,j'} r_{j'}, Y = D_{i,j} r_j$$

$$\begin{aligned}
\Pr[D_i r \neq 0] &= \Pr[Y \neq -X] \\
&= \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \Pr[Y \neq -x | X = x] \\
&= \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \Pr[D_{i,j} r_j \neq -x | X = x] \\
&\geq \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

□

- probabilities:

	true	false
$AB = C$	1	0
$AB \neq C$	$\leq 1/2$	$\geq 1/2$

Freivald's Algorithm: k experiments

1: **for** $t \leftarrow 1$ to k **do**
2: randomly choose a vector $r \in \{0,1\}^n$
3: **if** $ABr \neq Cr$ **then return false**
4: **return true**

- probabilities with k experiments:

	true	false
$AB = C$	1	0
$AB \neq C$	$\leq 1/2^k$	$\geq 1 - 1/2^k$

- to achieve δ probability of mistake, need $\log_2 \frac{1}{\delta} = O(\log \frac{1}{\delta})$ experiments.

- Frievald's algorithm is a Monta Carlo algorithm.

Def. A Monta Carlo algorithm is a randomized algorithm whose output may be incorrect with some probability.

- For a Monta Carlo algorithm that outputs true/false, we say the algorithm has one-sided error if it makes error only if the correct output is true (or false).

Def. A Las-Vegas algorithm is a randomized algorithm that always outputs a correct solution but has randomized running time.

Table: Comparisons between Monta Carlo and Las Vegas Algorithms.

	correctness	running time
Monta Carlo	may be wrong	usually has good worst-case running time
Las Vegas	always correct	may take a long time and usually only has good "expected running time"

Lemma Given a Las Vegas algorithm $\mathcal{A}$ with expected running time at most $T(n)$, we can design a Monta Carlo algorithm $\mathcal{A}'$ with worst-case running time $O(T(n))$ and error at most 0.99.

- 0.99 can be changed to any $c < 1$

Proof.

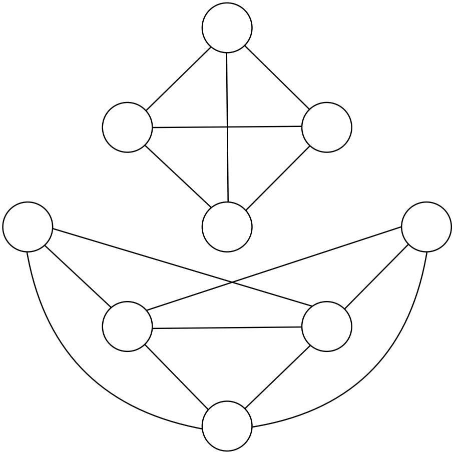
- run $\mathcal{A}$ for $100T(n)$ time
- if $\mathcal{A}$ terminated, output what $\mathcal{A}$ outputs
- otherwise, declare failure
- Markov Inequality:
  $\Pr[\mathcal{A}$ runs for more than $100T(n)$ time$] \leq 1/100$ $\square$

Outline

Global Min-Cut Problem

Input: a connected graph $G = (V, E)$

Output: the minimum number of edges whose removal will disconnect G



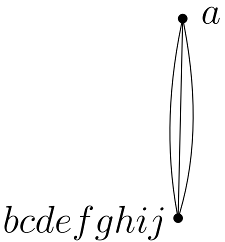
Solving Global Min-Cut Using s-t Min-Cut

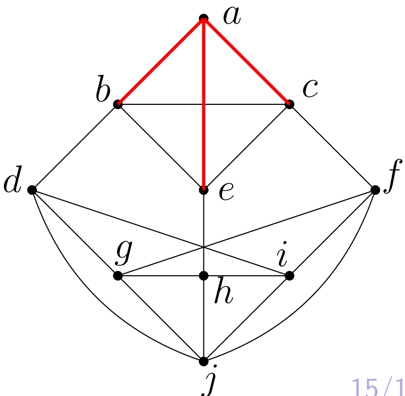
1: let G' be the directed graph obtained from G by replacing every edge with two anti-parallel edges
2: **for** a fixed $s \in V$ and every pair $t \in V \setminus \{s\}$ **do**
3: obtain the minimum cut separating s and t in G, by solving the maximum flow instance with graph G',source s and sink t
4: output the smallest minimum cut we found

- Time $= O(n) \times$ (Time for Maximum Flow)

Karger's Randomized Algorithm for Min-Cut

1: $G' = (V', E') \leftarrow G$
2: **while** $|V'| > 2$ **do**
3: pick $uv \in E'$ uniformly at random
4: contract uv in G', keeping parallel edges, but not self-loops
5: **return** the cut in G correspondent to E'





Obs. Contraction does not decrease size of min-cut.

Lemma If $G' = (V', E')$ has size of min-cut being c, then $|E'| \geq |V'|c/2$

Proof.

Every vertex will have degree at least c, and thus $2|E'| \geq |V'|c$. □

- let $C \subseteq E$ be a fixed min-cut of G
- an iteration fails if we chose some edge $e \in C$ to contract.

Coro. Focus on some iteration where we have the graph $G' = (V', E')$ with $n' = |V'|$ at the beginning. Suppose all previous iterations succeed. Then the probability this iteration fails is at most $\frac{c}{n'c/2} = \frac{2}{n'}$.

- The probability that the algorithm succeeds is at least

$$\left(1-\frac{2}{n}\right)\left(1-\frac{2}{n-1}\right)\left(1-\frac{2}{n-2}\right)\cdots\left(1-\frac{2}{3}\right)$$
$$=\frac{n-2}{n}\times\frac{n-3}{n-1}\times\frac{n-4}{n-2}\times\cdots\times\frac{1}{3}=\frac{2}{n(n-1)}$$

Coro. Any graph G has at most $\frac{n(n-1)}{2}$ distinct minimum cuts.

- $A := \frac{n(n-1)}{2}$: algorithm succeeds with probability at least $\frac{1}{A}$
- Running the algorithm for Ak times will increase the probability to

$$1-(1-\frac{1}{A})^{Ak} \geq 1-e^{-k}.$$

- To get a success probability of $1-\delta$, run the algorithm for $O(n^2 \log \frac{1}{\delta})$ times.

Equivalent Algorithm

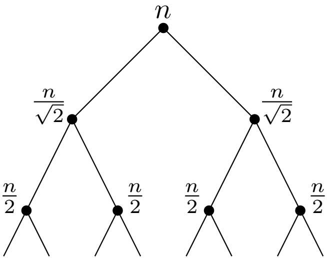
1: give every edge a weight in $[0,1]$ uniformly at random.
2: solve the MST on the graph G with the weights, using either Kruskal or Prim's algorithm
3: remove the heaviest edge in the MST,
4: let U and $V \setminus U$ be the vertex sets of two components
5: **return** the cut in G between U and $V \setminus U$

- run it once: time $= O(m + n\log n)$
- to get success probability $1-\delta$: time $= O(n^2(m + n\log n)\log\frac{1}{\delta})$

Karger-Stein: A Faster Algorithm

Karger-Stein($G = (V, E)$)

if $|V| \le 6$ **then return** min cut of G directly
repeat twice and return the smaller cut:
    run Karger(G) down to $\lceil n/\sqrt{2} \rceil$ vertices, to obtain G'
    consider the candidate cut returned by Karger-Stein(G')



- Running time:
  $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2)$
- $T(n) = O(n^2 \log n)$

Karger-Stein$(G=(V,E))$

if $|V| \le 6$ **then return** min cut of G directly
repeat twice and return the smaller cut:
    run Karger(G) down to $\lceil n/\sqrt{2} \rceil + 1$ vertices, to obtain G'
    consider the candidate cut returned by Karger-Stein(G')

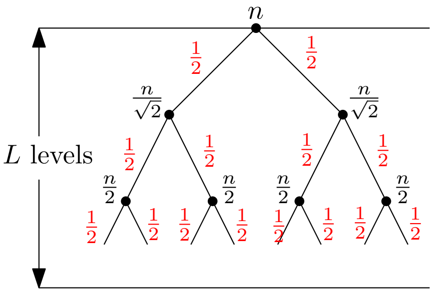
Analysis of Probability of Success

- running Karger(G) down to $\lceil n/\sqrt{2} \rceil + 1$ vertices, success probability is at least

$$\frac{n-2}{n} \times \frac{n-3}{n-1} \times \cdots \times \frac{\lceil n/\sqrt{2} \rceil}{\lceil n/\sqrt{2} \rceil + 2} = \frac{(\lceil n/\sqrt{2} \rceil + 1)\lceil n/\sqrt{2} \rceil}{n(n-1)}$$

$$\ge \frac{n^2/2 + n/\sqrt{2}}{n^2 - n} \ge \frac{1}{2}$$

- recursion for Probability: $P(n) \ge 1 - \left(1 - \frac{1}{2}P(\frac{n}{\sqrt{2}})\right)^2$



- every edge is chosen w.p $1/2$
- success if we choose some root-to-leaf path
- what is the success probability in terms of L?

Lemma $P_L \geq \frac{1}{L+1}$.

Proof.

- $L = 0$: a singleton, holds trivially.
- induction:

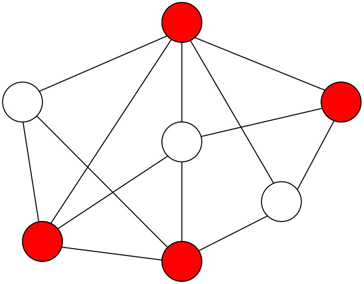
$$P_L = 1 - \left(1 - \frac{1}{2}P_{L-1}\right)^2 \geq 1 - \left(1 - \frac{1}{2L}\right)^2 = \frac{1}{L} - \frac{1}{4L^2}$$
$$= \frac{4L-1}{4L^2} \geq \frac{1}{L+1} \qquad \square$$

Karger-Stein$(G = (V, E))$

```
if |V| ≤ 6 then return min cut of G directly
repeat twice and return the smaller cut:
    run Karger(G) down to ⌈n/√2⌉ + 1 vertices, to obtain G'
    consider the candidate cut returned by Karger-Stein(G')
```

- Running time: $O(n^2 \log n)$
- Success probability: $\Omega\left(\frac{1}{\log n}\right)$
- Repeat $O(\log n)$ times can increase the probability to a constant

Outline



Vertex-Cover Problem

Input: $G = (V, E)$

Output: a vertex cover C with minimum $|C|$

- (The decision version of) vertex-cover is NP-complete.

Q: What if we are only interested in a vertex cover of size at most k, for some small number k?

Q: What if we are only interested in a vertex cover of size at most k, for some constant k?

- Motivation: if the minimum vertex cover is too big, then the solution becomes meaningless.
- Enumeration gives a $O(kn^{k+1})$-time algorithm.
- For moderately large k (e.g., $n = 1000, k = 10$), algorithm is impractical.

Lemma There is an algorithm with running time $O(2^k \cdot kn)$ to check if G contains a vertex cover of size at most k or not.

- Remark: m does not appear in the running time. Indeed, if $m > kn$, then there is no vertex cover of size k.

Vertex-Cover$(G' = (V', E'), k)$

if $|E'| = \emptyset$ **then return true**
if $k = 0$ **then return false**
pick any edge $(u, v) \in E'$
return Vertex-Cover$(G' \setminus u, k-1)$ or Vertex-Cover$(G' \setminus v, k-1)$

- $G' \setminus u$: the graph obtained from G' by removing u and its incident edges
- Correctness: if $(u, v) \in E'$, we must choose u or choose v to cover (u, v).
- Running time: 2^k recursions and each recursion has running time $O(kn)$.

Def. An problem is fixed parameterized tractable (FPT) with respect to a parameter k, if it can be solved in $f(k) \cdot \text{poly}(n)$ time, where n is the size of its input and $\text{poly}(n) = \bigcup_{t=0}^{\infty} O(n^t)$.

- Vertex cover is fixed parameterized tractable with respect to the size k of the optimum solution.

Outline

1. Randomized Algorithms

2. Finding Small Vertex Covers: Fixed Parameterized Tractability

3. Approximation Algorithms using Greedy
   - 2-Approximation Algorithm for Vertex Cover
   - f-Approximation for Set-Cover with Frequency f
   - $(\ln n + 1)$-Approximation for Set-Cover
   - $\left(1 - \frac{1}{e}\right)$-Approximation for Maximum Coverage

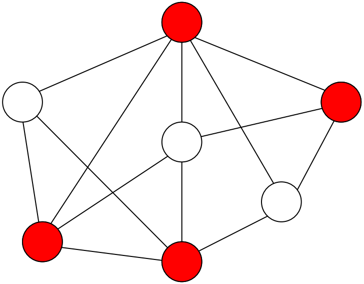
4. Approximation Using LP Rounding and Primal Dual

5. More Advanced Algorithms *

Outline

Vertex Cover Problem

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $C \subseteq V$ such that for every $(u, v) \in E$ then $u \in C$ or $v \in C$.



Vertex-Cover Problem

Input: $G = (V, E)$

Output: a vertex cover C with minimum $|C|$

First Try: A "Natural" Greedy Algorithm

Natural Greedy Algorithm for Vertex-Cover

$E' \leftarrow E, C \leftarrow \emptyset$
while $E' \neq \emptyset$ **do**
    let v be the vertex of the maximum degree in (V, E')
    $C \leftarrow C \cup \{v\}$,
    remove all edges incident to v from E'
return C

Theorem Greedy algorithm is an $(\ln n + 1)$-approximation for vertex-cover.

- We prove it for the more general set cover problem
- The logarithmic factor is tight for this algorithm

2-Approximation Algorithm for Vertex Cover

$E' \leftarrow E, C \leftarrow \emptyset$
while $E' \neq \emptyset$ **do**
    let (u, v) be any edge in E'
    $C \leftarrow C \cup \{u, v\}$
    remove all edges incident to u and v from E'
return C

- counter-intuitive: adding both u and v to C seems wasteful
- intuition for the 2-approximation ratio:
  - optimum solution C^* must cover edge (u, v), using either u or v
  - we select both, so we are always ahead of the optimum solution
  - we use at most 2 times more vertices than C^* does

2-Approximation Algorithm for Vertex Cover

1: $E' \leftarrow E, C \leftarrow \emptyset$
2: **while** $E' \neq \emptyset$ **do**
3:     let (u, v) be any edge in E'
4:     $C \leftarrow C \cup \{u, v\}$
5:     remove all edges incident to u and v from E'
6: **return** C

Theorem The algorithm is a 2-approximation algorithm for vertex-cover.

Proof.

- Let E' be the set of edges (u, v) considered in Step 3
- Observation: E' is a matching and $|C| = 2|E'|$
- To cover E', the optimum solution needs $|E'|$ vertices $\square$

Outline

Set Cover with Bounded Frequency f

Input: $U, |U| = n$: ground set

$S_1, S_2, \cdots, S_m \subseteq U$

every $j \in U$ appears in at most f subsets in $\{S_1, S_2, \cdots, S_m\}$

Output: minimum size set $C \subseteq [m]$ such that $\bigcup_{i \in C} S_i = U$

Vertex Cover $=$ Set Cover with Frequency 2

- edges $\Leftrightarrow$ elements
- vertices $\Leftrightarrow$ sets
- every edge (element) can be covered by 2 vertices (sets)

f-Approximation Algorithm for Set Cover with Frequency f

1: $C \leftarrow \emptyset$
2: **while** $\bigcup_{i \in C} S_i \neq U$ **do**
3: let e be any element in $U \setminus \bigcup_{i \in C} S_i$
4: $C \leftarrow C \cup \{i \in [m] : e \in S_i\}$
5: **return** C

Theorem The algorithm is a f-approximation algorithm.

Proof.

- Let U' be the set of all elements e considered in Step 3
- Observation: no set S_i contains two elements in U'
- To cover U', the optimum solution needs $|U'|$ sets
- $C \leq f \cdot |U'|$ $\square$

Outline

Set Cover

Input: $U, |U| = n$: ground set

$S_1, S_2, \cdots, S_m \subseteq U$

Output: minimum size set $C \subseteq [m]$ such that $\bigcup_{i \in C} S_i = U$

Greedy Algorithm for Set Cover

1: $C \leftarrow \emptyset, U' \leftarrow U$
2: **while** $U' \neq \emptyset$ **do**
3: choose the i that maximizes $|U' \cap S_i|$
4: $C \leftarrow C \cup \{i\}, U' \leftarrow U' \setminus S_i$
5: **return** C

- g: minimum number of sets needed to cover U

Lemma Let $u_t, t \in \mathbb{Z}_{\geq 0}$ be the number of uncovered elements after t steps. Then for every $t \geq 1$, we have

$$u_t \leq \left(1 - \frac{1}{g}\right) \cdot u_{t-1}.$$

Proof.

- Consider the g sets $S_1^*, S_2^*, \cdots, S_g^*$ in optimum solution
- $S_1^* \cup S_2^* \cup \cdots \cup S_g^* = U$
- at beginning of step t, some set in $S_1^*, S_2^*, \cdots, S_g^*$ must contain $\geq \frac{u_{t-1}}{g}$ uncovered elements
- $u_t \leq u_{t-1} - \frac{u_{t-1}}{g} = \left(1 - \frac{1}{g}\right) u_{t-1}$. $\square$

Proof of $(\ln n + 1)$-approximation.

- Let $t = \lceil g \cdot \ln n \rceil$. $u_0 = n$. Then
$$u_t \le \left(1 - \frac{1}{g}\right)^{g \cdot \ln n} \cdot n < e^{-\ln n} \cdot n = n \cdot \frac{1}{n} = 1.$$
- So $u_t = 0$, approximation ratio $\le \frac{\lceil g \cdot \ln n \rceil}{g} \le \ln n + 1$. $\square$

- A more careful analysis gives a H_n-approximation, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the n-th harmonic number.
- $\ln(n+1) < H_n < \ln n + 1$.

$(1-c)\ln n$-hardness for any $c = \Omega(1)$

Let $c > 0$ be any constant. There is no polynomial-time $(1-c)\ln n$-approximation algorithm for set-cover, unless

- NP $\subseteq$ quasi-poly-time, [Lund, Yannakakis 1994; Feige 1998]
- P $=$ NP. [Dinur, Steuer 2014]

Outline

- set cover: use smallest number of sets to cover all elements.
- maximum coverage: use k sets to cover maximum number of elements

Maximum Coverage

Input: $U, |U| = n$: ground set,
$S_1, S_2, \cdots, S_m \subseteq U$, $\qquad k \in [m]$

Output: $C \subseteq [m], |C| = k$ with the maximum $\bigcup_{i \in C} S_i$

Greedy Algorithm for Maximum Coverage

$C \leftarrow \emptyset, U' \leftarrow U$
for $t \leftarrow 1$ to k **do**
    choose the i that maximizes $|U' \cap S_i|$
    $C \leftarrow C \cup \{i\}, U' \leftarrow U' \setminus S_i$
return C

Theorem Greedy algorithm gives $(1-\frac{1}{e})$-approximation for maximum coverage.

Proof.

- o: max. number of elements that can be covered by k sets.
- p_t: #(covered elements) by greedy algorithm after step t
- $p_t \geq p_{t-1} + \dfrac{o - p_{t-1}}{k}$
- $o - p_t \leq o - p_{t-1} - \frac{o-p_{t-1}}{k} = \left(1 - \frac{1}{k}\right)(o - p_{t-1})$
- $o - p_k \leq \left(1 - \frac{1}{k}\right)^k (o - p_0) \leq \frac{1}{e} \cdot o$
- $p_k \geq \left(1 - \frac{1}{e}\right) \cdot o$ $\square$

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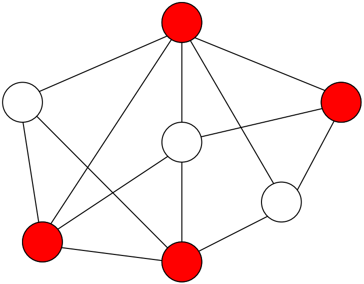
Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$.



Weighted Vertex-Cover Problem

Input: $G = (V, E)$ with vertex weights $\{w_v\}_{v \in V}$

Output: a vertex cover S with minimum $\sum_{v \in S} w_v$

Integer Programming for Weighted Vertex Cover

- For every $v \in V$, let $x_v \in \{0,1\}$ indicate whether we select v in the vertex cover S
- The integer programming for weighted vertex cover:

$$
\begin{array}{lrll}
(\text{IP}_{\text{WVC}}) & \min & \displaystyle\sum_{v \in V} w_v x_v & \text{s.t.} \\
& & x_u + x_v \geq 1 & \forall (u,v) \in E \\
& & x_v \in \{0,1\} & \forall v \in V
\end{array}
$$

- $(\text{IP}_{\text{WVC}}) \Leftrightarrow$ weighted vertex cover
- Thus it is NP-hard to solve integer programmings in general

- Integer programming for WVC:

$$(\mathsf{IP}_{\mathsf{WVC}}) \qquad \min \quad \sum_{v \in V} w_v x_v \qquad \text{s.t.}$$
$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$
$$x_v \in \{0,1\} \qquad \forall v \in V$$

- Linear programming relaxation for WVC:

$$(\mathsf{LP}_{\mathsf{WVC}}) \qquad \min \quad \sum_{v \in V} w_v x_v \qquad \text{s.t.}$$
$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$
$$x_v \in [0,1] \qquad \forall v \in V$$

- let IP $=$ value of $(\mathsf{IP}_{\mathsf{WVC}})$, LP $=$ value of $(\mathsf{LP}_{\mathsf{WVC}})$
- Then, LP $\leq$ IP

Algorithm for Weighted Vertex Cover

Algorithm for Weighted Vertex Cover

1: Solving (LP_{WVC}) to obtain a solution $\{x^*_u\}_{u\in V}$
2: Thus, $\text{LP} = \sum_{u\in V} w_u x^*_u \leq \text{IP}$
3: Let $S = \{u \in V : x_u \geq 1/2\}$ and output S

Lemma S is a vertex cover of G.

Proof.

- Consider any edge $(u, v) \in E$: we have $x^*_u + x^*_v \geq 1$
- Thus, either $x^*_u \geq 1/2$ or $x^*_v \geq 1/2$
- Thus, either $u \in S$ or $v \in S$. $\square$

Algorithm for Weighted Vertex Cover

Algorithm for Weighted Vertex Cover

1: Solving (LP_{WVC}) to obtain a solution $\{x^*_u\}_{u \in V}$
2: Thus, $\text{LP} = \sum_{u \in V} w_u x^*_u \leq \text{IP}$
3: Let $S = \{u \in V : x_u \geq 1/2\}$ and output S

Lemma S is a vertex cover of G.

Lemma $\text{cost}(S) := \sum_{u \in S} w_u \leq 2 \cdot \text{LP}$.

Proof.

$$\begin{aligned} \text{cost}(S) &= \sum_{u \in S} w_u \leq \sum_{u \in S} w_u \cdot 2x^*_u = 2 \sum_{u \in S} w_u \cdot x^*_u \\ &\leq 2 \sum_{u \in V} w_u \cdot x^*_u = 2 \cdot \text{LP}. \end{aligned}$$

□

Algorithm for Weighted Vertex Cover

Algorithm for Weighted Vertex Cover

1: Solving (LP$_{\text{WVC}}$) to obtain a solution $\{x_u^*\}_{u \in V}$
2: Thus, LP $= \sum_{u \in V} w_u x_u^* \leq$ IP
3: Let $S = \{u \in V : x_u^* \geq 1/2\}$ and output S

Lemma S is a vertex cover of G.

Lemma cost$(S) := \sum_{u \in S} w_u \leq 2 \cdot$ LP.

Theorem Algorithm is a 2-approximation algorithm for WVC.

Proof.

cost$(S) \leq 2 \cdot$ LP $\leq 2 \cdot$ IP $= 2 \cdot$ cost(best vertex cover). □

Outline

LP Relaxation

$$\min \quad \sum_{v \in V} w_v x_v$$

$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$

$$x_v \geq 0 \qquad \forall v \in V$$

Dual LP

$$\max \quad \sum_{e \in E} y_e$$

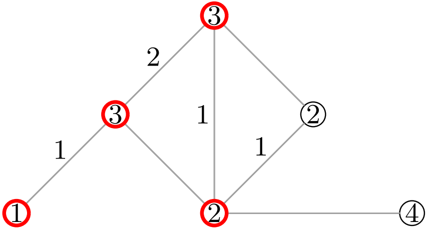
$$\sum_{e \in \delta(v)} y_e \leq w_v \qquad \forall v \in V$$

$$y_e \geq 0 \qquad \forall e \in E$$

- Algorithm constructs integral primal solution x and dual solution y simultaneously.

Primal-Dual Algorithm for Weighted Vertex Cover

$x \leftarrow 0, y \leftarrow 0$, all edges said to be uncovered
while there exists at least one uncovered edge **do**
    take such an edge e arbitrarily
    increasing y_e until the dual constraint for one end-vertex v of e becomes tight
    $x_v \leftarrow 1$, claim all edges incident to v are covered
return x



Lemma

1. x satisfies all primal constraints
2. y satisfies all dual constraints
3. $P \leq 2D \leq 2D^* \leq 2 \cdot \mathrm{opt}$

   $P := \sum_{v \in V} x_v$: value of x

   $D := \sum_{e \in E} y_e$: value of y

   D^* : dual LP value

Proof of $P \le 2D$.

$$P = \sum_{v \in V} w_v x_v \le \sum_{v \in V} x_v \sum_{e \in \delta(v)} y_e = \sum_{(u,v) \in E} y_{(u,v)} (x_u + x_v)$$
$$\le 2 \sum_{e \in E} y_e = 2D. \qquad \square$$

- a more general framework: construct an arbitrary maximal dual solution y; choose the vertices whose dual constraints are tight
- y is maximal: increasing any coordinate y_e makes y infeasible

- primal-dual algorithms do not need to solve LPs
- LPs are used in analysis only
- faster than LP-rounding algorithm in general

Outline

Unrelated Machine Scheduling

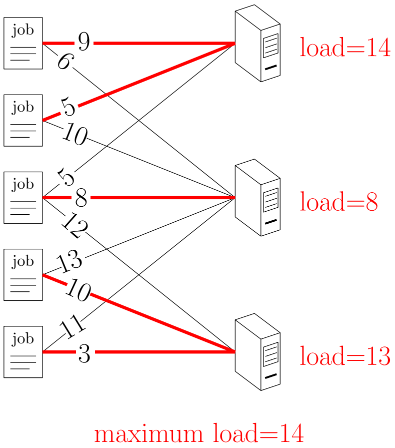
Input: $J, |J| = n$: jobs

$M, |M| = m$: machines

p_{ij}: processing time of job j on machine i

Output: assignment $\sigma : J \mapsto M$:, so as to minimize makespan:

$$\max_{i \in M} \sum_{j \in \sigma^{-1}(i)} p_{ij}$$

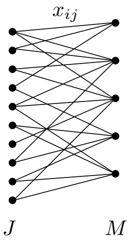


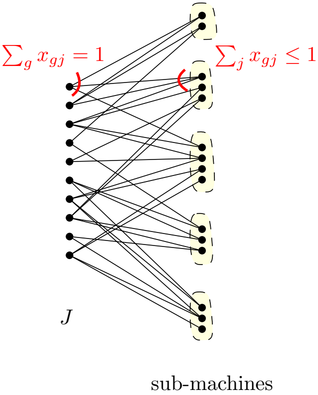
- Assumption: we are given a target makespan T, and $p_{ij} \in [0, T] \cup \{\infty\}$

- x_{ij}: fraction of j assigned to i

$$\begin{aligned} \sum_i x_{ij} &= 1 && \forall j \in J \\ \sum_j p_{ij} x_{ij} &\le T && \forall i \in M \\ x_{ij} &\ge 0 && \forall ij \end{aligned}$$

2-Approximate Rounding Algorithm of Shmoys-Tardos





Obs. x between J and sub-machines is a point in the bipartite-matching polytope, where all jobs in J are matched.

- Recall bipartite matching polytope is integral.
- x is a convex combination of matchings.
- Any matching in the combination covers all jobs J.

Lemma Any matching in the combination gives an schedule of makespan $\leq 2T$.

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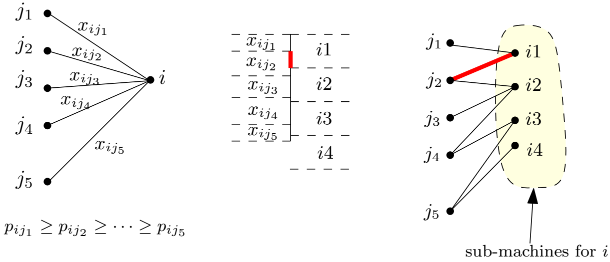


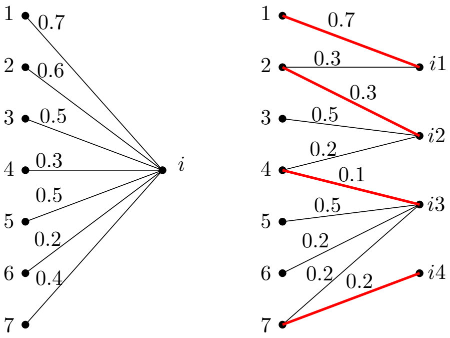
Proof.

- focus on machine i, let $i_1, i_2, \cdots, i_a$ be the sub-machines for i
- assume job k_t is assigned to sub-machine i_t.

$$\text{(load on } i\text{)} = \sum_{t=1}^{a} p_{ik_t} \leq p_{ik_1} + \sum_{t=2}^{a} \sum_{j} x_{i_{t-1}j} \cdot p_{ij}$$

$$\leq p_{ik_1} + \sum_{j} x_{ij} p_{ij} \leq T + T = 2T. \qquad \square$$

- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:
  - $1 \to i1, 2 \to i2$
  - $4 \to i3, 7 \to i4$



$$p_1 \leq T$$
$$p_2 \leq 0.7p_1 + 0.3p_2$$
$$p_4 \leq 0.3p_2 + 0.5p_3 + 0.2p_4$$
$$p_7 \leq 0.1p_4 + 0.5p_5 + 0.2p_6 + 0.2p_7$$

$$\begin{aligned} & p_1 + p_2 + p_4 + p_7 \leq T + (0.7p_1 + 0.3p_2) + (0.3p_2 + 0.5p_3 + 0.2p_4) \\ & \quad + (0.1p_4 + 0.5p_5 + 0.2p_6 + 0.2p_7) \\ & \leq T + (0.7p_1 + 0.6p_2 + 0.5p_3 + 0.3p_4 + 0.5p_5 + 0.2p_6 + 0.4p_7) \\ & \leq T + T = 2T \end{aligned}$$

Outline

Outline

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

29	82	75	64	38	45	94	69	25	76	15	92	37	17	85

29	38	45	25	15	37	17	64	82	75	94	92	69	76	85

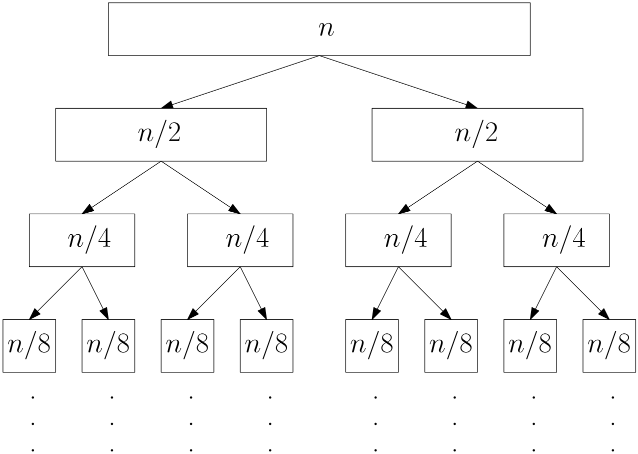
25	15	17	29	38	45	37	64	82	75	94	92	69	76	85

Quicksort

quicksort(A, n)

1: if $n \leq 1$ then return A
2: $x \leftarrow$ lower median of A
3: $A_L \leftarrow$ elements in A that are less than x \\ Divide
4: $A_R \leftarrow$ elements in A that are greater than x \\ Divide
5: $B_L \leftarrow$ quicksort$(A_L, A_L.\text{size})$ \\ Conquer
6: $B_R \leftarrow$ quicksort$(A_R, A_R.\text{size})$ \\ Conquer
7: $t \leftarrow$ number of times x appear A
8: return the array obtained by concatenating B_L, the array containing t copies of x, and B_R

- Recurrence $T(n) \leq 2T(n/2) + O(n)$
- Running time $= O(n \log n)$



- Each level has total running time $O(n)$
- Number of levels $= O(\log n)$
- Total running time $= O(n \log n)$

Randomized Quicksort Algorithm

quicksort(A, n)

```
if n ≤ 1 then return A
x ← a random element of A (x is called a pivot)
A_L ← elements in A that are less than x                    \\ Divide
A_R ← elements in A that are greater than x                 \\ Divide
B_L ← quicksort(A_L, A_L.size)                              \\ Conquer
B_R ← quicksort(A_R, A_R.size)                              \\ Conquer
t ← number of times x appear A
return the array obtained by concatenating B_L, the array
containing t copies of x, and B_R
```

Variant of Randomized Quicksort Algorithm

quicksort(A, n)

if $n \leq 1$ then return A
repeat
    $x \leftarrow$ a random element of A (x is called a pivot)
    $A_L \leftarrow$ elements in A that are less than x \\ Divide
    $A_R \leftarrow$ elements in A that are greater than x \\ Divide
until A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$
$B_L \leftarrow$ quicksort$(A_L, A_L$.size$)$ \\ Conquer
$B_R \leftarrow$ quicksort$(A_R, A_R$.size$)$ \\ Conquer
$t \leftarrow$ number of times x appear A
return the array obtained by concatenating B_L, the array containing t copies of x, and B_R

Analysis of Variant

1: $x \leftarrow$ a random element of A
2: $A_L \leftarrow$ elements in A that are less than x
3: $A_R \leftarrow$ elements in A that are greater than x

Q: What is the probability that A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$?

A: At least $1/2$

Analysis of Variant

repeat
    $x \leftarrow$ a random element of A
    $A_L \leftarrow$ elements in A that are less than x
    $A_R \leftarrow$ elements in A that are greater than x
until A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$

Q: What is the expected number of iterations the above procedure takes?

A: At most 2

- Suppose an experiment succeeds with probability $p \in (0, 1]$, independent of all previous experiments.

```
1: repeat
2:     run an experiment
3: until the experiment succeeds
```

Lemma The expected number of experiments we run in the above procedure is $1/p$.

Lemma The expected number of experiments we run in the above procedure is $1/p$.

Proof

$$\begin{aligned}
\text{Expectation} &= p + (1-p)p \times 2 + (1-p)^2 p \times 3 + (1-p)^3 p \times 4 \\
&\quad + \cdots \\
&= p \sum_{i=1}^{\infty} (1-p)^{i-1} i \quad = \quad p \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} (1-p)^{i-1} \\
&= p \sum_{j=1}^{\infty} (1-p)^{j-1} \frac{1}{1-(1-p)} \quad = \quad \sum_{j=1}^{\infty} (1-p)^{j-1} \\
&= (1-p)^0 \frac{1}{1-(1-p)} = 1/p
\end{aligned}$$

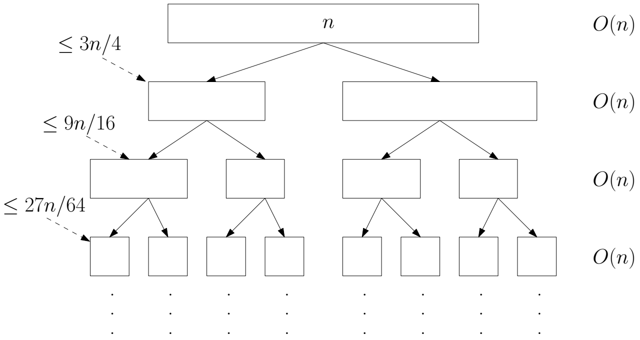
Variant Randomized Quicksort Algorithm

quicksort(A, n)

if $n \le 1$ then return A
repeat
    $x \leftarrow$ a random element of A (x is called a pivot)
    $A_L \leftarrow$ elements in A that are less than x \\ Divide
    $A_R \leftarrow$ elements in A that are greater than x \\ Divide
until A_L.size $\le 3n/4$ and A_R.size $\le 3n/4$
$B_L \leftarrow$ quicksort$(A_L, A_L$.size$)$ \\ Conquer
$B_R \leftarrow$ quicksort$(A_R, A_R$.size$)$ \\ Conquer
$t \leftarrow$ number of times x appear A
return the array obtained by concatenating B_L, the array containing t copies of x, and B_R

Analysis of Variant

- Divide and Combine: takes $O(n)$ time
- Conquer: break an array of size n into two arrays, each has size at most $3n/4$. Recursively sort the 2 sub-arrays.



- Number of levels $\le \log_{4/3} n = O(\log n)$

Randomized Quicksort Algorithm

quicksort(A, n)

if $n \leq 1$ then return A
$x \leftarrow$ a random element of A (x is called a pivot)
$A_L \leftarrow$ elements in A that are less than x \\ Divide
$A_R \leftarrow$ elements in A that are greater than x \\ Divide
$B_L \leftarrow$ quicksort$(A_L, A_L.\text{size})$ \\ Conquer
$B_R \leftarrow$ quicksort$(A_R, A_R.\text{size})$ \\ Conquer
$t \leftarrow$ number of times x appear A
return the array obtained by concatenating B_L, the array containing t copies of x, and B_R

- Intuition: the quicksort algorithm should be better than the variant.

Analysis of Randomized Quicksort Algorithm

- $T(n)$: an upper bound on the expected running time of the randomized quicksort algorithm on n elements
- Assuming we choose the element of rank i as the pivot.
- The left sub-instance has size at most $i-1$
- The right sub-instance has size at most $n-i$
- Thus, the expected running time in this case is $\big(T(i-1)+T(n-i)\big)+O(n)$
- Overall, we have

$$
\begin{aligned}
T(n) &= \frac{1}{n}\sum_{i=1}^{n}\big(T(i-1)+T(n-i)\big)+O(n) \\
&= \frac{2}{n}\sum_{i=0}^{n-1}T(i)+O(n)
\end{aligned}
$$

- Can prove $T(n) \le c(n\log n)$ for some constant c by reduction

Analysis of Randomized Quicksort Algorithm

The induction step of the proof:

$$
\begin{aligned}
T(n) &\le \frac{2}{n}\sum_{i=0}^{n-1} T(i) + c'n \le \frac{2}{n}\sum_{i=0}^{n-1} ci\log i + c'n \\
&\le \frac{2c}{n}\left(\sum_{i=0}^{\lfloor n/2\rfloor -1} i\log\frac{n}{2} + \sum_{i=\lfloor n/2\rfloor}^{n-1} i\log n\right) + c'n \\
&\le \frac{2c}{n}\left(\frac{n^2}{8}\log\frac{n}{2} + \frac{3n^2}{8}\log n\right) + c'n \\
&= c\left(\frac{n}{4}\log n - \frac{n}{4} + \frac{3n}{4}\log n\right) + c'n \\
&= cn\log n - \frac{cn}{4} + c'n \le cn\log n \qquad \text{if } c \ge 4c'
\end{aligned}
$$

Indirect Analysis Using Number of Comparisons

- Running time $= O(\text{number of comparisons})$
- $\forall 1 \leq i < j \leq n$, $D_{i,j}$ indicates if we compared the i-th smallest element with the j-th smallest element
- number of comparisons $= \sum_{1 \leq i < j \leq n} D_{i,j}$

Lemma $\mathbb{E}[D_{i,j}] = \frac{2}{j-i+1}$.

Proof.

- A': sorted array for A. Focus on $A'[i..j]$.
- pivot outside $A'[i]$: $A'[i \cdots j]$ will be passed to left or right recursion; go to that recursion
- pivot inside $A'[i]$: $A'[i]$ and $A'[j]$ will be separated; call this critical recursion
- $A[i]$ and $A[j]$ are compared in the critical recursion with probability $\frac{2}{j-i+1}$.

□

Randomized Selection Algorithm

selection(A, n, i)

if $n = 1$ **then return** A
$x \leftarrow$ random element of A (called pivot)
$A_L \leftarrow$ elements in A that are less than x ▷ Divide
$A_R \leftarrow$ elements in A that are greater than x ▷ Divide
if $i \leq A_L$.size **then**
    return selection$(A_L, A_L.\text{size}, i)$ ▷ Conquer
else if $i > n - A_R$.size **then**
    return selection$(A_R, A_R.\text{size}, i - (n - A_R.\text{size}))$ ▷ Conquer
else
    return x

- expected running time $= O(n)$

Randomized Selection

- $X_j, j = 0, 1, 2, \cdots$: the size of A in the j-th recursion

$$
\begin{aligned}
\mathbb{E}[X_{j+1}|X_j = n'] &\leq \frac{1}{n'}\sum_{k=1}^{n'} \max\{k-1, n'-k\} \\
&\leq \frac{1}{n'}\left(\int_{k=0}^{n'/2}(n'-k)\mathrm{d}k + \int_{k=n'/2}^{n'} k\mathrm{d}k\right) \\
&= \frac{1}{n'}\left(\left(n'k - \frac{k^2}{2}\right)\Big|_0^{n'/2} + \frac{k^2}{2}\Big|_{n'/2}^{n'}\right) \\
&= \frac{1}{n'}\left(\frac{n'^2}{2} - \frac{n'^2}{8} + \frac{n'^2}{2} - \frac{n'^2}{8}\right) = \frac{3n'}{4}.
\end{aligned}
$$

- $\mathbb{E}[X_{j+1}] \leq \frac{3}{4}\mathbb{E}[X_j]$
- $X_0 = n \quad \Longrightarrow \quad \mathbb{E}[X_j] \leq \left(\frac{3}{4}\right)^j n$

$$\begin{aligned}
&\mathbb{E}[\text{running time of randomized selection}] \\
&\leq \mathbb{E}\left[O(1)\sum_{j=0}^{\infty} X_j\right] \leq O(1)\sum_{j=0}^{\infty}\mathbb{E}[X_j] \\
&\leq O(1)\sum_{j=0}^{\infty}\left(\frac{3}{4}\right)^j n = O(1)\cdot 4n = O(n).
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left[\text{number of comparisons}\right] &= \mathbb{E}\left[\sum_{1 \le i < j \le n} D_{i,j}\right] \\
&= \sum_{1 \le i < j \le n} \mathbb{E}\left[D_{i,j}\right] = 2 \sum_{1 \le i < j \le n} \frac{1}{j - i + 1} \\
&\le 2n \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \\
&= \Theta\left(n \log n\right).
\end{aligned}
$$

Outline

Approximation Algorithms

An algorithm for an optimization problem is an α-approximation algorithm, if it runs in polynomial time, and for any instance to the problem, it outputs a solution whose cost (or value) is within an α-factor of the cost (or value) of the optimum solution.

- opt: cost (or value) of the optimum solution
- sol: cost (or value) of the solution produced by the algorithm
- α: approximation ratio
- For minimization problems:
  - $\alpha \geq 1$ and we require $\text{sol} \leq \alpha \cdot \text{opt}$
- For maximization problems, there are two conventions:
  - $\alpha \leq 1$ and we require $\text{sol} \geq \alpha \cdot \text{opt}$
  - $\alpha \geq 1$ and we require $\text{sol} \geq \text{opt}/\alpha$

Max 3-SAT

Input: n boolean variables $x_1, x_2, \cdots, x_n$

m clauses, each clause is a disjunction of 3 literals from 3 distinct variables

Output: an assignment so as to satisfy as many clauses as possible

Example:

- clauses: $x_2 \vee \neg x_3 \vee \neg x_4, \quad x_2 \vee x_3 \vee \neg x_4,$
  $\neg x_1 \vee x_2 \vee x_4, \quad x_1 \vee \neg x_2 \vee x_3, \quad \neg x_1 \vee \neg x_2 \vee \neg x_4$
- We can satisfy all the 5 clauses: $x = (1, 1, 1, 0, 1)$

Randomized Algorithm for Max 3-SAT

- Simple idea: randomly set each variable $x_u = 1$ with probability $1/2$, independent of other variables

Lemma Let m be the number of clauses. Then, in expectation, $\frac{7}{8}m$ number of clauses will be satisfied.

Proof.

- for each clause C_j, let $Z_j = 1$ if C_j is satisfied and 0 otherwise
- $Z = \sum_{j=1}^{m} Z_j$ is the total number of satisfied clauses
- $\mathbb{E}[Z_j] = 7/8$: out of 8 possible assignments to the 3 variables in C_j, 7 of them will make C_j satisfied
- $\mathbb{E}[Z] = \mathbb{E}\left[\sum_{j=1}^{m} Z_j\right] = \sum_{j=1}^{m} \mathbb{E}[Z_j] = \sum_{j=1}^{m} \frac{7}{8} = \frac{7}{8}m$, by linearity of expectation. $\square$

Randomized Algorithm for Max 3-SAT

Lemma Let m be the number of clauses. Then, in expectation, $\frac{7}{8}m$ number of clauses will be satisfied.

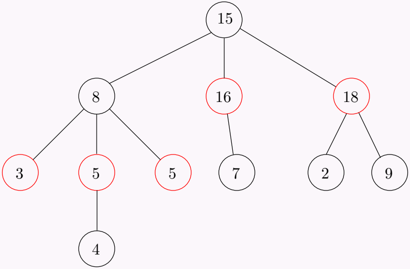
- Since the optimum solution can satisfy at most m clauses, lemma gives a randomized $7/8$-approximation for Max-3-SAT.

Theorem ([Hastad 97]) Unless P $=$ NP, there is no ρ-approximation algorithm for MAX-3-SAT for any $\rho > 7/8$.

Outline

- Many NP-hard problems on general graphs are easy on trees.
- Greedy algorithms: independent set, vertex cover, dominating set,
- Dynamic programming: weighted versions of above problems

Example: Maximum-Weight Independent Set



- dynamic programming:
- $f[i, 0]$: optimum value in tree i when i is not chosen
- $f[i, 1]$: optimum value in tree i

- Reason why many problems can be solved using DP on trees: the child-trees of a vertex i are only connected through i.

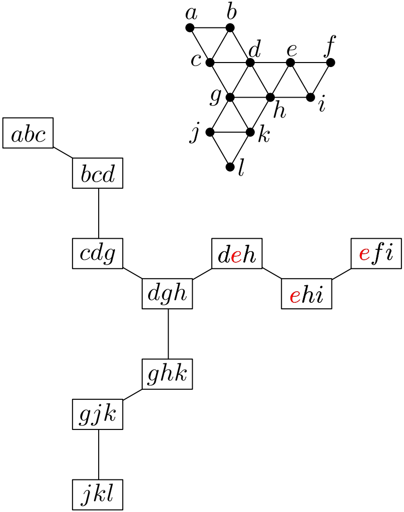
Bounded-Tree-Width Graphs

Def. A tree decomposition of a graph $G = (V, E)$ consists of

- a tree T with node set U, and
- a subset $V_t \subseteq V$ for every $t \in U$, which we call the bag for t,

satisfying the following properties:

- (Vertex Coverage) Every $v \in V$ appears in at least one bag.
- (Edge Coverage) For every $(u, v) \in E$, some bag contains both u and v.
- (Coherence) For every $u \in V$, the nodes $t \in U : u \in V_t$ induce a connected sub-graph of T.

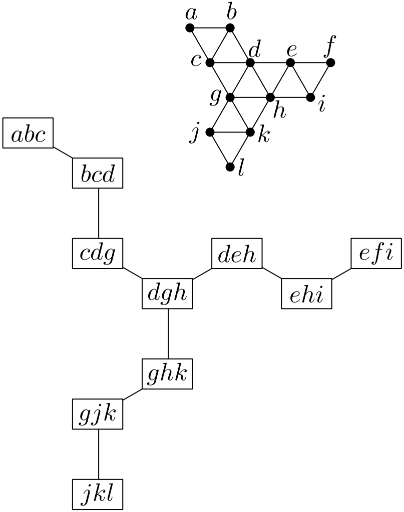


Bounded-Tree-Width Graphs

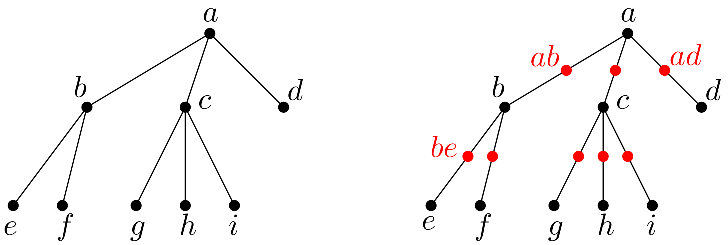
Def. The tree-width of the tree-decomposition $(T, (V_t)_{t \in U})$ is defined as $\max_{t \in U} |V_t| - 1$.

Def. The tree-width of a graph $G = (V, E)$, denoted as $\mathrm{tw}(G)$, is the minimum tree-width of a tree decomposition $(T, (V_t)_{t \in U})$ of G.

- The graph on the top right has tree-width 2.



Obs. A (non-empty) tree has tree-width 1.



Lemma A graph has tree-width 1 if and only if it is a (non-empty) forest.

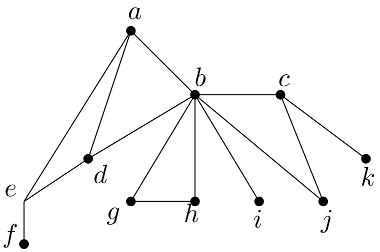
- Many problems on graphs with small tree-width can be solved using dynamic programming.
- Typically, the running time will be exponential in $\mathrm{tw}(G)$.

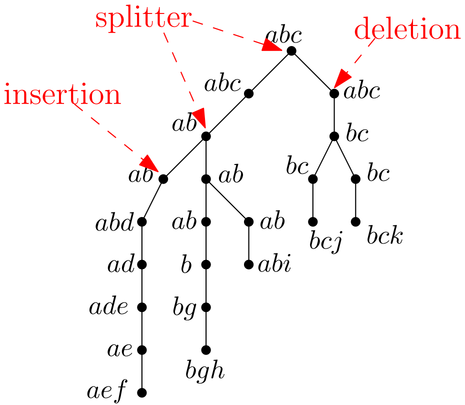
Example: Maximum Weight Independent Set

- given $G = (V, E)$, a tree-decomposition $(T, (V_t)_{t \in U})$ of G with tree-width tw.
- vertex weights $w \in \mathbb{R}^V_{>0}$.
- find an independent set S of G with the maximum total weight.

Assumption: every node in T has at most 2 children. Moreover, every internal nodes in T is one of the following types:

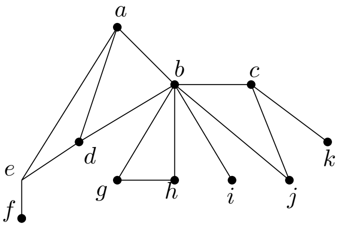
- Splitter: a node t with two children t' and t'', $V_t = V_{t'} = V_{t''}$
- Insertion node: a node t with one child t', $\exists u \notin V_t, V_{t'} = V_t \cup \{u\}$
- Deletion node: a node t with one child t', $\exists u \in V_t, V_{t'} = V_t \setminus \{u\}$

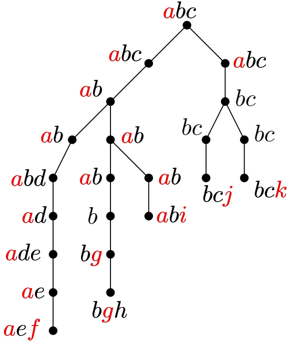




Def. Given a graph $G = (V, E)$, and a tree decomposition $(T, (V_t)_{t \in U})$, a valid labeling of T is a vector $(R_t)_{t \in U}$ of sets, one for every node t, such that the following conditions hold.

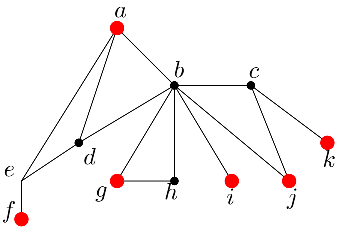
- $R_t \subseteq V_t, \forall t \in U$, and R_t is an independent set in G
- $R_t = R_{t'} = R_{t''}$ for a S-node t, and its two children t', t''.
- $R_{t'} \setminus \{u\} = R_t$ for an I-node t and its child t' with $V_{t'} = V_t \cup \{u\}$.
- $R_{t'} = R_t \setminus \{u\}$ for a D-node t and its child t' with $V_{t'} = V_t \setminus \{u\}$.

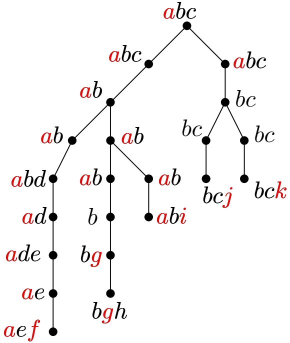




Lemma If S is an IS of G, then $(R_t := S \cap V_t)_{t \in U}$ is a valid labeling.

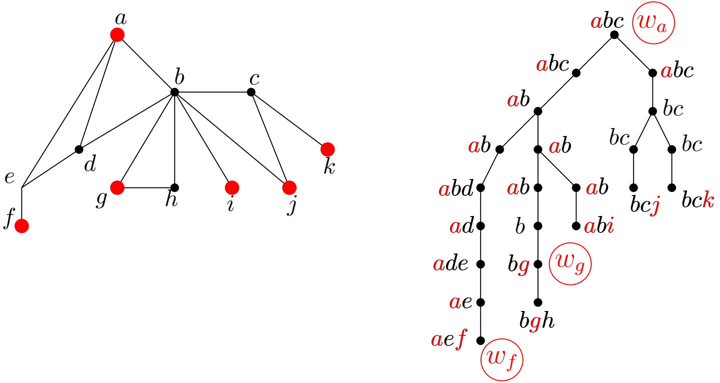
Lemma If $(R_t)_{t \in U}$ is a valid labeling, then $\bigcup_t R_t$ is an IS.





- Therefore, there is an one-to-one mapping between independent sets and valid labelings.

- For every $t \in U$, every $R \subseteq V_t$ that is an IS in G (we call R a label for t), we define a weight $w_t(R)$.
- for the root t and a label R for t, $w_t(R) = \sum_{v \in R} w_r$.
- for an insertion node t with the child t' such that $V_{t'} = V_t \cup \{u\}$, a label R for t', we define $w_{t'}(R) = w_u$ if $u \in R$ and 0 otherwise.
- For all other cases, the weights are defined as 0



- Problem: find a valid labeling for T with maximum weight

Dynamic Programming

- $\forall t \in U$, a label R for t: let $f(t, R)$ be the maximum weight of a valid (partial) labeling for the sub-tree of T rooted at t.

$$f(t,R) := \begin{cases} w_t(R) & t \text{ is a leaf} \\ w_t(R) + f(t', R) + f(t'', R) & \\ \quad t \text{ is an S-node with children } t' \text{ and } t'' & \\ w_t(R) + \max\{f(t', R), f(t', R \cup \{u\})\} & \\ \quad t \text{ is I-node w. child } t',\ V_{t'} = V_t \cup \{u\} & \\ w_t(R) + f(t', R \setminus \{u\}) & \\ \quad t \text{ is D-node w. child } t',\ V_{t'} = V_t \setminus \{u\} & \end{cases}$$

- In I-node case, if $R \cup \{u\}$ is an invalid label, then $f(t, R \cup \{u\}) = -\infty$.

- The running time of the dynamic programming: $O\big(2^{\text{tw}} \cdot \text{tw} \cdot n\big)$.
- It is efficient when tw is $O(\log n)$.

Q: Suppose we are only given G with tree-width tw, how can we find a tree-decomposition of width tw?

- This is an NP-hard problem.
- We can achieve a weaker goal: find a tree-decomposition of with at most 4tw in time $f(\text{tw}) \cdot \text{poly}(n)$, where $f(\text{tw})$ is a function of tw.
- If $\text{tw} = O(1)$, the algorithm runs in polynomial time.
- The constant 4 is acceptable.

Outline

Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes} \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

- Motivation: you have budget W, and want to buy a subset of items of maximum total value

Greedy Algorithm

sort items according to non-increasing order of v_i/w_i
for each item in the ordering **do**
    take the item if we have enough budget

- Bad example: $W = 100, n = 2, w = (1, 100), v = (1.1, 100)$.
- Optimum takes item 2 and greedy takes item 1.

DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \cdots, i\}$.

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max \left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W' - w_i] + v_i \end{array} \right\} & i > 0, w_i \le W' \end{cases}$$

- Running time of the algorithm is $O(nW)$.

Q: Is this a polynomial time?

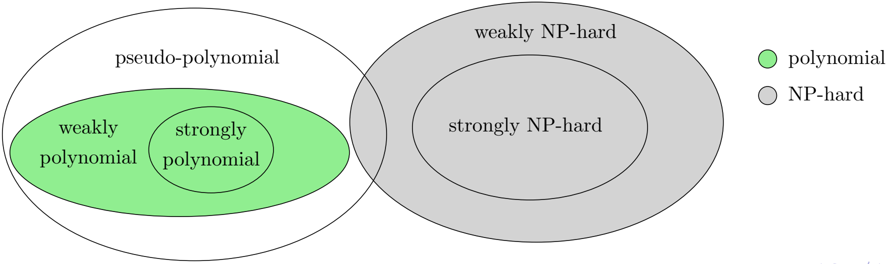
A: No.

- The input size is polynomial in n and $\log W$; running time is polynomial in n and W.
- The running time is pseudo-polynomial.

- n: number of integers W: maximum value of all integers

- pseudo-polynomial time: $\mathrm{poly}(n, W)$ (e.g., DP for Knapsack)
- weakly polynomial time: $\mathrm{poly}(n, \log W)$ (e.g., Euclidean Algorithm for Greatest Common Divisor)
- strongly polynomial time: $\mathrm{poly}(n)$ time, assuming basic operations on integers taking $O(1)$ time (e.g., Kruskal's)

- weakly NP-hard: NP-hard to solve in time $\mathrm{poly}(n, \log W)$
- strongly NP-hard: NP-hard even if $W = \mathrm{poly}(n)$



Idea for improving the running time to polynomial

- If we make weights upper bounded by $\text{poly}(n)$, then pseudo-polynomial time becomes polynomial time
- Coarsening the weights: $w'_i = \left\lfloor \frac{w_i}{A} \right\rfloor$ for some appropriately defined integer A.
- However, coarsening weights will change the problem.
- $$\frac{\text{weight budget constraint} \quad : \quad \text{hard}}{\text{maximum value requirement} \quad : \quad \text{soft}}$$
- We coarsen the values instead
- In the DP, we use values as parameters

- Let A be some integer to be defined later
- $v'_i := \left\lfloor \frac{v_i}{A} \right\rfloor$ be the scaled value of item i
- Definition of DP cells: $f[i, V'] = \min_{S \subseteq [i]: v'(S) \geq V'} w(S)$

$$f[i, V'] = \begin{cases} 0 & V' \leq 0 \\ \infty & i = 0, V' > 0 \\ \min \left\{ \begin{array}{c} f[i-1, V'] \\ f[i-1, V' - v'_i] + w_i \end{array} \right\} & i > 0, V' > 0 \end{cases}$$

- Output A times the largest V' such that $f[n, V'] \leq W$.

- Instance $\mathcal{I}$: $(v_1, v_2, \cdots, v_n)$ $\qquad$ opt: optimum value of $\mathcal{I}$
- Instance $\mathcal{I}'$: $(Av'_1, \cdots, AV'_n)$ $\qquad$ opt': optimum value of $\mathcal{I}'$

$$
\begin{aligned}
& v_i - A < Av'_i \le v_i, \qquad && \forall i \in [n] \\
\implies \quad & \mathrm{opt} - nA < \mathrm{opt}' \le \mathrm{opt}
\end{aligned}
$$

- $\mathrm{opt} \ge v_{\max} := \max_{i\in[n]} v_i$ (assuming $w_i \le W, \forall i$)
- setting $A := \left\lfloor \frac{\epsilon \cdot v_{\max}}{n} \right\rfloor$: $(1-\epsilon)\mathrm{opt} \le \mathrm{opt}' \le \mathrm{opt}$

- $\forall i, v'_i = O(\frac{n}{\epsilon})$ $\qquad \implies \qquad$ running time $= O(\frac{n^3}{\epsilon})$

Theorem There is a $(1+\epsilon)$-approximation for the knapsack problem in time $O(\frac{n^3}{\epsilon})$.

Def. A polynomial-time approximation scheme (PTAS) is a family of algorithms A_ϵ, where A_ϵ for every $\epsilon > 0$ is a (polynomial-time) $(1 \pm \epsilon)$-approximation algorithm.

- Remark: the approximation ratio is $1 + \epsilon$ or $1 - \epsilon$, depending on whether the problem is a minimization/maximization problem

Def. A fully polynomial-time approximation scheme (FPTAS) is an approximation scheme A_ϵ such that the running time of A_ϵ is $\mathrm{poly}(n, \frac{1}{\epsilon})$ for input instances of n.

- So, Knapsack admits an FPTAS.

Q: Assume P $\neq$ NP. What is a neccesary condition for a NP-hard problem to admit an FPTAS?

- Vertex cover? Maximum independent set?

Outline

Makespan Minimization on Identical Machines

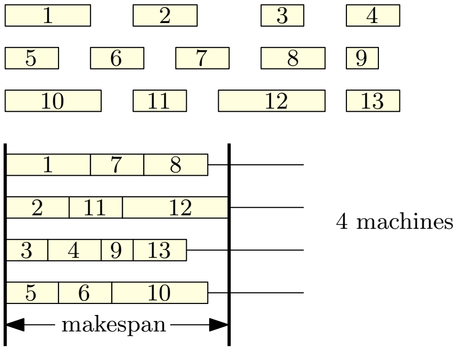
Input: n jobs index as $[n]$

each job $j \in [n]$ has a processing time $p_j \in \mathbb{Z}_{>0}$

m machines

Output: schedule of jobs on machines with minimum makespan

$\sigma : [n] \to [m]$ with minimum $\max_{i \in [m]} \sum_{j \in \sigma^{-1}(i)} p_j$



Greedy Algorithm

start from an empty schedule
for $j = 1$ to n **do**
    put job j on the machine with the smallest load

Analysis of $(2 - \frac{1}{m})$-Approximation for Greedy Algorithm

$$p_{\max} := \max_{j \in [n]} p_j$$

$$\mathrm{alg} \leq p_{\max} + \frac{1}{m} \cdot \Big(\sum_{j \in [n]} p_j - p_{\max}\Big) = \Big(1 - \frac{1}{m}\Big) p_{\max} + \frac{1}{m} \sum_{j \in [n]} p_j$$

$$\left.\begin{array}{rcl} \mathrm{opt} & \geq & p_{\max} \\ \mathrm{opt} & \geq & \frac{1}{m} \sum_{j \in [n]} p_j \end{array}\right\} \Longrightarrow \quad \mathrm{alg} \leq \Big(2 - \frac{1}{m}\Big)\mathrm{opt}$$

Q: What happens if all items have size at most $\epsilon \cdot \mathrm{opt}$?

A: $\mathrm{alg} \leq \frac{1}{m}\sum_{j\in[n]} p_j + p_{\max} \leq \mathrm{opt} + \epsilon \cdot \mathrm{opt} = (1+\epsilon)\mathrm{opt}.$

Q: What can we do if all items have size at least $\epsilon \cdot \mathrm{opt}$?

A: We can round the sizes, so that #(distinct sizes) is small

Overview of Algorithm

1: declare j small if $p_j < \epsilon \cdot p_{\max}$ and big otherwise
2: use trunction + DP to solve the instance defined by big jobs
3: use DP for instance $(p'_j)_{j \text{ big}}$ to schedule big jobs
4: add small jobs to schedule greedily

Dynamic Programming for Big Jobs

- $B := \{j \in [n] : p_j \geq \epsilon p_{\max}\}$: set of big jobs
- $p'_j := \max\{\epsilon p_{\max}(1+\epsilon)^t \leq p_j : t \in \mathbb{Z}\}, \forall j \in B$

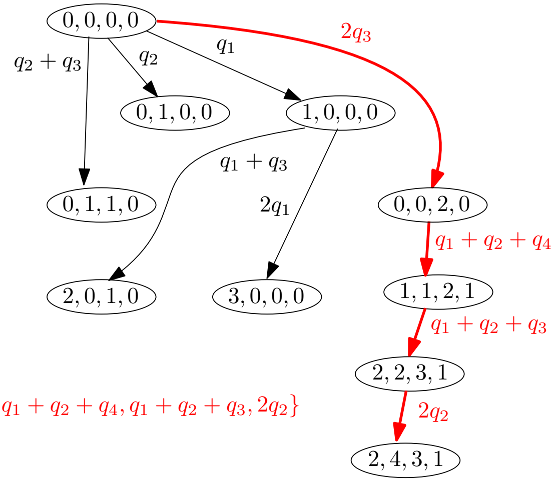
  p'_j is the rounded size of j
- $k := |\{p'_j : j \in B\}|$: #(distinct rounded sizes)

  $k \leq 1 + \log_{1+\epsilon} \frac{p_{\max}}{\epsilon p_{\max}} = O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)$
- $\{q_1, q_2, \cdots, q_k\} := \{p'_j : j \in B\}$: the k distinct rounded sizes
- $n_1, \cdots, n_k$: #(big jobs) with rounded sizes being $q_1, \cdots, q_k$

Constructing a Directed Acyclic Graph $G = (V, E)$

- a vertex $(a_1, \cdots, a_k)$, $a_i \in [0, n_i], \forall i \in [k]$
  - denotes the instance with a_1 jobs of size q_1, a_2 jobs of size q_2, $\cdots$, a_k jobs of size q_k
- an arc $(a_1, \cdots, a_k) \to (b_1, \cdots b_k)$ of weight $\sum_{i=1}^{k}(b_i - a_i)q_i$, if $a_i \leq b_i, \forall i \in [k]$, and $a_i < b_i$ for some $i \in [k]$
  - reducing instance $(b_1, \cdots b_k)$ to $(a_1, \cdots, a_k)$ requires 1 machine of load $\sum_{i=1}^{k}(b_i - a_i)q_i$

- Goal: find a path from $(0, \cdots, 0)$ to $(n_1, \cdots, n_k)$ of at most m edges, so as to minimize the maximum weight on the path.
- problem can be solved in $O(m \cdot |E|)$ time using DP
- $O(m \cdot |E|) = O(m \cdot n^{2k}) = n^{O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)}$.



$\text{cost} = \max\{2q_3, q_1 + q_2 + q_4, q_1 + q_2 + q_3, 2q_2\}$

Analysis of Algorithm for Big Jobs

- $\mathcal{I}_B$: instance $(p_j)_{j\in B}$ $\quad$ opt_B: its optimum makespan
- $\mathcal{I}'_B$: instance $(p'_j)_{j\in B}$ $\quad$ opt'_B: its optimum makespan
- $\mathrm{opt}'_B \leq \mathrm{opt}_B$
- schedule for $\mathcal{I}'_B \Rightarrow$ schedule for $\mathcal{I}_B$:

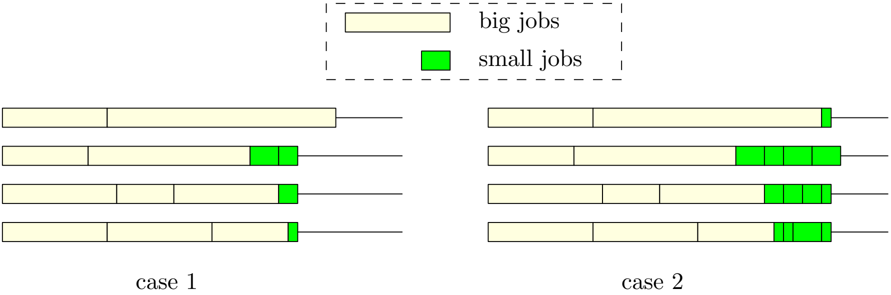
  $(1+\epsilon)$-blowup in makespan

Theorem The dynamic programming algorithm gives a schedule of makespan at most $(1+\epsilon)\mathrm{opt}_B$ in time $n^{O\left(\frac{1}{\epsilon}\log\frac{1}{\epsilon}\right)}$.

Adding small jobs to schedule

1: starting from the schedule for big jobs
2: **for** every small job j **do**
3: add j to the machine with the smallest load

Analysis of the Final Algorithm



- Case 1: makespan is not increased by small jobs

$$\text{alg} \le (1+\epsilon)\text{opt}_B \le (1+\epsilon)\text{opt}.$$

- Case 2: makespan is increased by small jobs
  - loads between any two machines differ by at most size of a small job, which is at most $\epsilon \cdot p_{\max}$

$$\text{alg} \le \epsilon \cdot p_{\max} + \frac{1}{m}\sum_{j \in [n]} p_j \le \epsilon \cdot \text{opt} + \text{opt} = (1+\epsilon) \cdot \text{opt}.$$