算法设计与分析(2026年春季学期)

Data Structures

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Examples of Data Structures

- array, queue, stack, linked list (linear data structures)
- set, map, hash map, priority queue
- binary search tree (BST), self-balancing BST
- union-find

- abstract data structure: a model defined by the set of supported operations, without an implementation. It is not associated with time complexities for operations, or space usage.
- (concrete) data structure: a concrete implementation of some abstract data structure. So, it has a time complexity for each operation, and space usage.

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- data structures are often used as tools to design (more) efficient algorithms.
- for a given problem, define operations needed, and choose a data structure that suits the need.

Outline

- Let V be a ground set of size n.

Def. A priority queue is an abstract data structure that maintains a set $U \subseteq V$ of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element $v \in V \setminus U$, with associated key value key_value.
- decrease_key(v, new_key_value): decrease the key value of an element $v \in U$ to new_key_value
- extract_min$()$: return and remove the element in U with the smallest key value
- remove, increase_key, get_min, $\cdots$

Example:

- insert(a, 9), insert(b, 3), insert(c, 5), insert(d, 10), $\triangleright$ {a:9, b:3, c:5, d:10}
- extract_min(), return b $\triangleright$ {a:9, c:5, d:10}
- decrease_key(d, 4) $\triangleright$ {a:9, c:5, d:4}
- extract_min(), return d $\triangleright$ {a:9, c:5}
- insert(e, 12), $\triangleright$ {a:9, c:5, e:12}
- extract_min(), return c $\triangleright$ {a:9, e:12}
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Algorithms in this course that use priority queue

- Greedy Algorithm for Offline Cache
- Prim's Algorithm for Minimum Spanning Tree
- Dijkstra's Algorithm for Shortest Path

Simple Implementations for Priority Queue

- n = size of ground set V

data structures	**insert**	**extract_min**	**decrease_key**
array			
sorted array			

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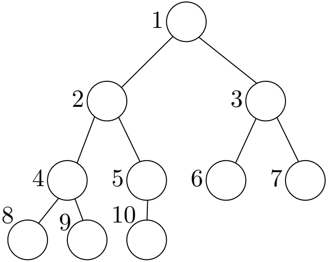
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
(binary) heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Heap

The elements in a heap is organized using a complete binary tree:

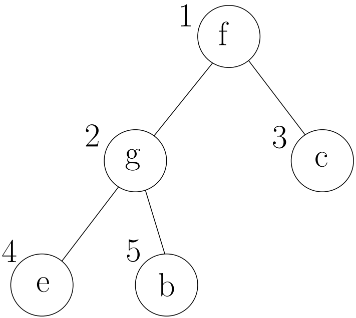


- Nodes are indexed as $\{1, 2, 3, \cdots, s\}$
- Parent of node i: $\lfloor i/2 \rfloor$
- Left child of node i: $2i$
- Right child of node i: $2i + 1$

(Binary) Heap

A heap H contains the following fields

- s: size of U (number of elements in the heap)
- $A[i], 1 \leq i \leq s$: the element at node i of the tree
- $p[v], v \in U$: the index of node containing v
- $key[v], v \in U$: the key value of element v

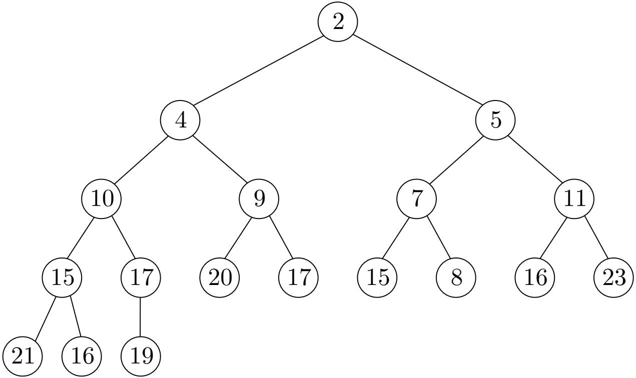


- $s = 5$
- $A = (\text{f}, \text{g}, \text{c}, \text{e}, \text{b})$
- $p[\text{f}] = 1, p[\text{g}] = 2, p[\text{c}] = 3,$
  $p[\text{e}] = 4, p[\text{b}] = 5$

Heap

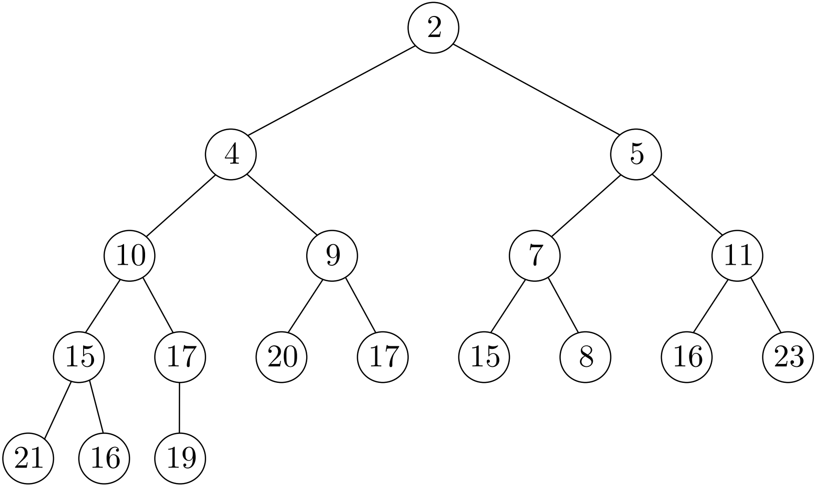
The following heap property is satisfied:

- for any two nodes i, j such that i is the parent of j, we have $key[A[i]] \leq key[A[j]]$.



A heap. Numbers in the circles denote key values of elements.

insert(v, key_value)



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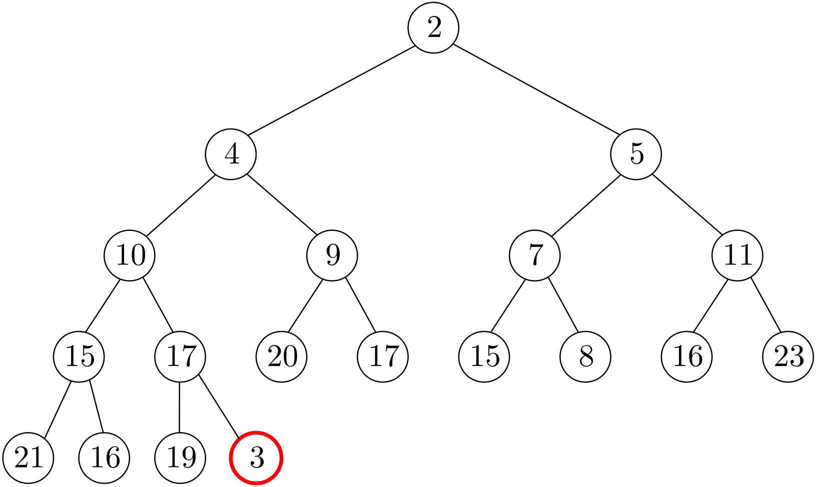

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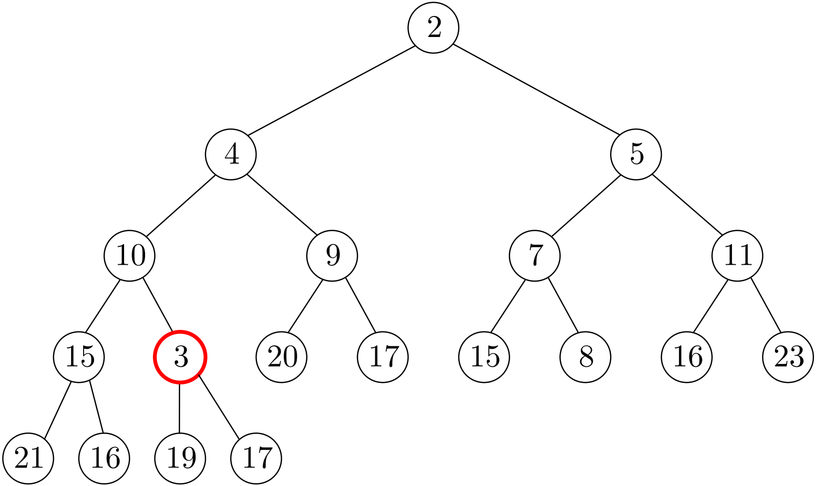

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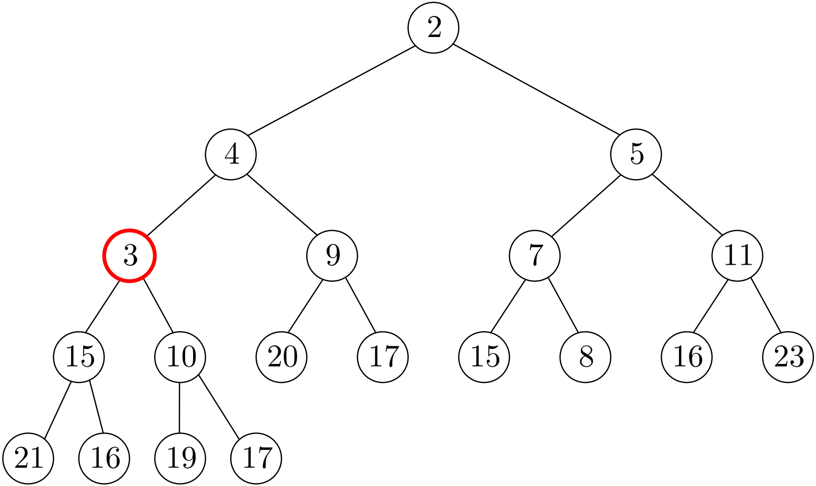

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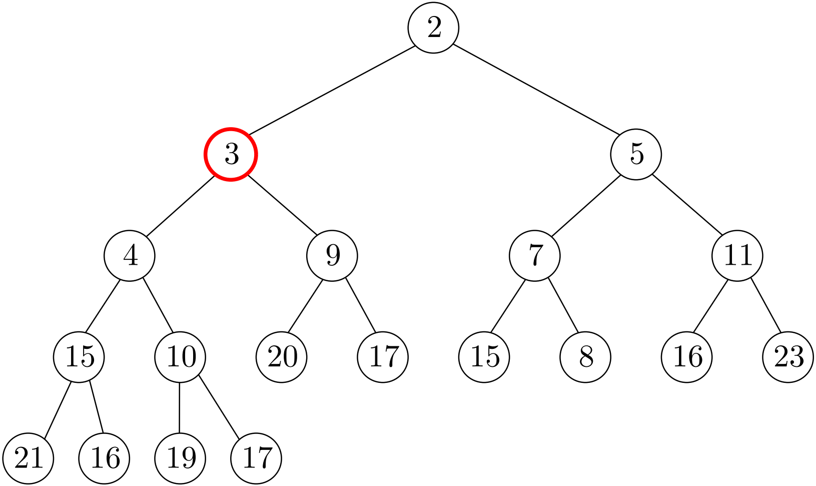


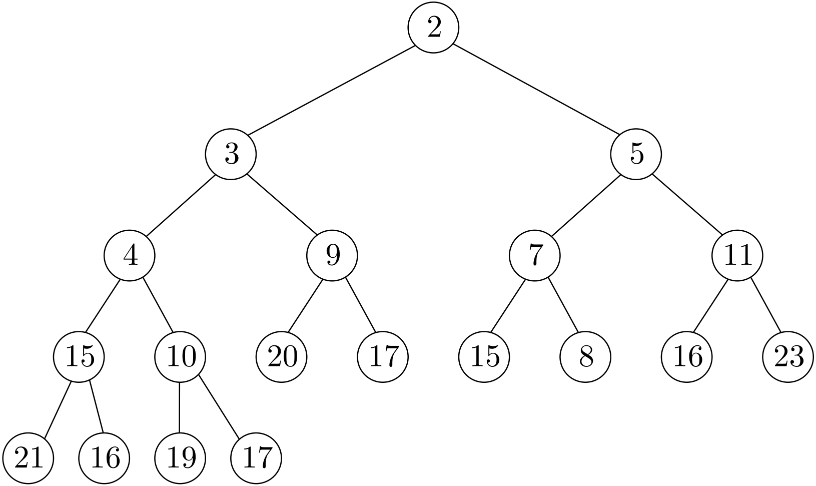
insert(v, key_value)

1. $s \leftarrow s + 1$
2. $A[s] \leftarrow v$
3. $p[v] \leftarrow s$
4. $key[v] \leftarrow key\_value$
5. heapify_up(s)

heapify-up(i)

```
1: while i > 1 do
2:     j ← ⌊i/2⌋
3:     if key[A[i]] < key[A[j]] then
4:         swap A[i] and A[j]
5:         p[A[i]] ← i, p[A[j]] ← j
6:         i ← j
7:     else break
```

extract_min()



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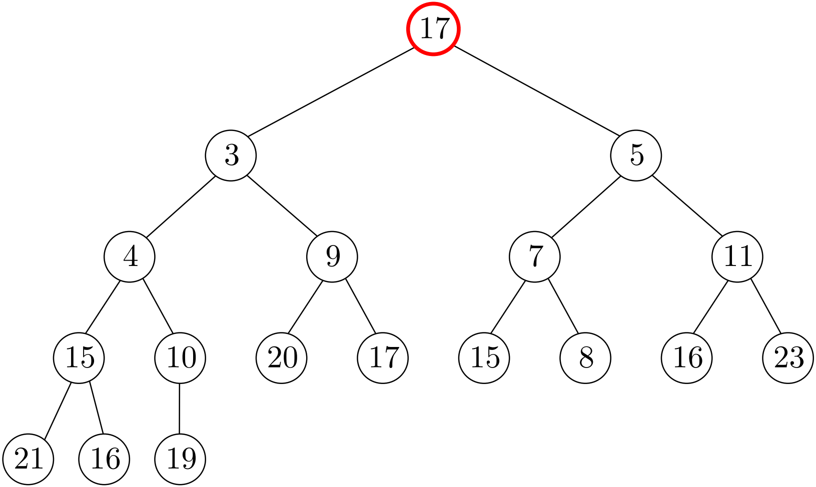

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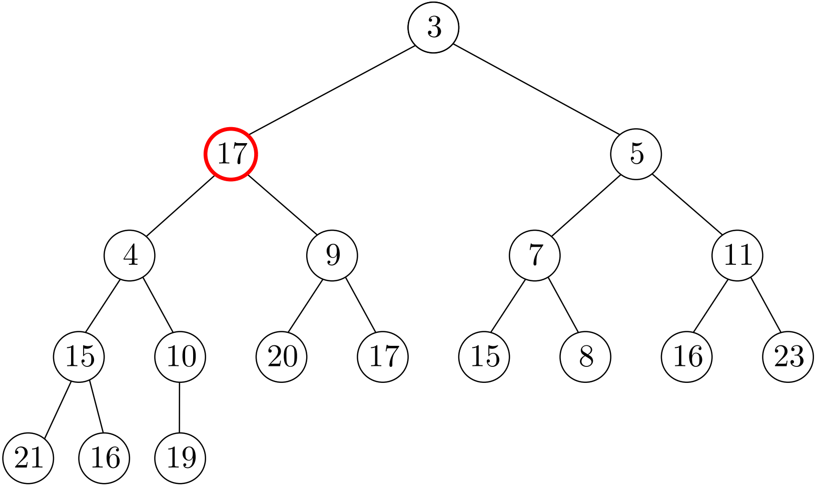

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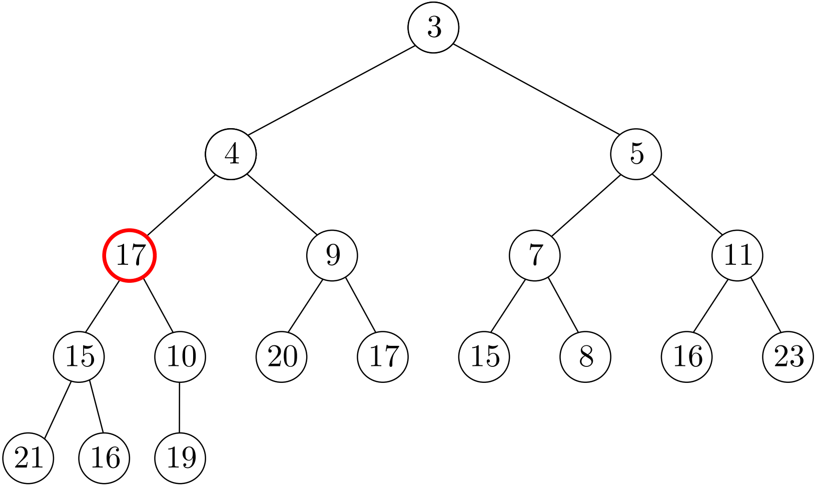

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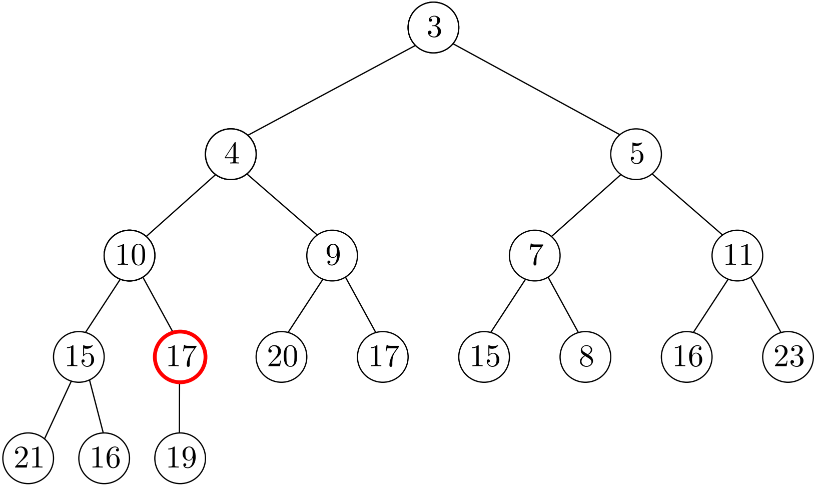

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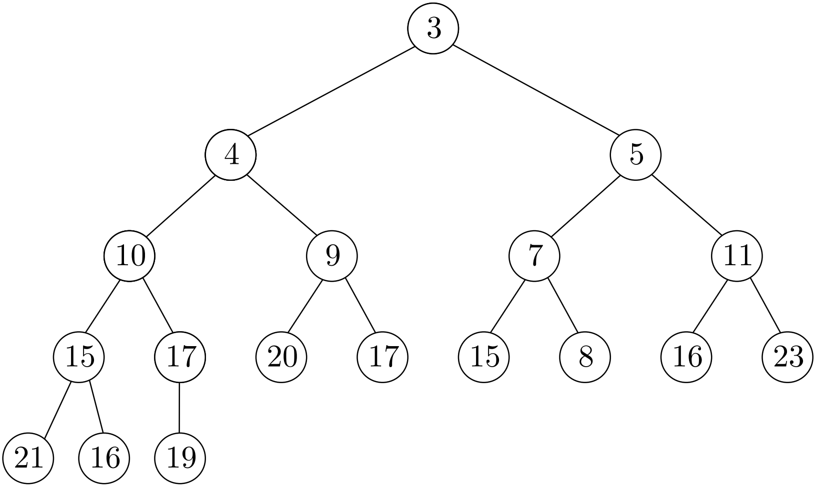

extract_min()

$ret \leftarrow A[1]$
$A[1] \leftarrow A[s]$
$p[A[1]] \leftarrow 1$
$s \leftarrow s - 1$
if $s \geq 1$ **then**
    heapify_down(1)
return ret

decrease_key(v, key_value)

$key[v] \leftarrow key\_value$
heapify-up$(p[v])$

heapify-down(i)

while $2i \leq s$ **do**
    if $2i = s$ or $key[A[2i]] \leq key[A[2i+1]]$ **then**
        $j \leftarrow 2i$
    else
        $j \leftarrow 2i + 1$
    if $key[A[j]] < key[A[i]]$ **then**
        swap $A[i]$ and $A[j]$
        $p[A[i]] \leftarrow i$, $p[A[j]] \leftarrow j$
        $i \leftarrow j$
    else break

- Running time of heapify_up and heapify_down: $O(\lg n)$

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- Note: running times for Fibonacci heap are amortized running time:

Def. One operation has amortized running time T, if for any integer $s \geq 1$, the first s executions of the operation have total running time at most Ts.

Two Definitions Needed to Prove that the Procedures Maintain Heap Property

Def. We say that H is almost a heap except that $key[A[i]]$ is too small if we can increase $key[A[i]]$ to make H a heap.

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Lemma At the beginning of any iteration of the while loop in heapify_up, H is almost a heap except that $key[A[i]]$ is too small.

Lemma At the beginning of any iteration of the while loop in heapify_down, H is almost a heap except that $key[A[i]]$ is too big.

Outline

A self-balancing binary search tree T maintains a set of comparable elements and supports:

- Insertion of an element to T
- Deletion of an element from T
- Whether an element exists in T
- Return the rank of an element in T (i.e, 1 plus number of elements in T smaller than the element)
- Return the i-th smallest element in T

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- Each operation takes time $O(\lg n)$
- A self-balancing BST supports more operations than a priority queue does, thus is harder to implement.

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Def. Given an array A of n integers, an inversion in A is a pair (i, j) of indices such that $i < j$ and $A[i] > A[j]$.

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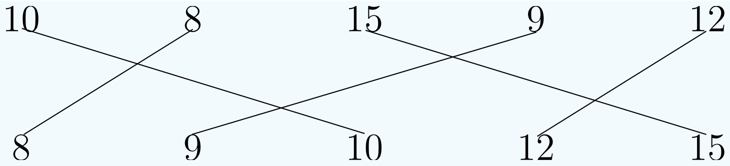

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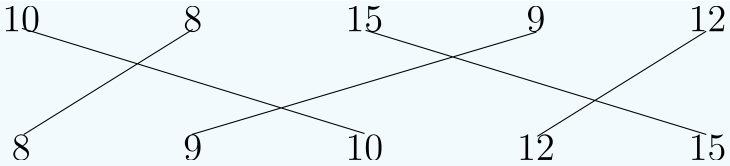


- 4 inversions (for convenience, using numbers, not indices): $(10, 8), (10, 9), (15, 9), (15, 12)$

Counting Inversions Using Self-Balancing BST

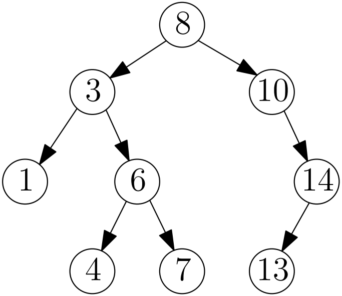
inversions(A, n)

$T \leftarrow$ empty Self-Balancing Binary Search Tree
$c \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    $c \leftarrow c + i - T.\text{rank}(A[i])$
    $T.\text{insert}(A[i])$
return c

Binary Search Trees

For any node v in tree:

- key in v must be greater than all keys on the left-sub-tree of v
- key in v must be smaller than all keys on the right-sub-tree of v



Binary Search Trees

For any node v in tree:

- key in v must be greater than all keys on the left-sub-tree of v
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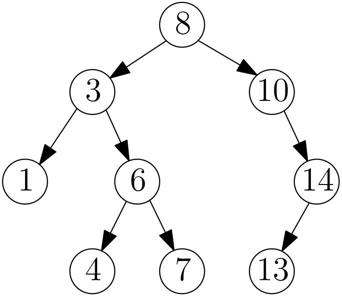
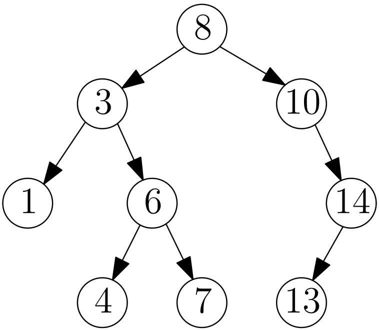
- in-order traversal of tree gives a sorted list of keys



Binary Search Trees: Insertion



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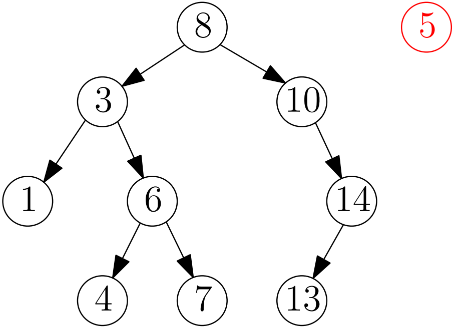

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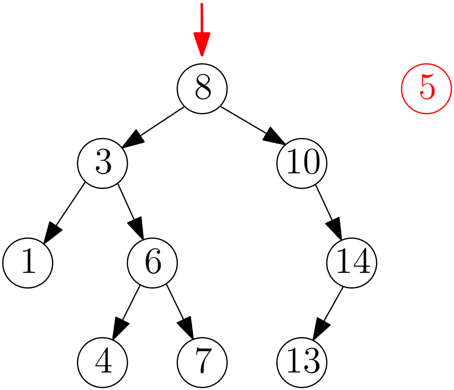

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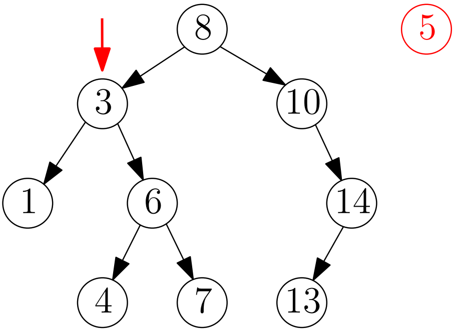

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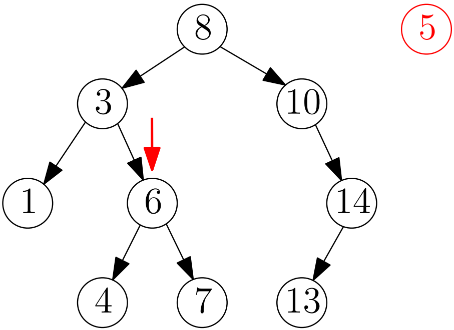

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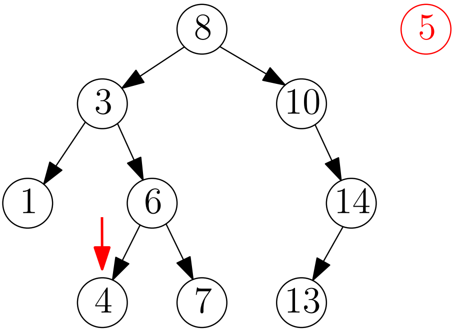

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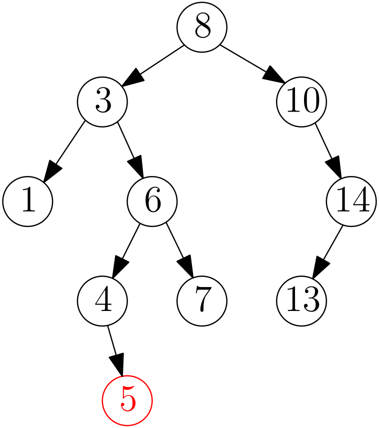


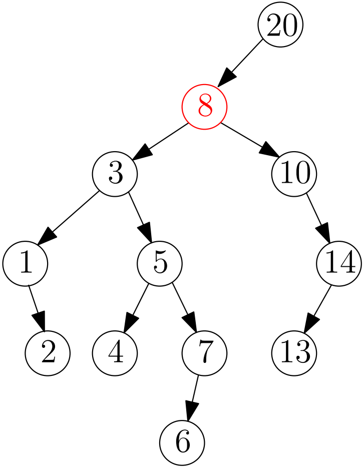
Binary Search Trees: Insertion

insert(v, u) $\triangleright$ u is the new node to be inserted

if $u.key < v.key$ **then**
    if $v.left = nil$ **then** $v.left \leftarrow u$
    else insert$(v.left, key)$
else
    if $v.right = nil$ **then** $v.right \leftarrow u$
    else insert$(v.right, key)$

- Call insert$(root, u)$ if $root \neq nil$, and $root = u$ otherwise.

Binary Search Trees: Deletion



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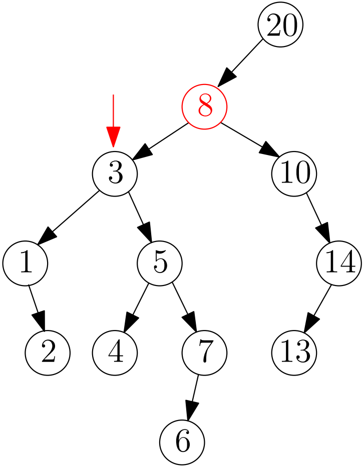

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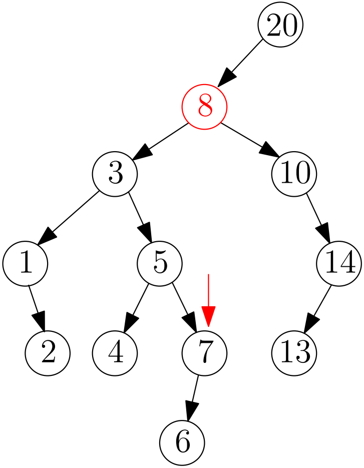

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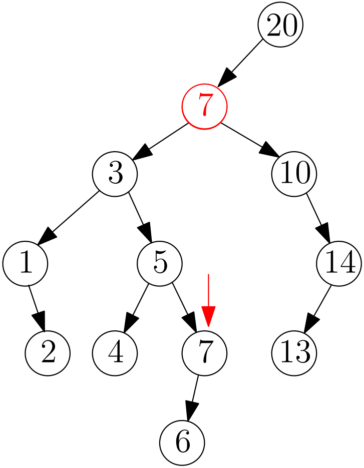

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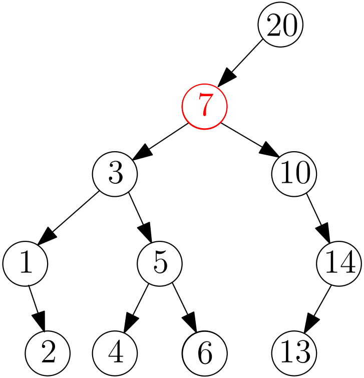


Binary Search Trees: Deletion

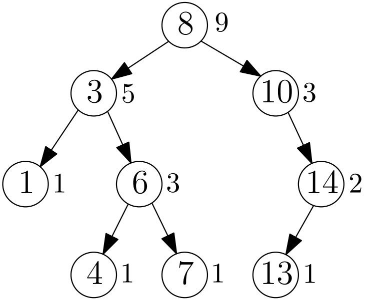
delete(v)

```
if v.left = nil then
    ret ← v.right, remove v, return ret
else
    u ← v.left
    if u.right = nil then
        u.right ← v.right, remove v, return u
    else
        w ← u.right
        while w.right ≠ nil do u ← w, w ← w.right
        u.right ← w.left, v.key ← w.key, remove w, return v
```

- procedure returns the new root of the sub-tree after removing v
- v is left child of p: $p.left \leftarrow$ delete(v), similar for right child case
- if v is the root: $root =$ delete(v)

Rank and Returning i-th Smallest Element

- Need to maintain a field "size"



- In both insertion and deletion operations, we need to adjust the size field accordingly.

Rank and Returning i-th Smallest Element

- Engineering trick: let nil be artificial node representing an empty node, with $nil.size = 0$

rank(v, key)

1: **if** $v = nil$ **then return** 0
2: **if** $key < v.key$ **then return** rank$(v.left, key)$
3: **else return** $v.left.size + 1 +$ rank$(v.right, key)$

selection(v, i)

1: **if** $i \le v.left.size$ **then return** selection$(v.left, i)$
2: **else if** $i = v.left.size + 1$ **then return** $v.key$
3: **else return** $selection(v.right, i - (v.left.size + 1))$

Running Time for Operations

- each operation takes time $O(d)$.
- $d =$ depth of tree
- best case:
- worst case:

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- worst case: $d = \Theta(n)$

Running Time for Operations

- each operation takes time $O(d)$.
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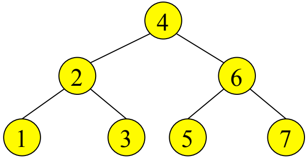

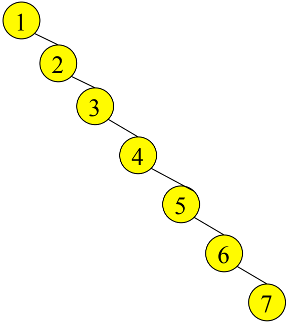

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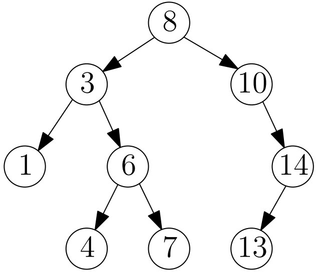

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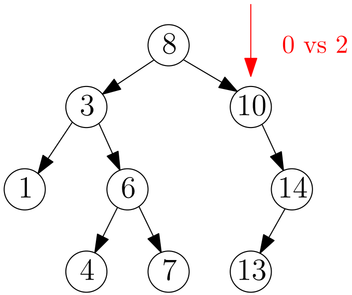

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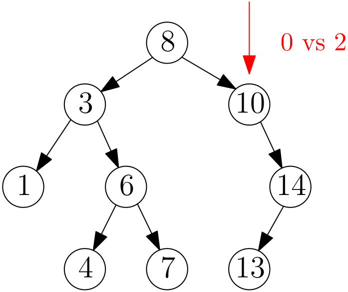

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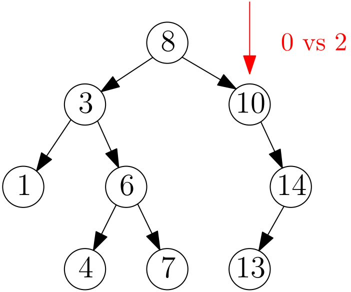

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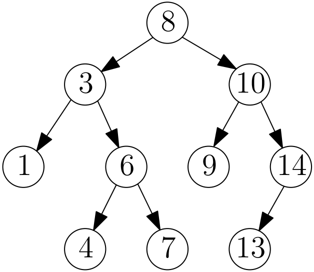

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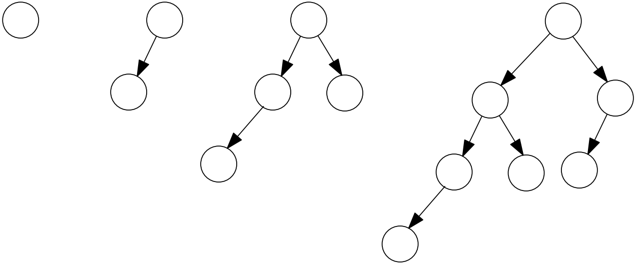

- $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 4, f(4) = 7 \cdots$

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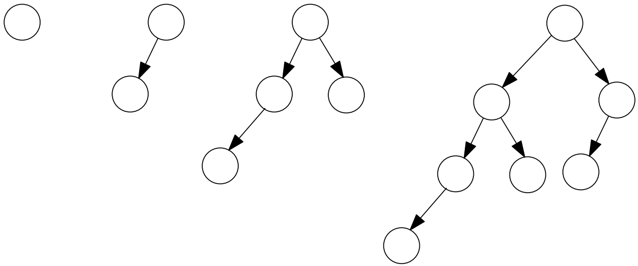

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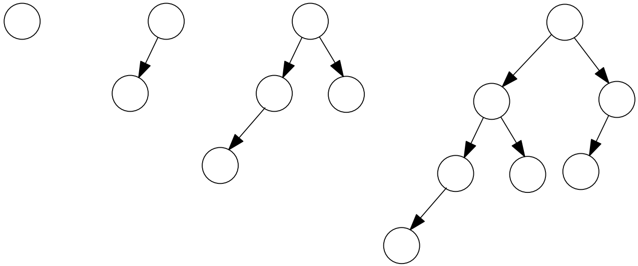

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- $f(d) = 2^{\Theta(d)}$
- If a AVL tree has size n and depth d, then

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- Thus, $d = O(\log n)$

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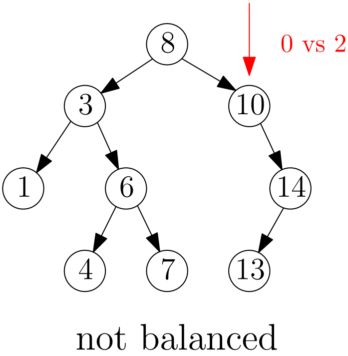

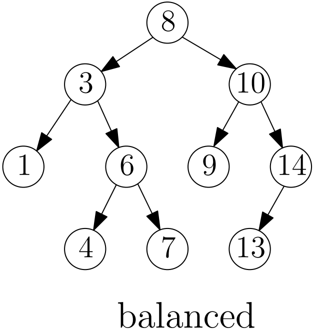

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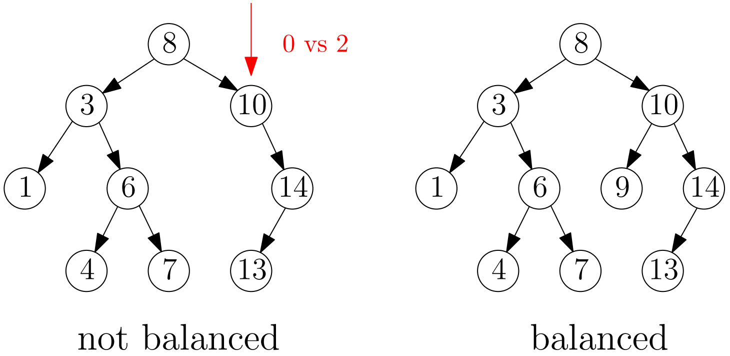

- How can we maintain the property?

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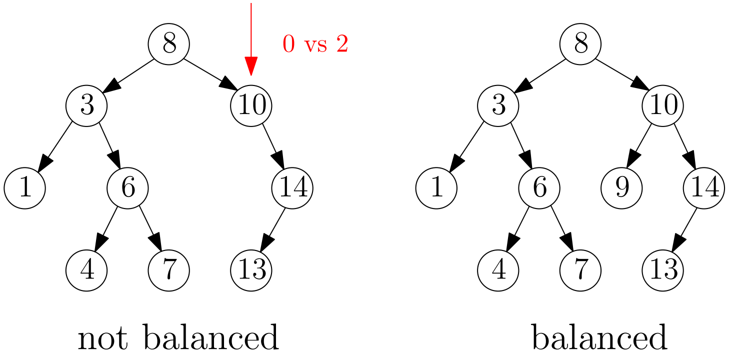

- How can we maintain the property?
- Assume we only do insertions; there are no deletions.

Maintain Balance Property

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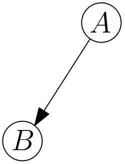

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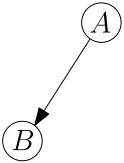

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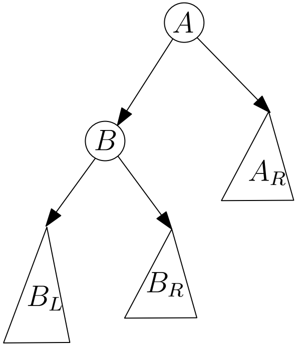

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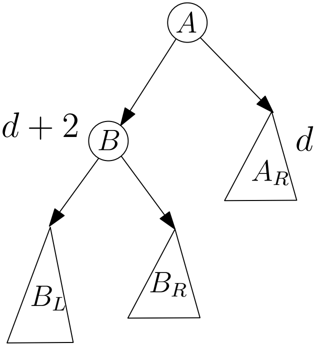

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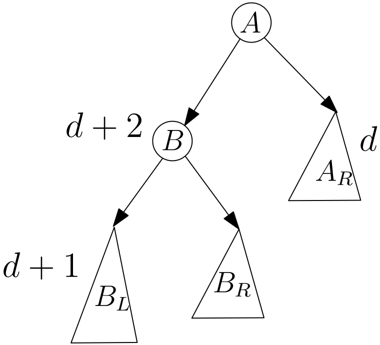

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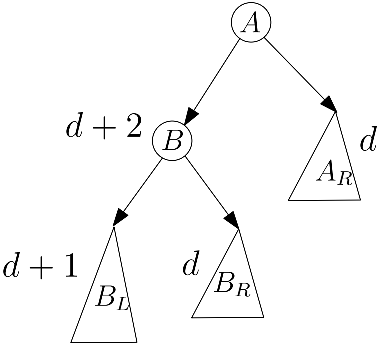

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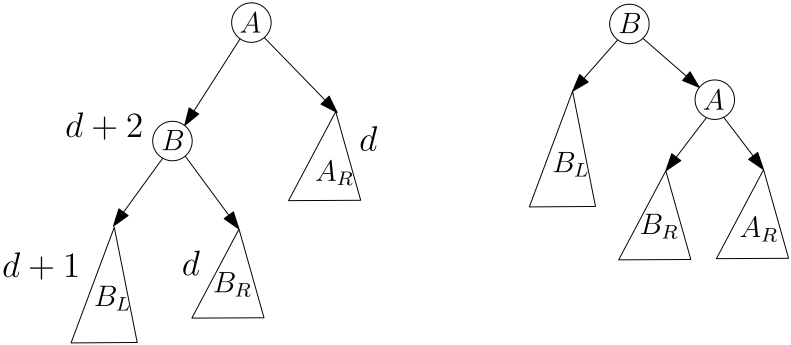

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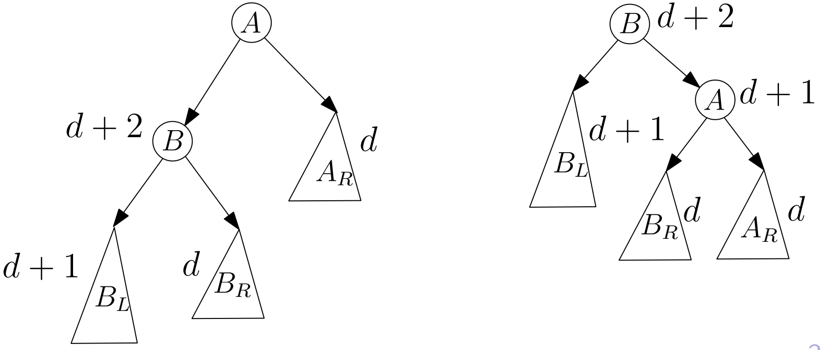

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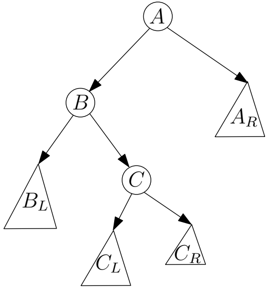

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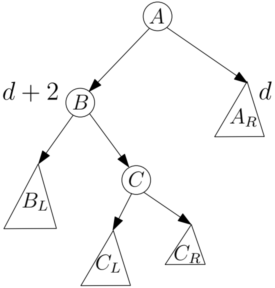

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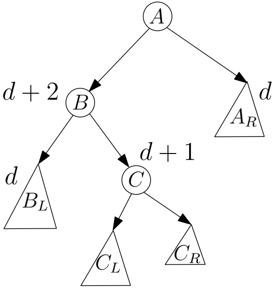

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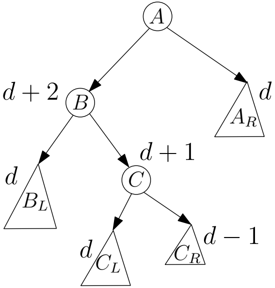

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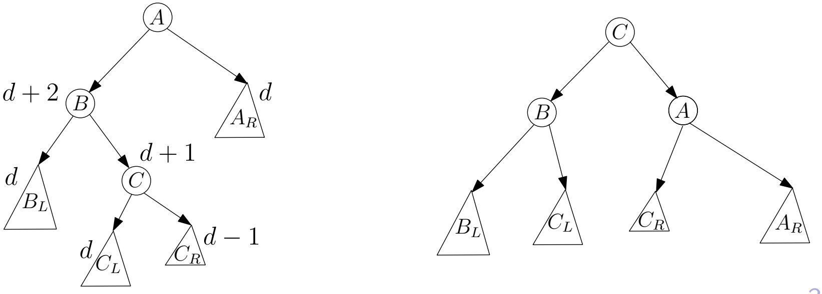

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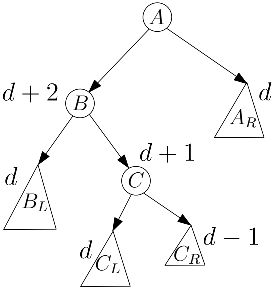

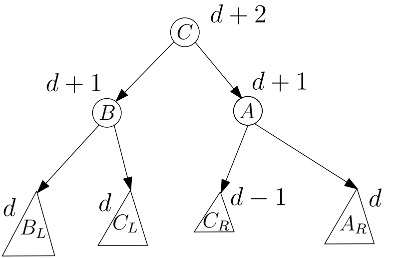

Outline

Union-Find Data Structure

- V: ground set
- We need to maintain a partition of V and support following operations:
  - Check if u and v are in the same set of the partition
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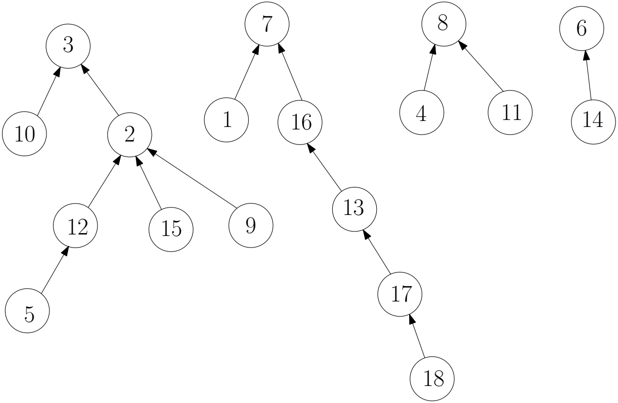
  - Merge two sets in partition
- Algorithm we shall learn Using the data structure: Kruskal's Algorithm for Minimum Spanning Tree

Example:

- init(6) ▷ {1}, {2}, {3}, {4}, {5}, {6}
- check$(1,5)$ return **false**
- merge$(2,3)$ ▷ {1}, {2, 3}, {4}, {5}, {6}
- merge$(1,6)$, ▷ {1, 6}, {2, 3}, {4}, {5}
- check$(2,3)$ return **true**
- check$(1,3)$ return **false**
- merge$(1,2)$ ▷ {1, 2, 3, 6}, {4}, {5}
- check$(3,6)$ return **true**
- check$(1,5)$ return **false**
- merge$(4,5)$ ▷ {1, 2, 3, 6}, {4, 5}
- merge$(2,5)$ ▷ {1, 2, 3, 4, 5, 6}
- check$(3,4)$ return **true**

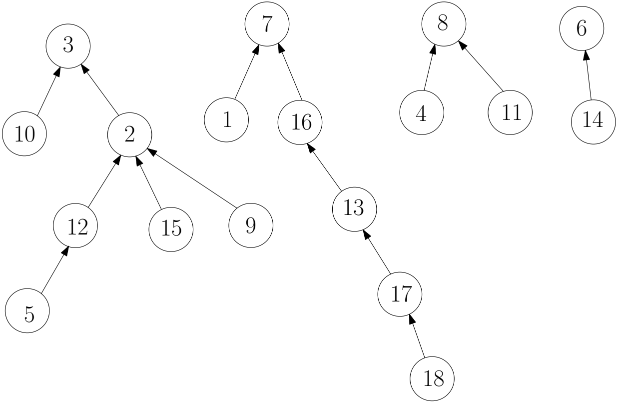
Using Rooted Trees to Store the Partition

- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$



- $par[i]$: parent of i, ($par[i] = nil$ if i is a root).

Union-Find Data Structure



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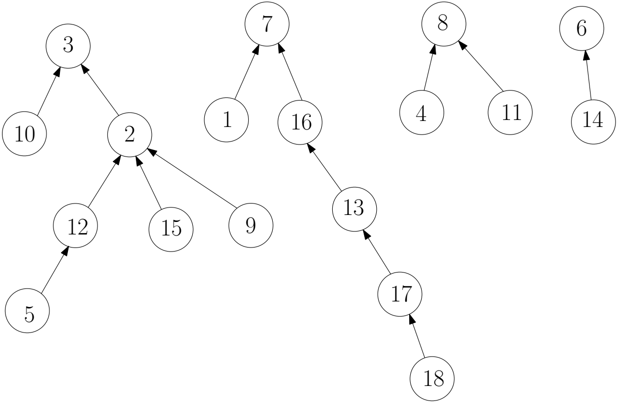

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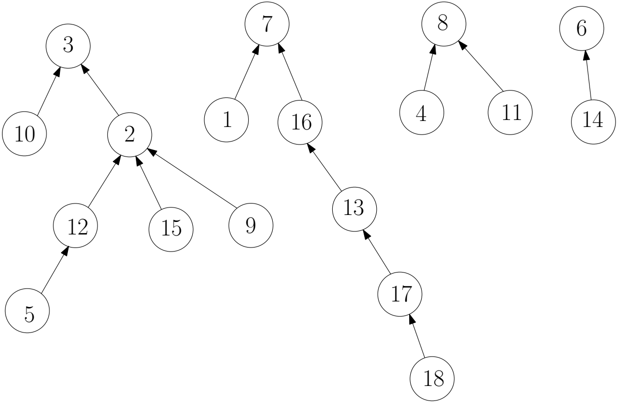

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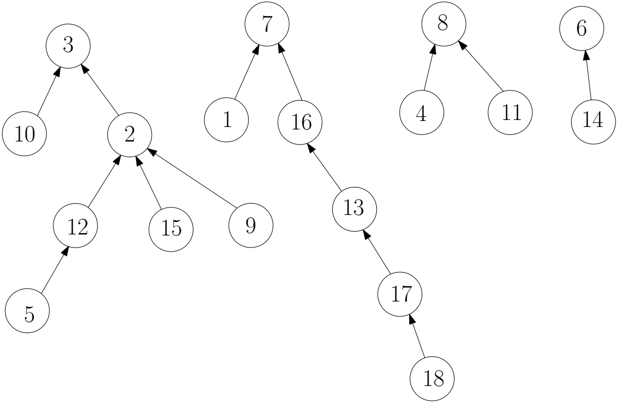

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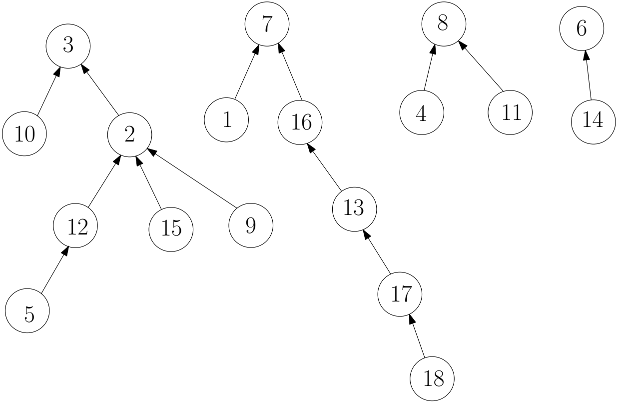

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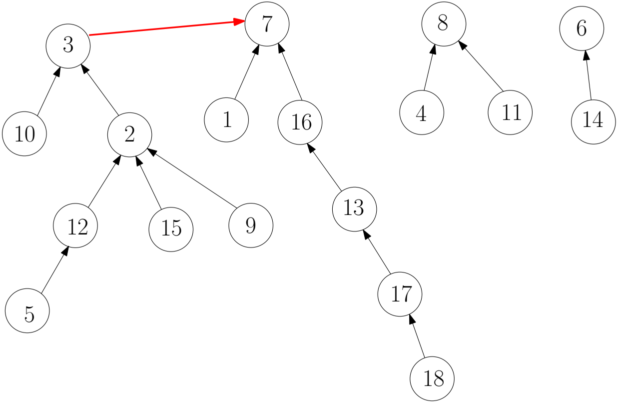

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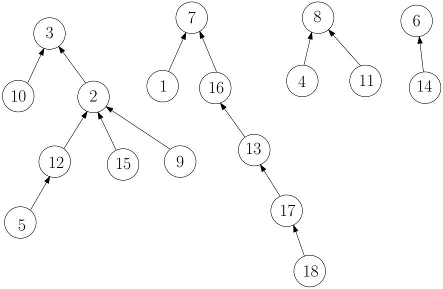

root(v)

1: **if** $par[v] = nil$ **then**
2: **return** v
3: **else**
4: **return** root($par[v]$)

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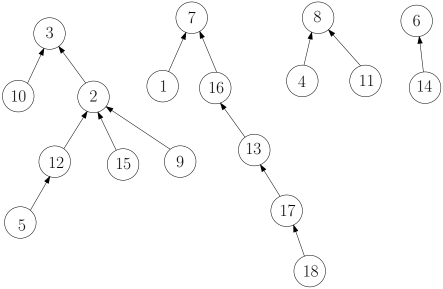

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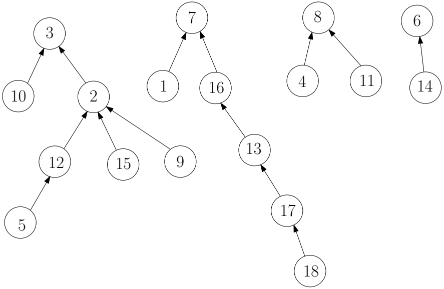

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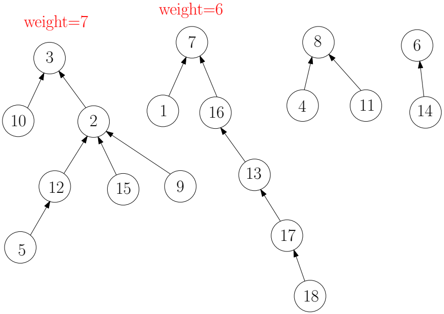

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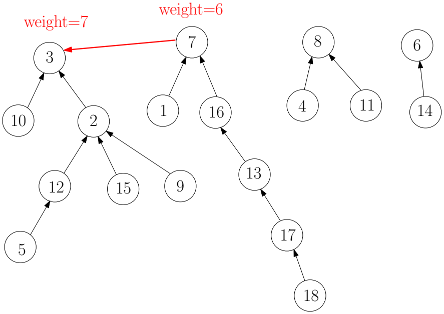

root(v)

1: **if** $par[v] = nil$ **then**
2: **return** v
3: **else**
4: **return** root($par[v]$)

- issue: the tree might be too deep; running time might be large
- ideas for improvements
  - union by size: lighter $\longrightarrow$ heavier

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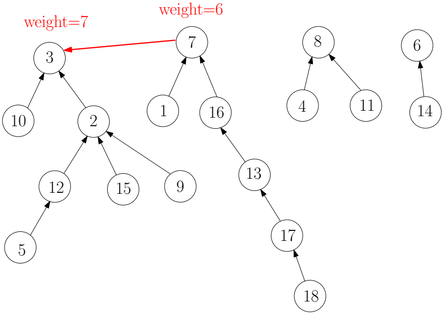

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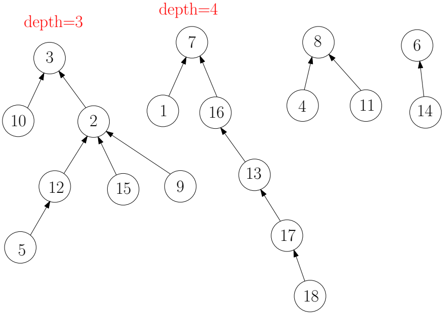

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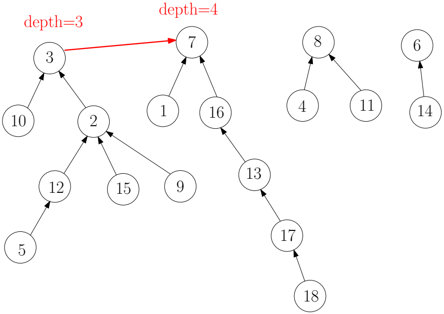

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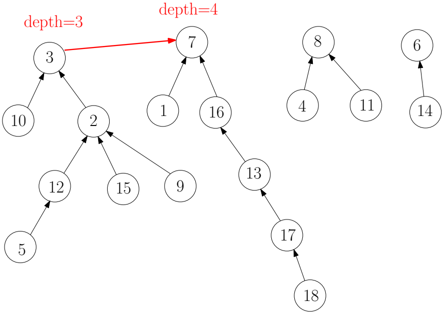

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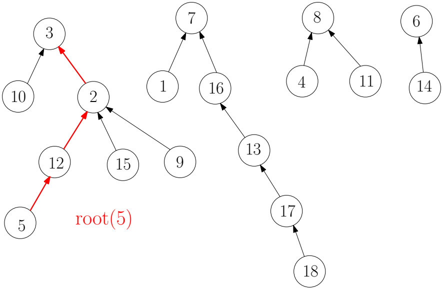

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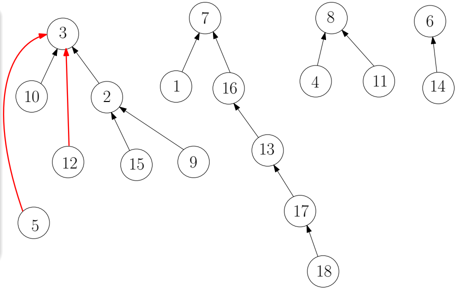

Time Complexity Using Different Operations

Optimization Method	Find/Union Operation
No Optimization	$O(n)$
Union by Rank or Size	$O(\log n)$
Path Compression Only	$O(\log n)$ (amortized)
Path Compression + Union by Rank	$O(\alpha(n))$

- $\alpha(n)$: Inverse Ackermann Function.
- When n is less than the number of atoms in the universe, we have $\alpha(n) \leq 4$. This makes the optimized Union-Find nearly $O(1)$-time in practice.