

算法设计与分析(2026年春季学期)

Divide-and-Conquer

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Outline

- 1 Divide-and-Conquer
- 2 Counting Inversions
- 3 Solving Recurrences
- 4 Quicksort and Selection
 - Quicksort
 - Lower Bound for Comparison-Based Sorting Algorithms
 - Selection Problem
- 5 Polynomial Multiplication
- 6 Strassen's Algorithm for Matrix Multiplication
- 7 FFT(Fast Fourier Transform): Polynomial Multiplication in $O(n \log n)$ Time
- 8 Finding Closest Pair of Points in 2D Euclidean Space
- 9 Computing n -th Fibonacci Number

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- mainly for combinatorial optimization problems
- trivial algorithm runs in exponential time
- greedy algorithm gives an efficient algorithm
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Divide-and-Conquer

- not necessarily for combinatorial optimization problems
- trivial algorithm already runs in polynomial time
- divide-and-conquer gives a more efficient algorithm
- main focus of analysis: running time

Divide-and-Conquer

- **Divide:** Divide instance into many smaller instances
- **Conquer:** Solve each of smaller instances recursively and separately
- **Combine:** Combine solutions to small instances to obtain a solution for the original big instance

merge-sort(A, n)

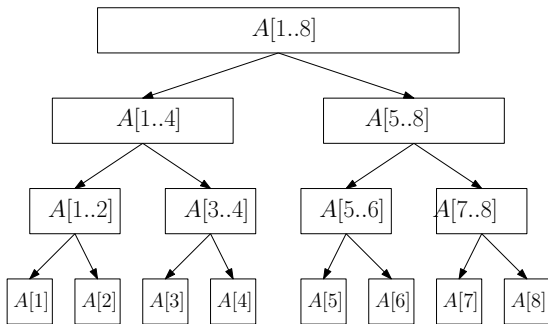
```
1: if  $n = 1$  then  
2:   return  $A$   
3: else  
4:    $B \leftarrow \text{merge-sort}(A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor)$   
5:    $C \leftarrow \text{merge-sort}(A[\lfloor n/2 \rfloor + 1..n], \lceil n/2 \rceil)$   
6:   return  $\text{merge}(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$ 
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- Divide: trivial
- Conquer: 4, 5
- Combine: 6

Running Time for Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\log n)$ levels
- Running time = $O(n \log n)$
- Better than insertion sort

Running Time for Merge-Sort Using Recurrence

- $T(n)$ = running time for sorting n numbers, then

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) & \text{if } n \geq 2 \end{cases}$$

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- Solving this recurrence, we have $T(n) = O(n \log n)$ (we shall show how later)

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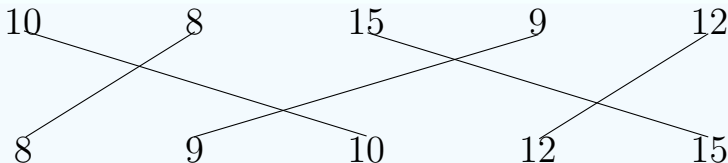
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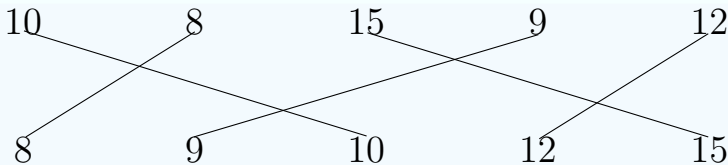
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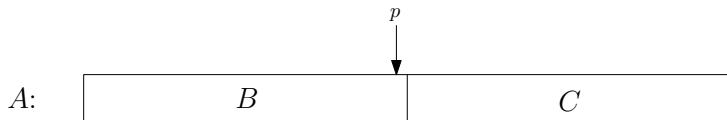
- 4 inversions (for convenience, using numbers, not indices):
(10, 8), (10, 9), (15, 9), (15, 12)

Naive Algorithm for Counting Inversions

count-inversions(A, n)

```
1:  $c \leftarrow 0$ 
2: for every  $i \leftarrow 1$  to  $n - 1$  do
3:   for every  $j \leftarrow i + 1$  to  $n$  do
4:     if  $A[i] > A[j]$  then  $c \leftarrow c + 1$ 
5: return  $c$ 
```

Divide-and-Conquer



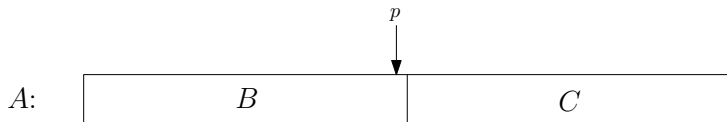
- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p+1..n]$
- $$\#invs(A) = \#invs(B) + \#invs(C) + m$$
$$m = |\{(i, j) : B[i] > C[j]\}|$$

Q: How fast can we compute m , via trivial algorithm?

A: $O(n^2)$

- Can not improve the $O(n^2)$ time for counting inversions.

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Lemma If both B and C are sorted, then we can compute m in $O(n)$ time!

Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:

B :

3	8	12	20	32	48
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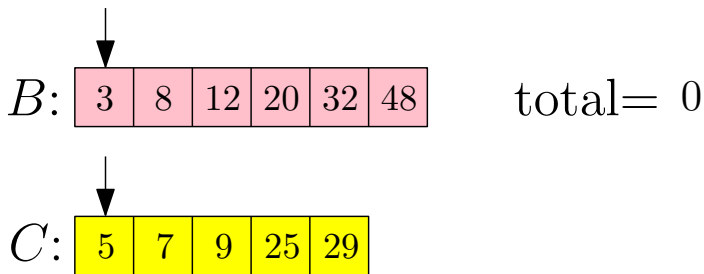
 total = 0

C :

5	7	9	25	29
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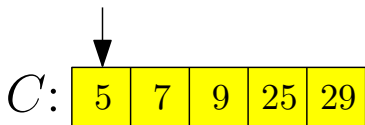
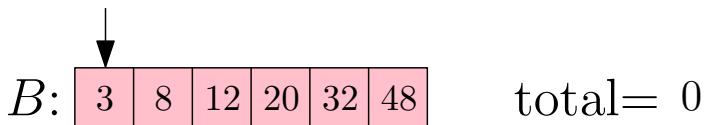
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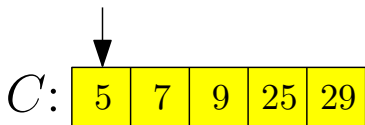
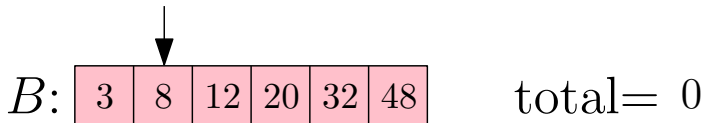


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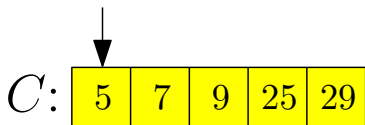
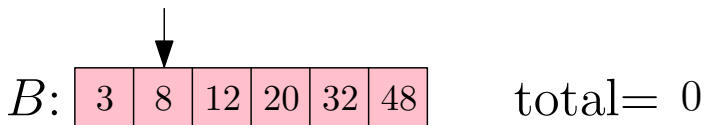


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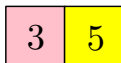


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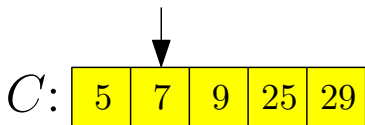
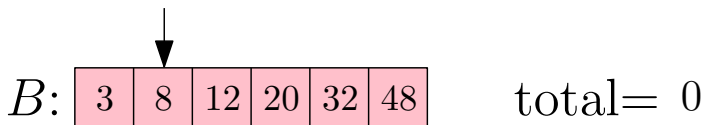


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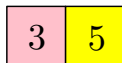


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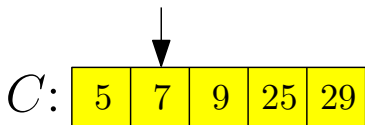
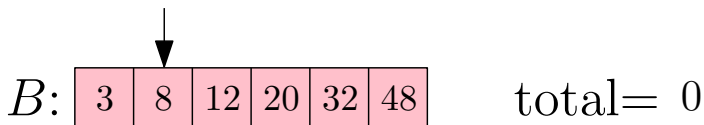


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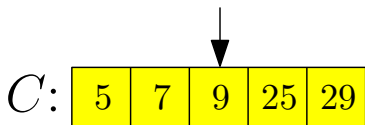
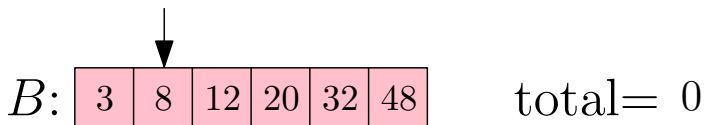


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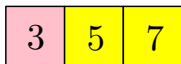


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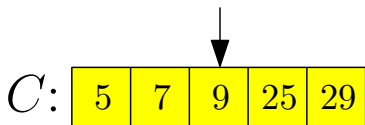
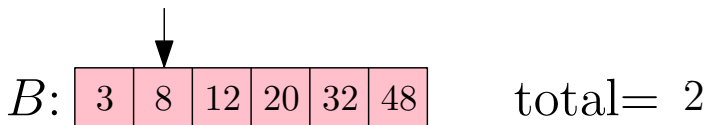


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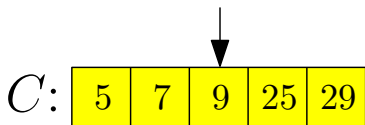
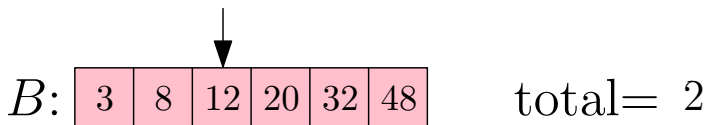


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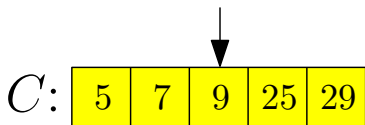
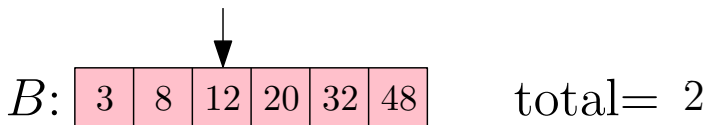


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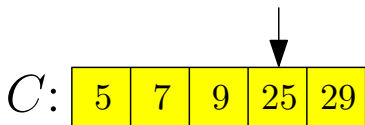
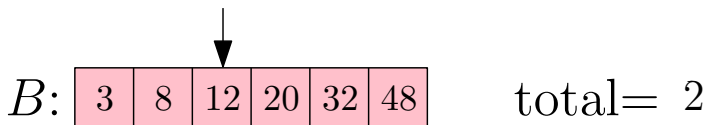
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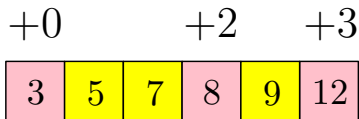
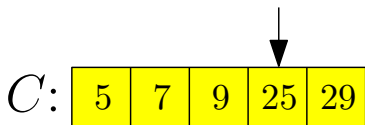
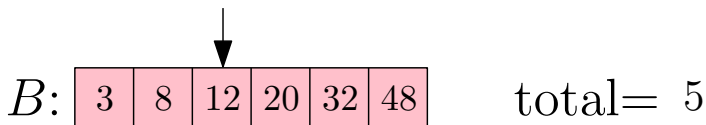
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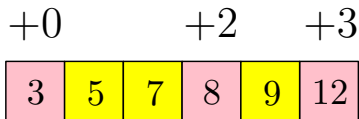
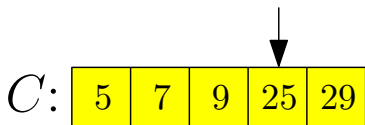
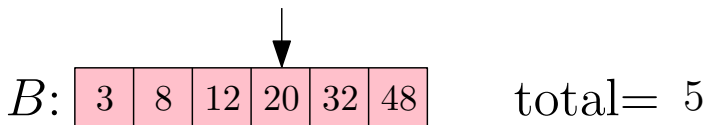
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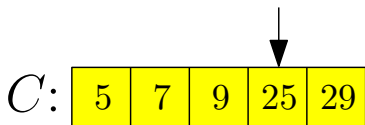
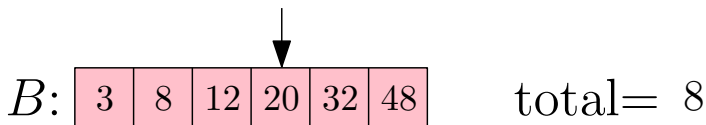
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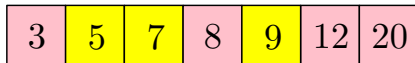


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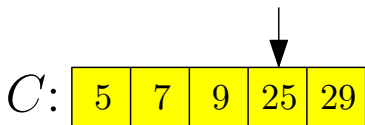
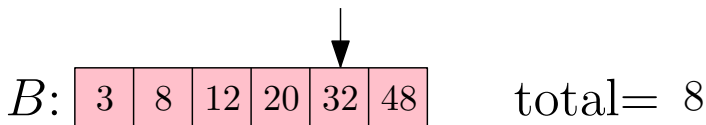


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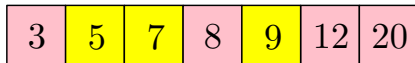


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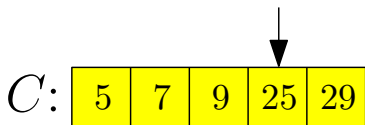
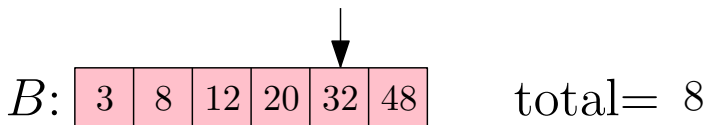


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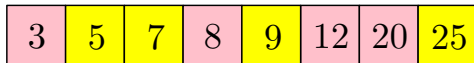


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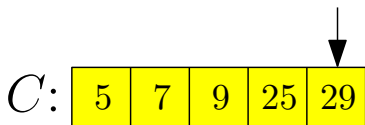
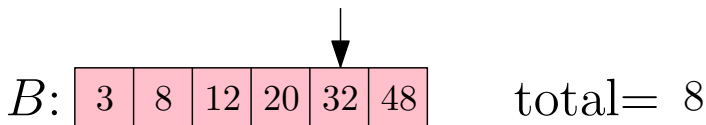


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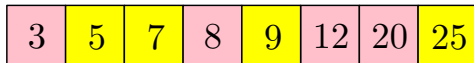


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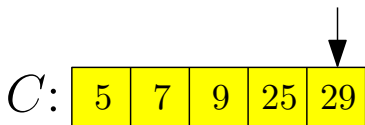
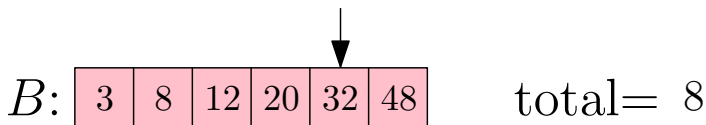


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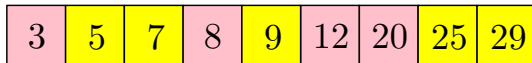


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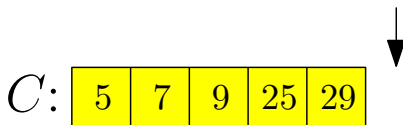
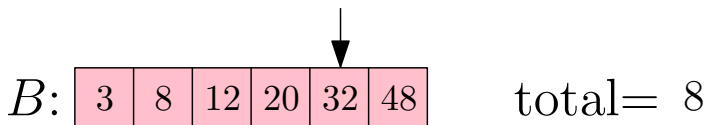


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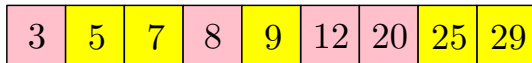


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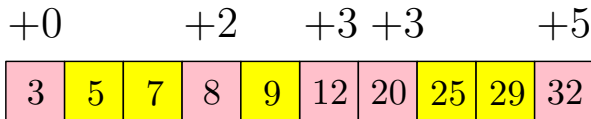
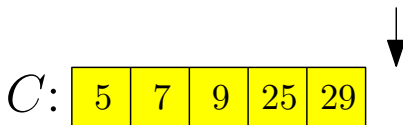
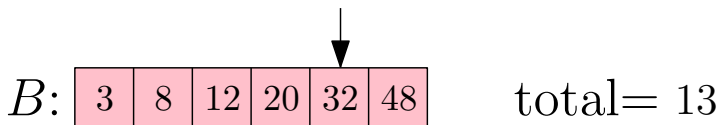


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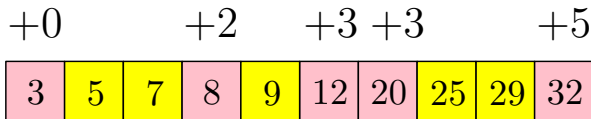
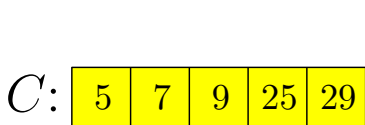
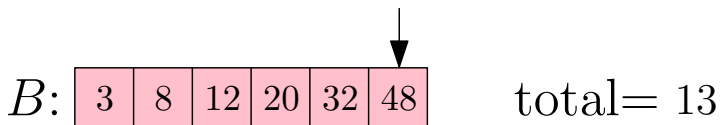
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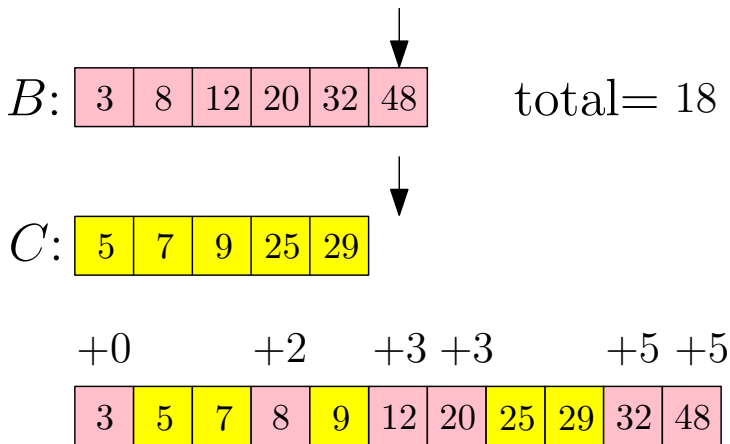
Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:



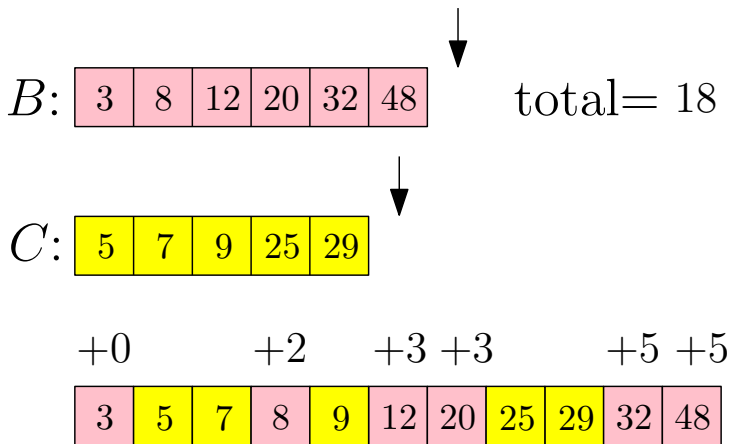
Counting Inversions between B and C

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Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:



Count Inversions between B and C

- Procedure that merges B and C and counts inversions between B and C at the same time

merge-and-count(B, C, n_1, n_2)

```
1:  $count \leftarrow 0$ ;  
2:  $A \leftarrow$  array of size  $n_1 + n_2$ ;  $i \leftarrow 1$ ;  $j \leftarrow 1$   
3: while  $i \leq n_1$  or  $j \leq n_2$  do  
4:   if  $j > n_2$  or ( $i \leq n_1$  and  $B[i] \leq C[j]$ ) then  
5:      $A[i + j - 1] \leftarrow B[i]$ ;  $i \leftarrow i + 1$   
6:      $count \leftarrow count + (j - 1)$   
7:   else  
8:      $A[i + j - 1] \leftarrow C[j]$ ;  $j \leftarrow j + 1$   
9: return ( $A, count$ )
```

Sort and Count Inversions in A

- A procedure that returns the sorted array of A and counts the number of inversions in A :

sort-and-count(A, n)

```
1: if  $n = 1$  then  
2:   return ( $A, 0$ )  
3: else  
4:    $(B, m_1) \leftarrow \text{sort-and-count}(A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor)$   
5:    $(C, m_2) \leftarrow \text{sort-and-count}(A[\lfloor n/2 \rfloor + 1..n], \lceil n/2 \rceil)$   
6:    $(A, m_3) \leftarrow \text{merge-and-count}(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$   
7:   return ( $A, m_1 + m_2 + m_3$ )
```


Sort and Count Inversions in A

- A procedure that returns the sorted array of A and counts the number of inversions in A :

sort-and-count(A, n)

1: **if** $n = 1$ **then**

2: **return** $(A, 0)$

3: **else**

4: $(B, m_1) \leftarrow \text{sort-and-count}(A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor)$

5: $(C, m_2) \leftarrow \text{sort-and-count}(A[\lfloor n/2 \rfloor + 1..n], \lceil n/2 \rceil)$

6: $(A, m_3) \leftarrow \text{merge-and-count}(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$

7: **return** $(A, m_1 + m_2 + m_3)$

- Divide: trivial

- Conquer: 4, 5

- Combine: 6, 7

sort-and-count(A, n)

```
1: if  $n = 1$  then  
2:   return ( $A, 0$ )  
3: else  
4:    $(B, m_1) \leftarrow \text{sort-and-count}(A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor)$   
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```

- Recurrence for the running time: $T(n) = 2T(n/2) + O(n)$

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```

- Recurrence for the running time: $T(n) = 2T(n/2) + O(n)$
- Running time = $O(n \log n)$

Example

`sort-and-count(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)`

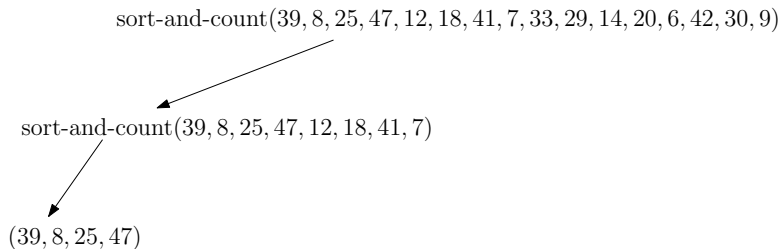
Example

sort-and-count(39, 8, 25, 47, 12, 18, 41, 7, 33, 29, 14, 20, 6, 42, 30, 9)

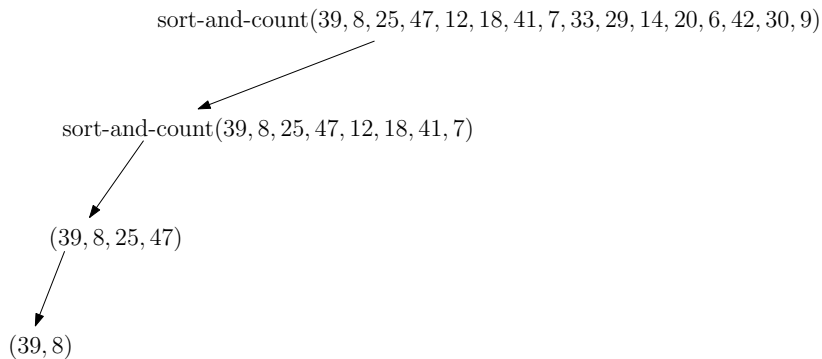


sort-and-count(39, 8, 25, 47, 12, 18, 41, 7)

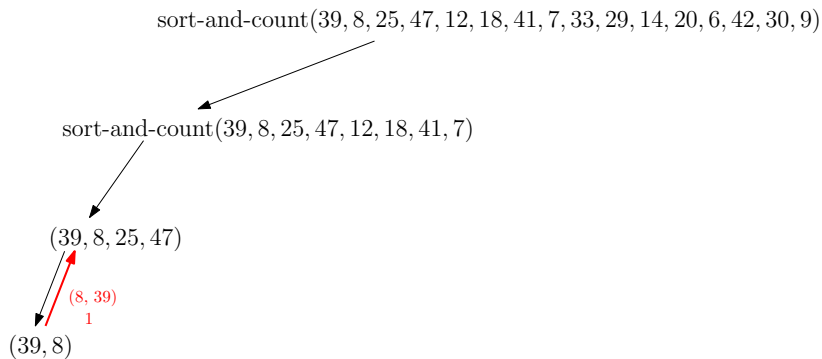
Example



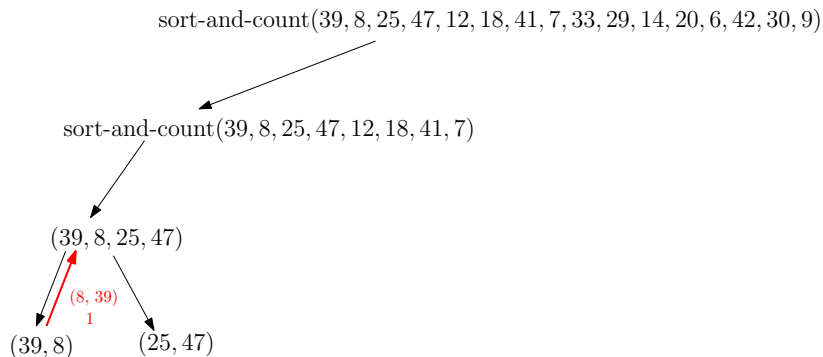
Example



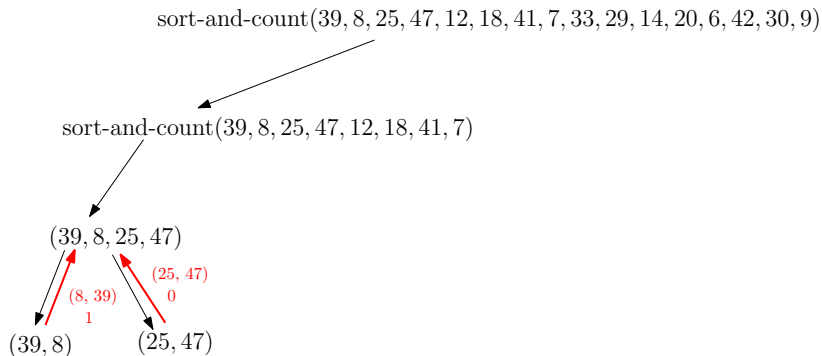
Example



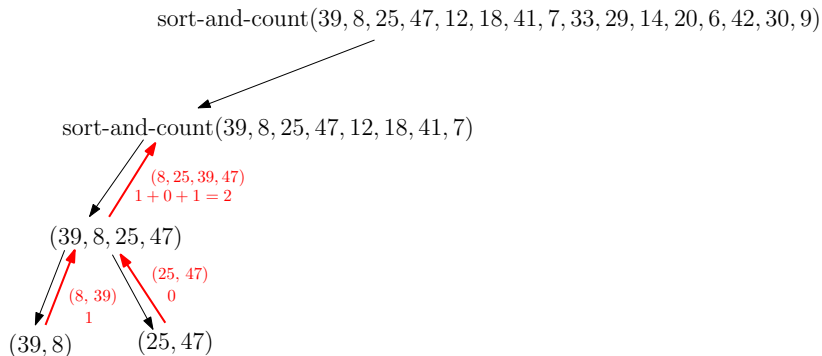
Example



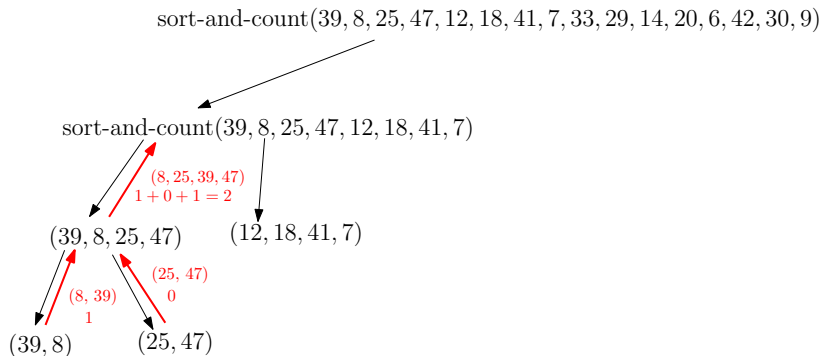
Example



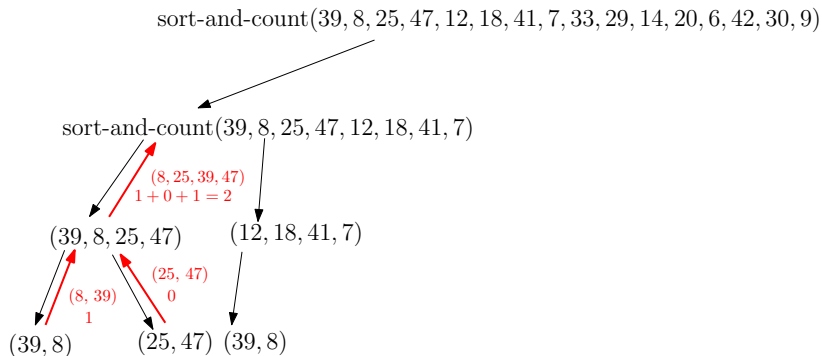
Example



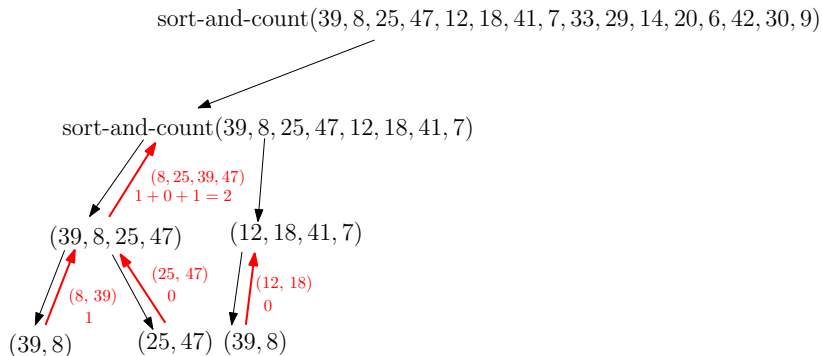
Example



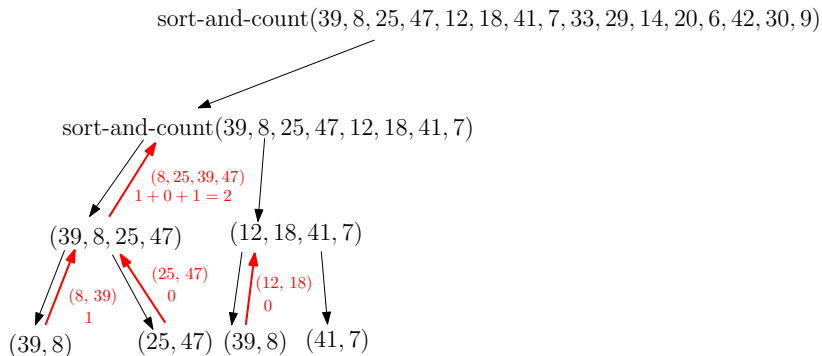
Example



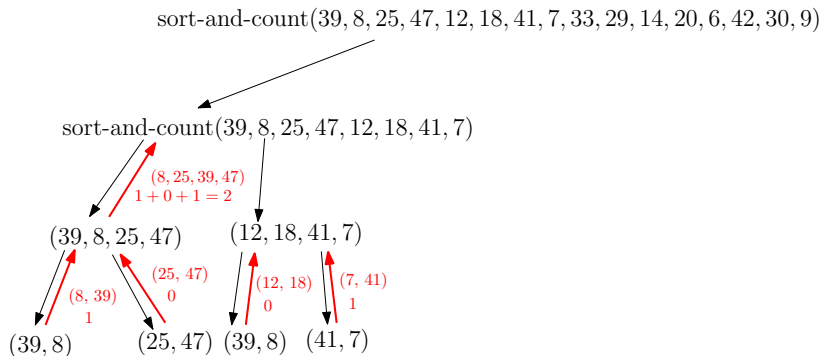
Example



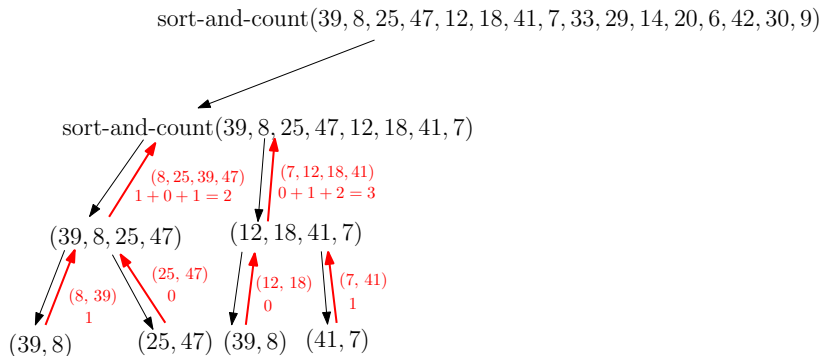
Example



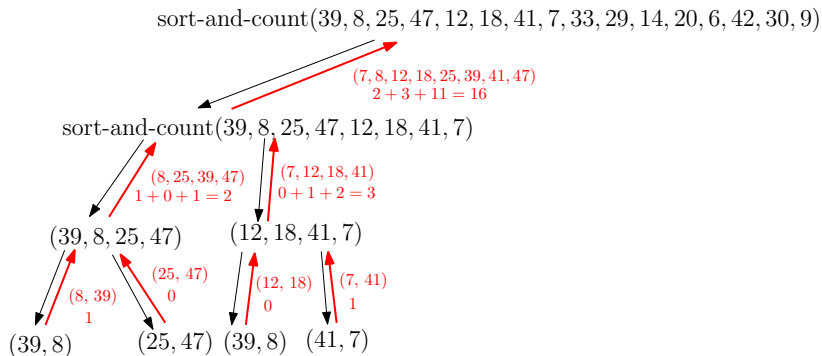
Example



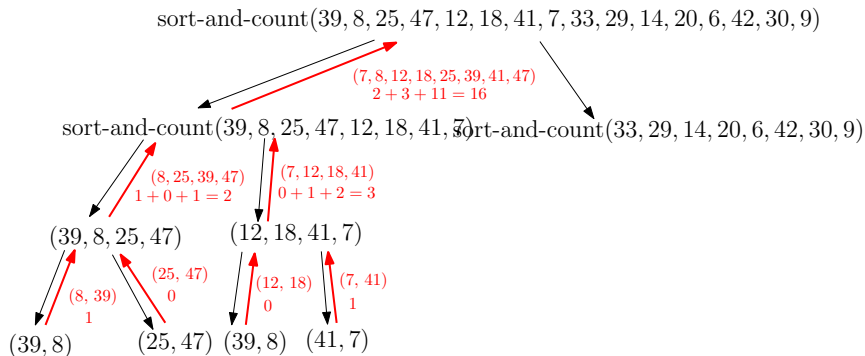
Example



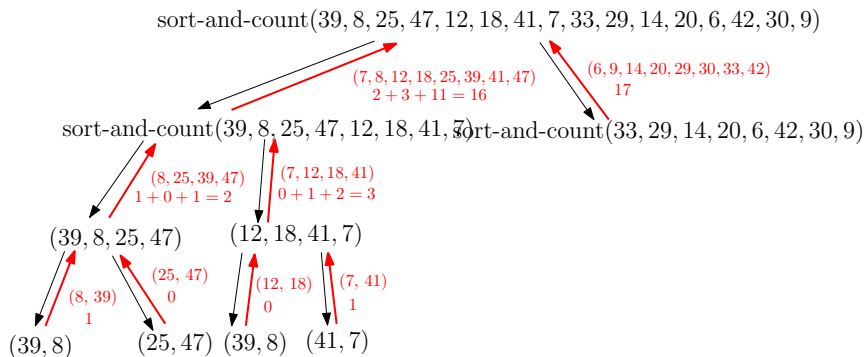
Example



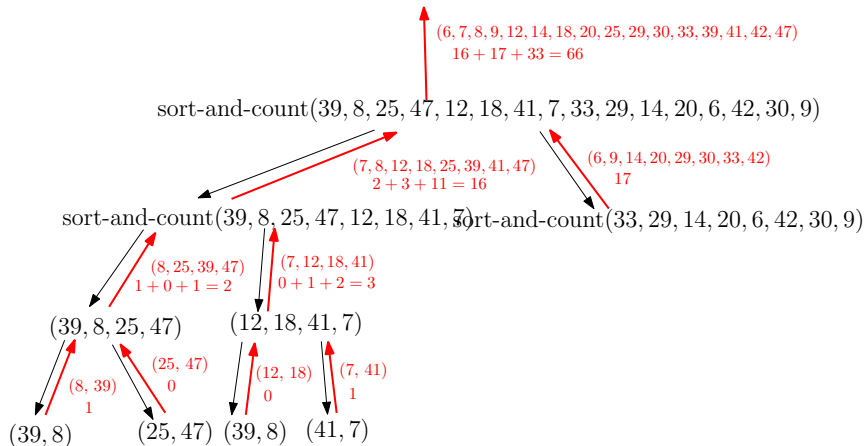
Example



Example



Example



Outline

- 1 Divide-and-Conquer
- 2 Counting Inversions
- 3 Solving Recurrences
- 4 Quicksort and Selection
 - Quicksort
 - Lower Bound for Comparison-Based Sorting Algorithms
 - Selection Problem
- 5 Polynomial Multiplication
- 6 Strassen's Algorithm for Matrix Multiplication
- 7 FFT(Fast Fourier Transform): Polynomial Multiplication in $O(n \log n)$ Time
- 8 Finding Closest Pair of Points in 2D Euclidean Space
- 9 Computing n -th Fibonacci Number

Methods for Solving Recurrences

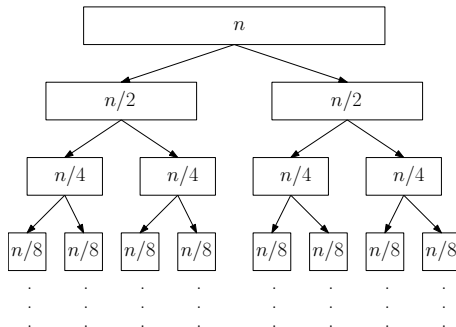
- The recursion-tree method
- The master theorem

Recursion-Tree Method

- $T(n) = 2T(n/2) + O(n)$

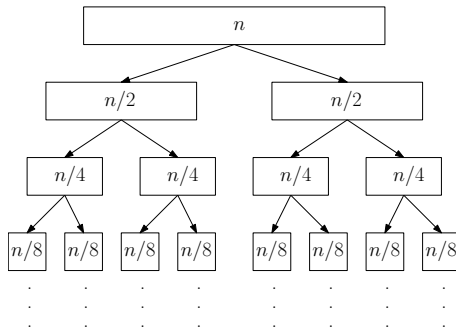
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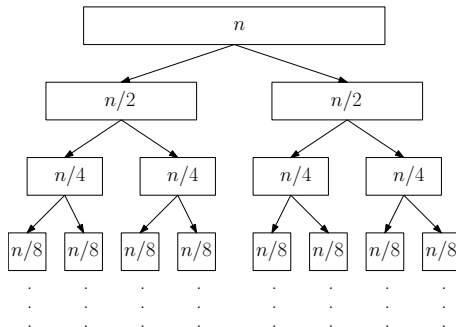
- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$

Recursion-Tree Method

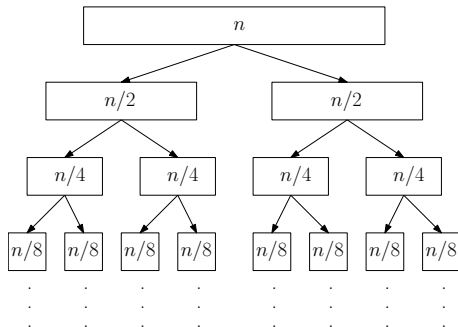
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- There are $O(\log n)$ levels

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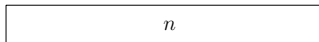
- Each level takes running time $O(n)$
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Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$

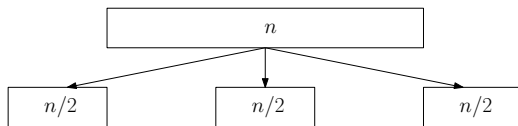
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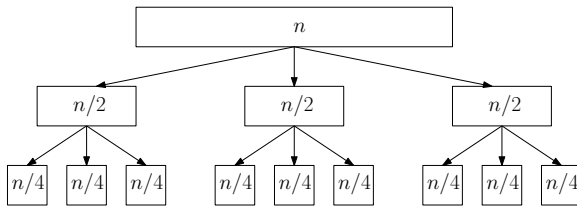
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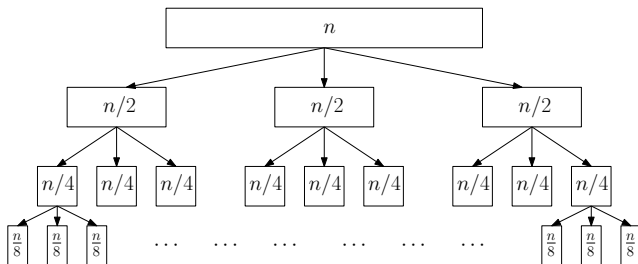
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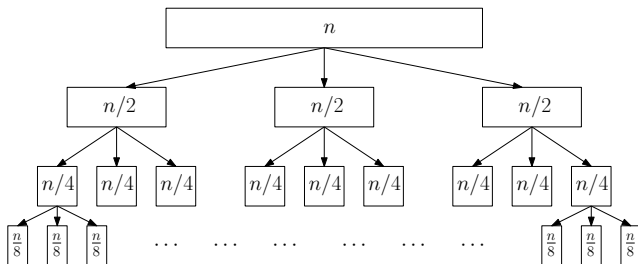
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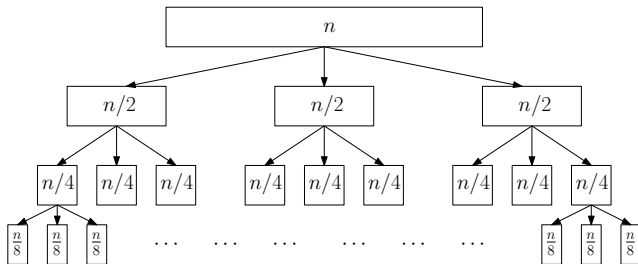
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- Total running time at level i ?

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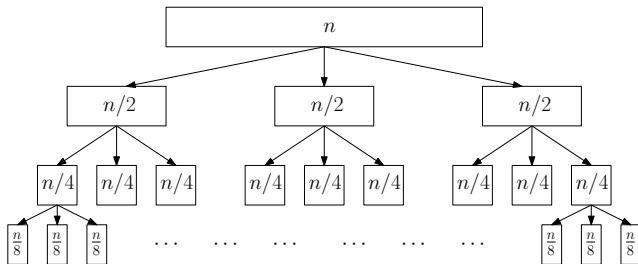
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Recursion-Tree Method

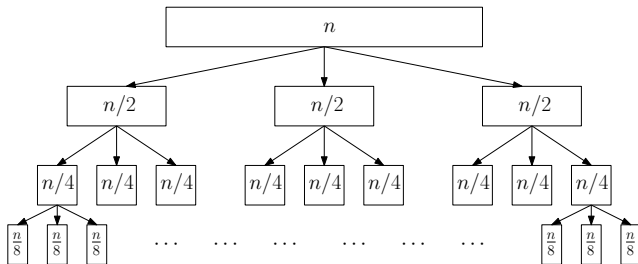
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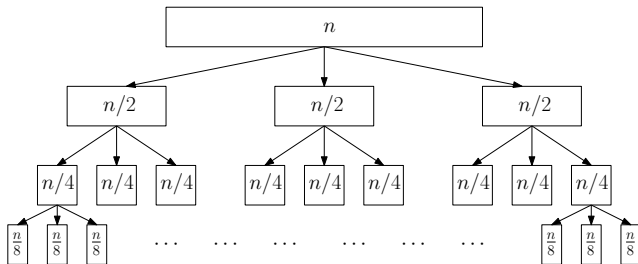
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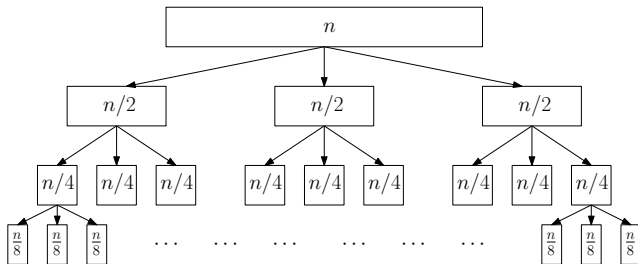
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$$\sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i n = O\left(n \left(\frac{3}{2}\right)^{\log_2 n}\right) = O(3^{\log_2 n}) = O(n^{\log_2 3}).$$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$

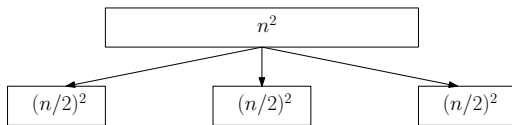
Recursion-Tree Method

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n^2

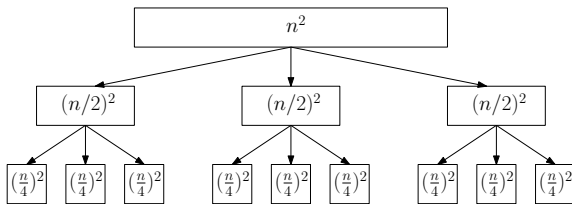
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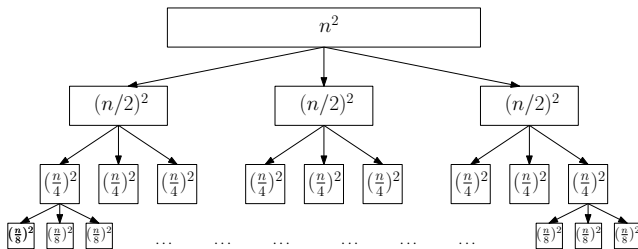
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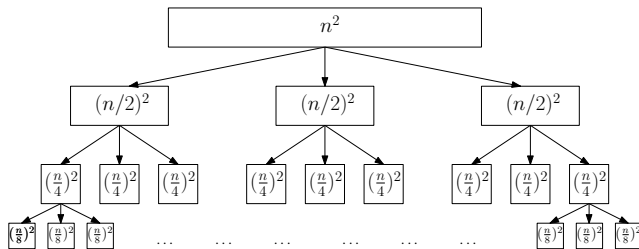
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Recursion-Tree Method

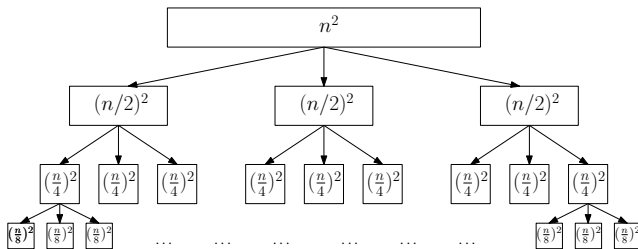
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Recursion-Tree Method

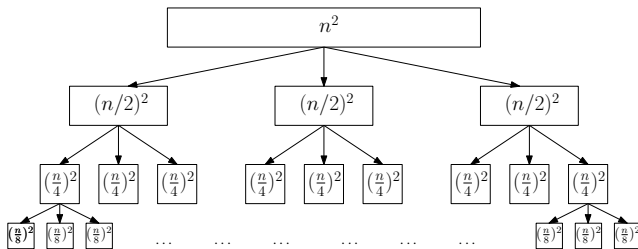
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Recursion-Tree Method

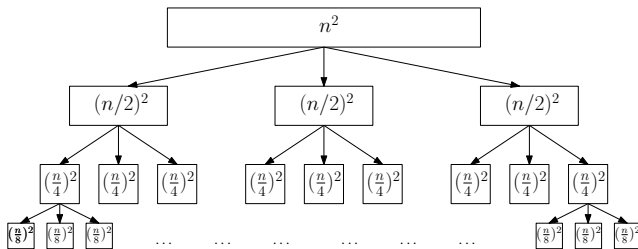
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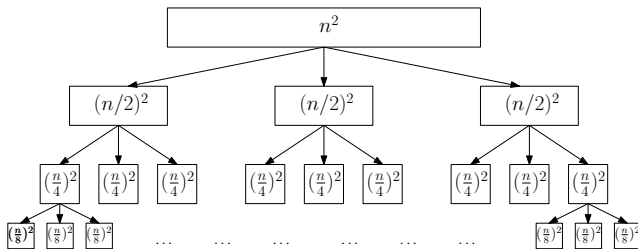
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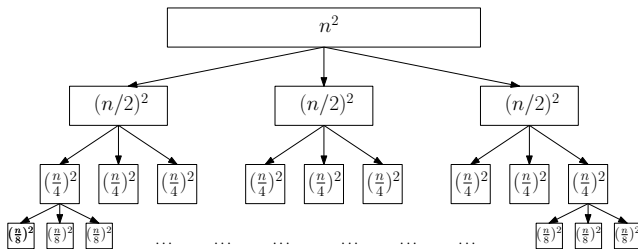
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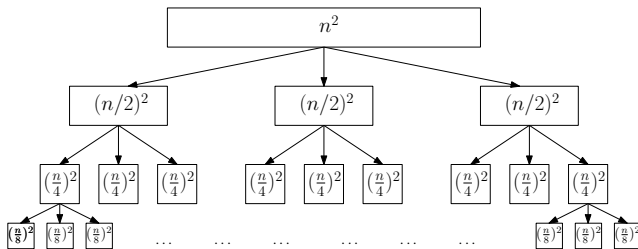


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Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$				$O(n \log n)$
$T(n) = 3T(n/2) + O(n)$				$O(n^{\log_2 3})$
$T(n) = 3T(n/2) + O(n^2)$				$O(n^2)$

Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \log n)$
$T(n) = 3T(n/2) + O(n)$				$O(n^{\log_2 3})$
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Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \log n)$
$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\log_2 3})$
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$$T(n) = \begin{cases} & \text{if } c < \log_b a \\ & \text{if } c = \log_b a \\ & \text{if } c > \log_b a \end{cases}$$

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$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\log_2 3})$
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$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\log_2 3})$
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$$T(n) = \begin{cases} O(n^{\log_b a}) & \text{if } c < \log_b a \\ & \text{if } c = \log_b a \\ & \text{if } c > \log_b a \end{cases}$$

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \log n)$
$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\log_2 3})$
$T(n) = 3T(n/2) + O(n^2)$	3	2	2	$O(n^2)$

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Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

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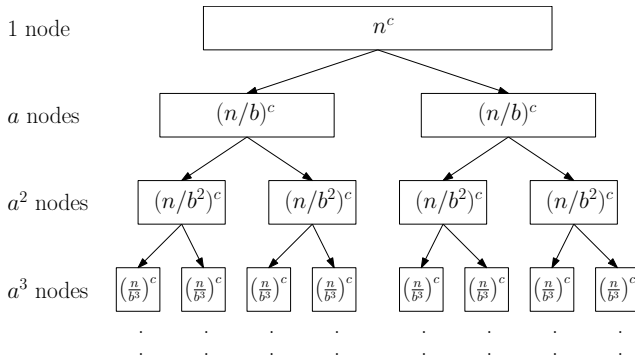
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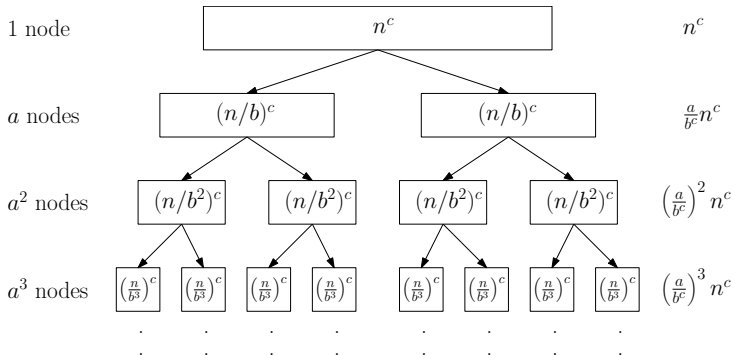
Proof of Master Theorem Using Recursion Tree

$$T(n) = aT(n/b) + O(n^c)$$



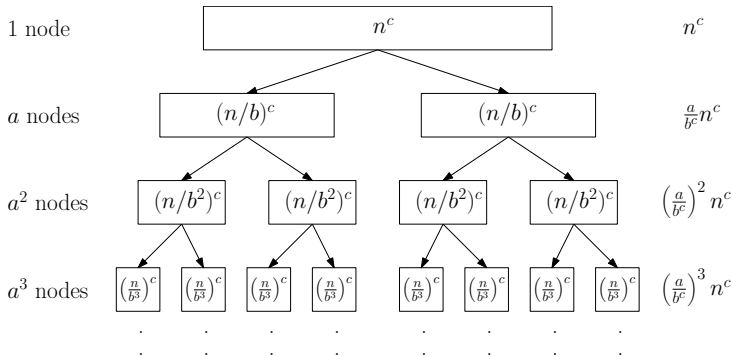
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Proof of Master Theorem Using Recursion Tree

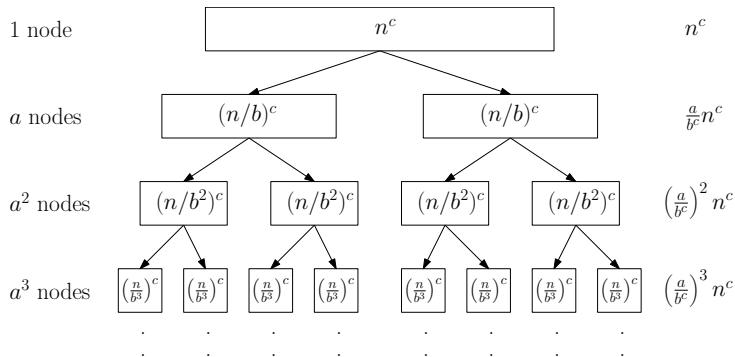
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- $c < \log_b a$: bottom-level dominates: $(\frac{a}{b^c})^{\log_b n} n^c = n^{\log_b a}$

Proof of Master Theorem Using Recursion Tree

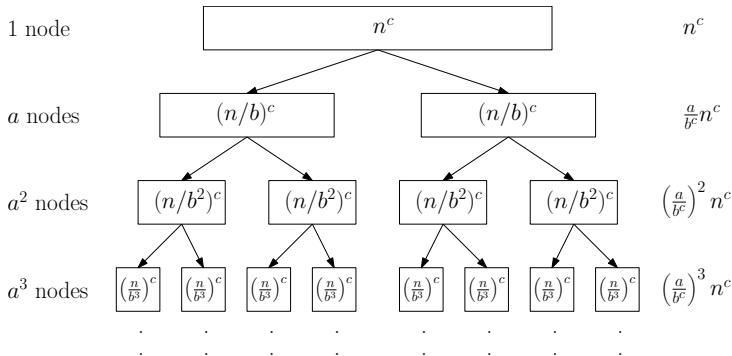
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- $c = \log_b a$: all levels have same time: $n^c \log_b n = O(n^c \log n)$
- $c > \log_b a$: top-level dominates: $O(n^c)$

Outline

- 1 Divide-and-Conquer
- 2 Counting Inversions
- 3 Solving Recurrences
- 4 Quicksort and Selection
 - Quicksort
 - Lower Bound for Comparison-Based Sorting Algorithms
 - Selection Problem
- 5 Polynomial Multiplication
- 6 Strassen's Algorithm for Matrix Multiplication
- 7 FFT(Fast Fourier Transform): Polynomial Multiplication in $O(n \log n)$ Time
- 8 Finding Closest Pair of Points in 2D Euclidean Space
- 9 Computing n -th Fibonacci Number

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Quicksort vs Merge-Sort

	Merge Sort	Quicksort
Divide	Trivial	Separate small and big numbers
Conquer	Recurse	Recurse
Combine	Merge 2 sorted arrays	Trivial

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

A :

29	82	75	64	38	45	94	69	25	76	15	92	37	17	85
----	----	----	----	----	----	----	----	----	----	----	----	----	----	----

`quicksort( $A$ , 1, 15)`

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quicksort(A , 1, 15)

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quicksort($A, 1, 15$)

quicksort($A, 1, 7$)



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quicksort($A, 1, 15$)

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Quicksort Example

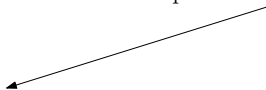
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quicksort($A, 5, 7$)

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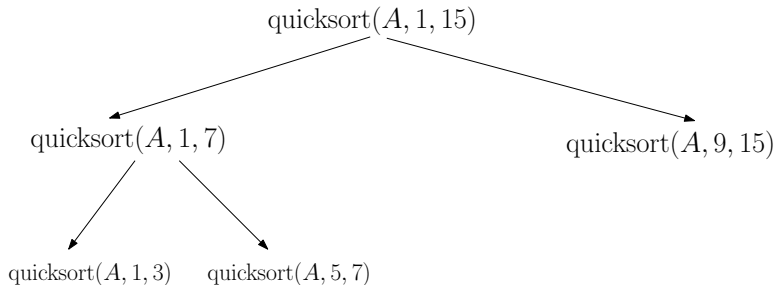
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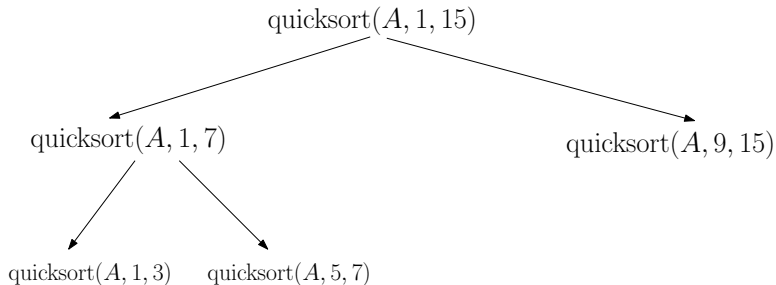


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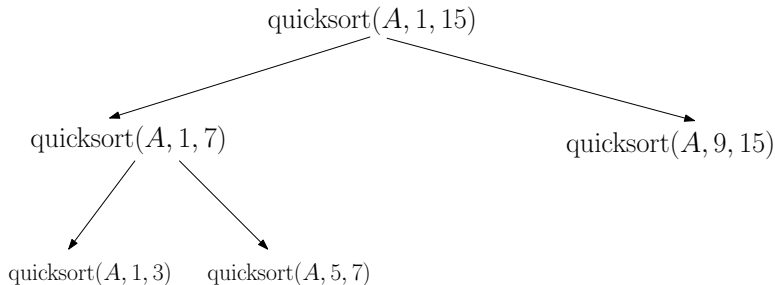


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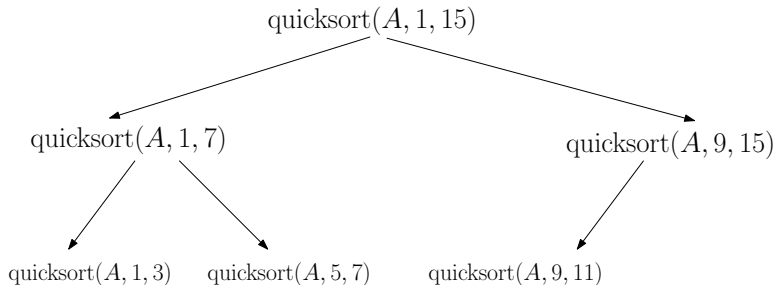


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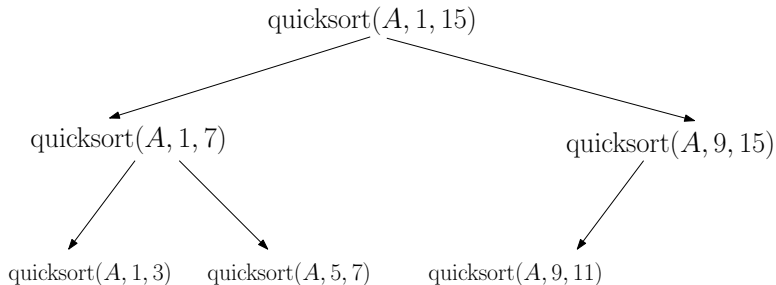


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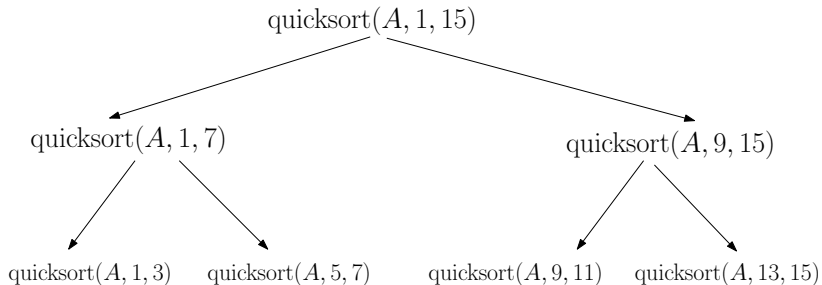


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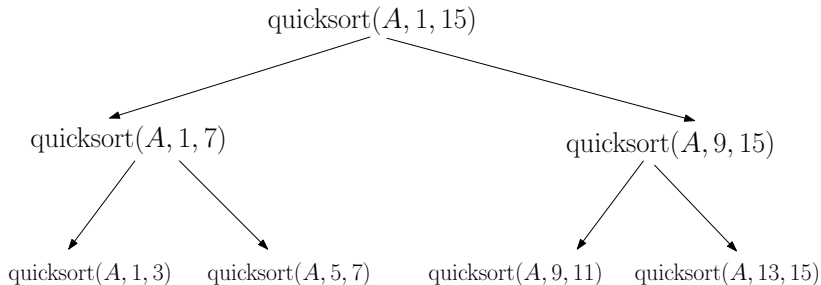


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Quicksort

quicksort(A, n)

- 1: **if** $n \leq 1$ **then return** A
- 2: $x \leftarrow$ lower median of A
- 3: $A_L \leftarrow$ array of elements in A that are less than x $\backslash\backslash$ Divide
- 4: $A_R \leftarrow$ array of elements in A that are greater than x $\backslash\backslash$ Divide
- 5: $B_L \leftarrow$ quicksort(A_L , length of A_L) $\backslash\backslash$ Conquer
- 6: $B_R \leftarrow$ quicksort(A_R , length of A_R) $\backslash\backslash$ Conquer
- 7: $t \leftarrow$ number of times x appear A
- 8: **return** concatenation of B_L , t copies of x , and B_R

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- Recurrence $T(n) \leq 2T(n/2) + O(n)$

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- Recurrence $T(n) \leq 2T(n/2) + O(n)$
- Running time = $O(n \log n)$

Assumption We can choose median of an array of size n in $O(n)$ time.

Q: How to remove this assumption?

Assumption We can choose median of an array of size n in $O(n)$ time.

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- 2 Choose a **pivot randomly** and pretend it is the median (it is practical)

Quicksort Using A Random Pivot

quicksort(A, n)

- 1: **if** $n \leq 1$ **then return** A
- 2: $x \leftarrow$ a random element of A (x is called a **pivot**)
- 3: $A_L \leftarrow$ array of elements in A that are less than x $\backslash\backslash$ Divide
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- In theory: assume they can.

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Lemma The **expected** running time of the algorithm is $O(n \log n)$.

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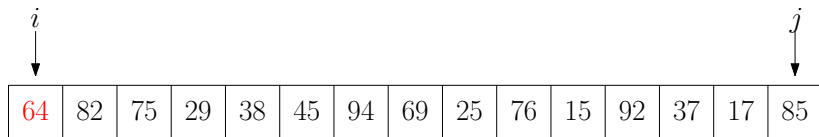
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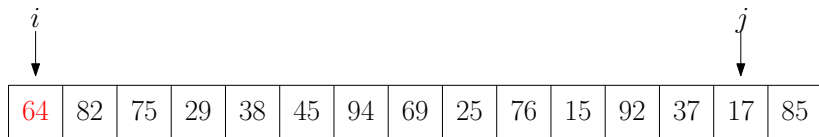
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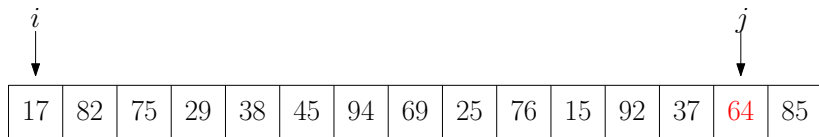
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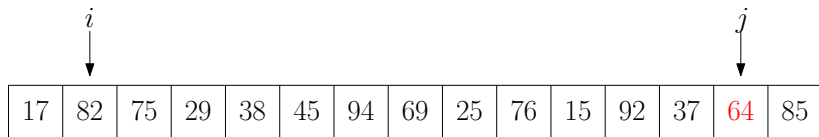
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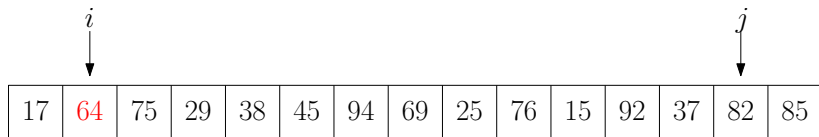
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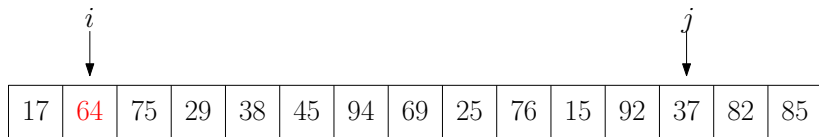
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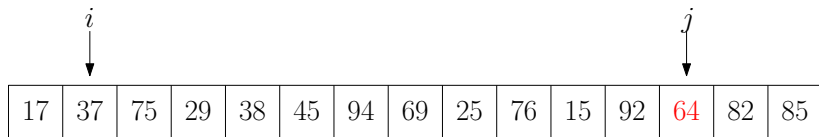
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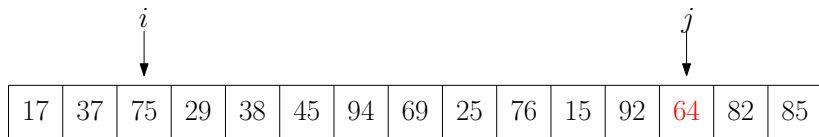
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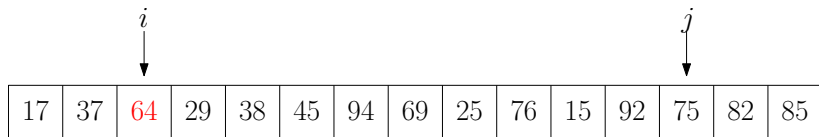
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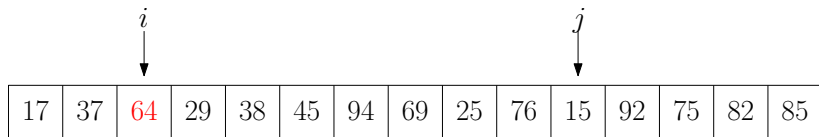
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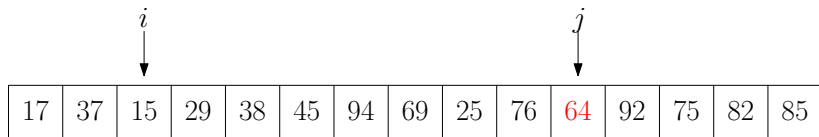
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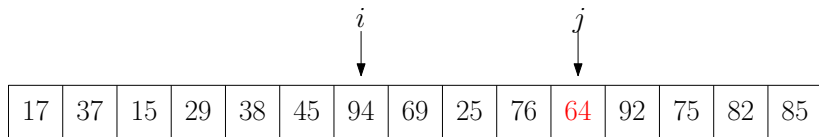
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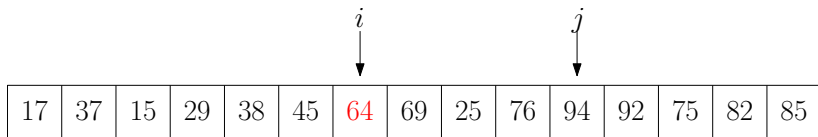
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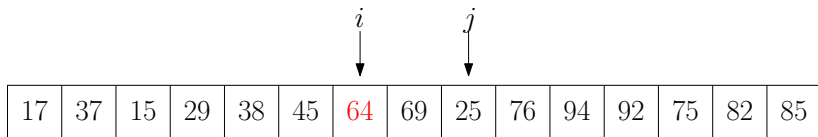
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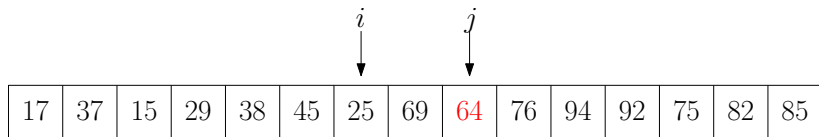
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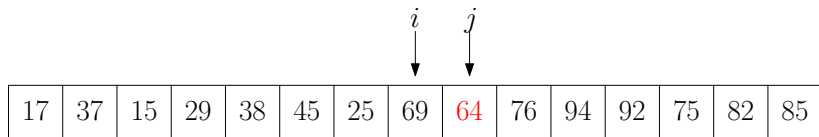
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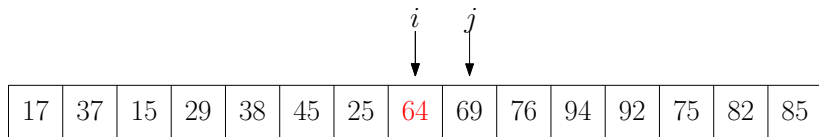
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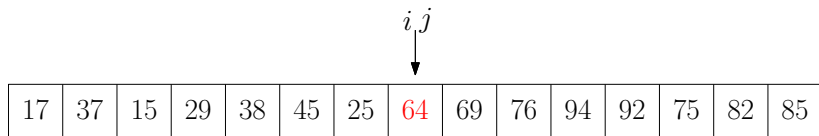
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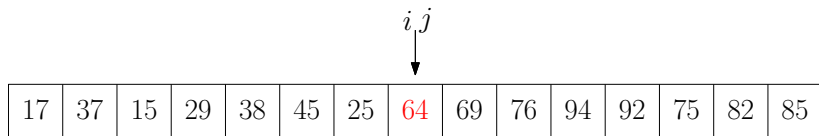
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- To partition the array into two parts, we only need $O(1)$ extra space.

partition(A, ℓ, r)

```
1:  $p \leftarrow$  random integer between  $\ell$  and  $r$ , swap  $A[p]$  and  $A[\ell]$ 
2:  $i \leftarrow \ell, j \leftarrow r$ 
3: while true do
4:   while  $i < j$  and  $A[i] < A[j]$  do  $j \leftarrow j - 1$ 
5:   if  $i = j$  then break
6:   swap  $A[i]$  and  $A[j]; i \leftarrow i + 1$ 
7:   while  $i < j$  and  $A[i] < A[j]$  do  $i \leftarrow i + 1$ 
8:   if  $i = j$  then break
9:   swap  $A[i]$  and  $A[j]; j \leftarrow j - 1$ 
10: return  $i$ 
```

In-Place Implementation of Quick-Sort

quicksort(A, ℓ, r)

- 1: **if** $\ell \geq r$ **then return**
- 2: $m \leftarrow \text{partition}(A, \ell, r)$
- 3: **quicksort**($A, \ell, m - 1$)
- 4: **quicksort**($A, m + 1, r$)

- To sort an array A of size n , call **quicksort**($A, 1, n$).

Note: We pass the array A by reference, instead of by copying.

Merge-Sort is Not In-Place

- To merge two arrays, we need a third array with size equaling the total size of two arrays

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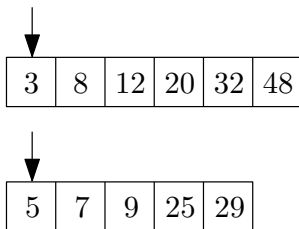
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3	8	12	20	32	48
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5	7	9	25	29
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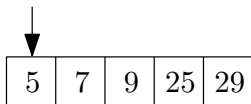
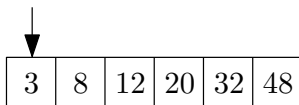
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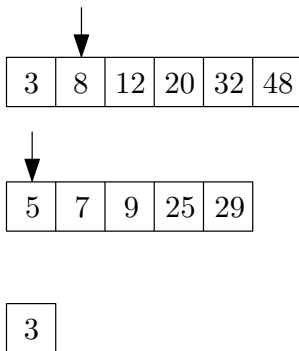
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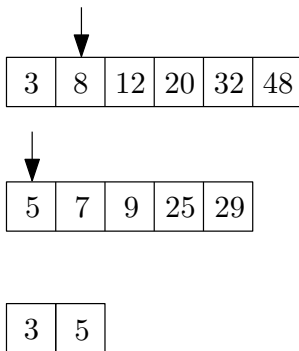
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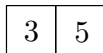
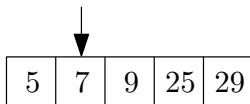
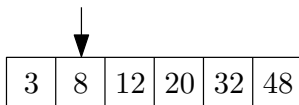
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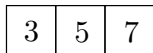
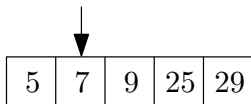
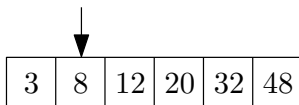
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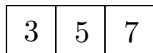
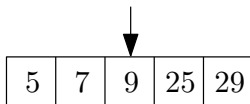
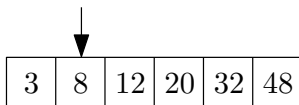
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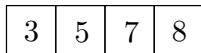
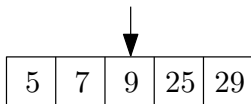
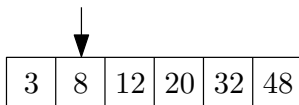
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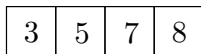
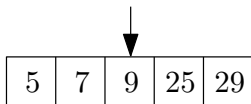
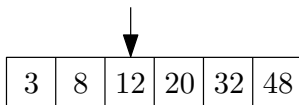
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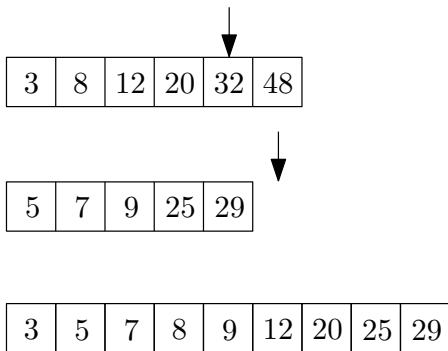
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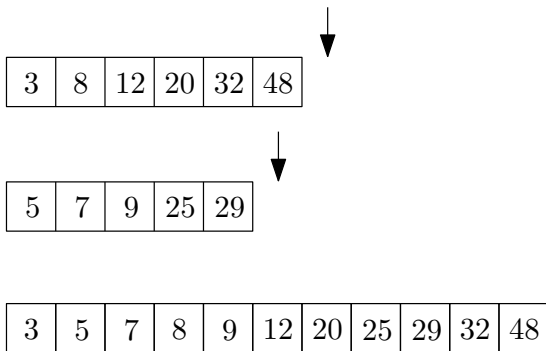
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- 1 Divide-and-Conquer
- 2 Counting Inversions
- 3 Solving Recurrences
- 4 **Quicksort and Selection**
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Comparison-Based Sorting Algorithms

- To sort, we are only allowed to **compare** two elements
- We can not use “internal structures” of the elements

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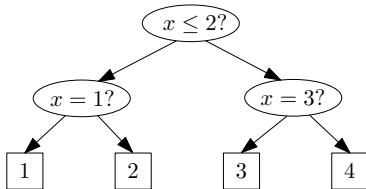
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- Our goal: $O(n)$ running time

Recall: Quicksort with Median Finder

quicksort(A, n)

- 1: **if** $n \leq 1$ **then return** A
- 2: $x \leftarrow$ lower median of A
- 3: $A_L \leftarrow$ elements in A that are less than x ▷ Divide
- 4: $A_R \leftarrow$ elements in A that are greater than x ▷ Divide
- 5: $B_L \leftarrow$ quicksort($A_L, A_L.size$) ▷ Conquer
- 6: $B_R \leftarrow$ quicksort($A_R, A_R.size$) ▷ Conquer
- 7: $t \leftarrow$ number of times x appear A
- 8: **return** the array obtained by concatenating B_L , the array containing t copies of x , and B_R

Selection Algorithm with Median Finder

selection(A, n, i)

- 1: **if** $n = 1$ **then return** A
- 2: $x \leftarrow$ lower median of A
- 3: $A_L \leftarrow$ elements in A that are less than x ▷ Divide
- 4: $A_R \leftarrow$ elements in A that are greater than x ▷ Divide
- 5: **if** $i \leq A_L.\text{size}$ **then**
- 6: **return** selection($A_L, A_L.\text{size}, i$) ▷ Conquer
- 7: **else if** $i > n - A_R.\text{size}$ **then**
- 8: **return** selection($A_R, A_R.\text{size}, i - (n - A_R.\text{size})$) ▷ Conquer
- 9: **else**
- 10: **return** x

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- Recurrence for selection: $T(n) = T(n/2) + O(n)$

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- Recurrence for selection: $T(n) = T(n/2) + O(n)$
- Solving recurrence: $T(n) = O(n)$

Randomized Selection Algorithm

selection(A, n, i)

- 1: **if** $n = 1$ **then return** A
- 2: $x \leftarrow$ **random element** of A (called **pivot**)
- 3: $A_L \leftarrow$ elements in A that are less than x ▷ Divide
- 4: $A_R \leftarrow$ elements in A that are greater than x ▷ Divide
- 5: **if** $i \leq A_L.\text{size}$ **then**
- 6: **return** selection($A_L, A_L.\text{size}, i$) ▷ Conquer
- 7: **else if** $i > n - A_R.\text{size}$ **then**
- 8: **return** selection($A_R, A_R.\text{size}, i - (n - A_R.\text{size})$) ▷ Conquer
- 9: **else**
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Randomized Selection Algorithm

selection(A, n, i)

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5: if  $i \leq A_L.size$  then
6:   return selection( $A_L, A_L.size, i$ )                                           ▷ Conquer
7: else if  $i > n - A_R.size$  then
8:   return selection( $A_R, A_R.size, i - (n - A_R.size)$ )                        ▷ Conquer
9: else
10:  return  $x$ 
```

- **expected** running time = $O(n)$

Outline

- 1 Divide-and-Conquer
- 2 Counting Inversions
- 3 Solving Recurrences
- 4 Quicksort and Selection
 - Quicksort
 - Lower Bound for Comparison-Based Sorting Algorithms
 - Selection Problem
- 5 Polynomial Multiplication**
- 6 Strassen's Algorithm for Matrix Multiplication
- 7 FFT(Fast Fourier Transform): Polynomial Multiplication in $O(n \log n)$ Time
- 8 Finding Closest Pair of Points in 2D Euclidean Space
- 9 Computing n -th Fibonacci Number

Polynomial Multiplication

Input: two polynomials of degree $n - 1$

Output: product of two polynomials

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Example:

$$(3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5)$$

Polynomial Multiplication

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Output: product of two polynomials

Example:

$$\begin{aligned} & (3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5) \\ &= 6x^6 - 9x^5 + 18x^4 - 15x^3 \\ &\quad + 4x^5 - 6x^4 + 12x^3 - 10x^2 \\ &\quad - 10x^4 + 15x^3 - 30x^2 + 25x \\ &\quad + 8x^3 - 12x^2 + 24x - 20 \\ &= 6x^6 - 5x^5 + 2x^4 + 20x^3 - 52x^2 + 49x - 20 \end{aligned}$$

Polynomial Multiplication

Input: two polynomials of degree $n - 1$

Output: product of two polynomials

Example:

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- **Input:** $(4, -5, 2, 3), (-5, 6, -3, 2)$
- **Output:** $(-20, 49, -52, 20, 2, -5, 6)$

Discrete Convolution on Finite Domain

- $f : \{0, 1, \dots, n-1\} \rightarrow \mathbb{R}, g : \{0, 1, \dots, m-1\} \rightarrow \mathbb{R}$
- the **convolution** of f and g , denoted as $h := f \times g$, is defined as

$$h(k) := \sum_{i,j:i+j=k} f(i)g(j) \quad \forall k \in \{0, 1, 2, \dots, m+n-2\}$$

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	0	1	2	3	4	5	6
f	4	-5	2	3			
g	-5	6	-3	2			
$f \times g$	-20	49	-52	20	2	-5	6

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Applications of Convolutions

- Polynomial and integer multiplication
- Signal and Image Processing
- Probability theory: Sum of two distributions
- Convolutional neural network

- Polynomial multiplication $\Leftrightarrow$ Convolution
- We shall focus on multiplication.

Big Integer Multiplication Using Polynomial Multiplication

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Big Integer Multiplication Using Polynomial Multiplication

- $16103416169 \times 424317167$

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Big Integer Multiplication Using Polynomial Multiplication

- $16103416169 \times 424317167$
- $(16x^3 + 103x^2 + 416x + 169) \times (424x^2 + 317x + 167)$

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Naïve Algorithm

polynomial-multiplication(A, B, n)

```
1: let  $C[k] \leftarrow 0$  for every  $k = 0, 1, 2, \dots, 2n - 2$ 
2: for  $i \leftarrow 0$  to  $n - 1$  do
3:   for  $j \leftarrow 0$  to  $n - 1$  do
4:      $C[i + j] \leftarrow C[i + j] + A[i] \times B[j]$ 
5: return  $C$ 
```

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```

Running time: $O(n^2)$

Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x + 2)x^2 + (-5x + 4)$$

$$q(x) = 2x^3 - 3x^2 + 6x - 5 = (2x - 3)x^2 + (6x - 5)$$

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- $p(x)$: degree of $n - 1$ (assume n is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

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$$\begin{aligned}pq &= (p_Hx^{n/2} + p_L)(q_Hx^{n/2} + q_L) \\ &= p_Hq_Hx^n + (p_Hq_L + p_Lq_H)x^{n/2} + p_Lq_L\end{aligned}$$

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$$\begin{aligned}\text{multiply}(p, q) &= \text{multiply}(p_H, q_H) \times x^n \\&\quad + (\text{multiply}(p_H, q_L) + \text{multiply}(p_L, q_H)) \times x^{n/2} \\&\quad + \text{multiply}(p_L, q_L)\end{aligned}$$

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- Recurrence: $T(n) = 4T(n/2) + O(n)$

Divide-and-Conquer for Polynomial Multiplication

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- Recurrence: $T(n) = 4T(n/2) + O(n)$
- $T(n) = O(n^2)$

Reduce Number from 4 to 3

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$$\begin{aligned}pq &= (p_H x^{n/2} + p_L)(q_H x^{n/2} + q_L) \\ &= p_H q_H x^n + (p_H q_L + p_L q_H) x^{n/2} + p_L q_L\end{aligned}$$

Reduce Number from 4 to 3

$$\begin{aligned}pq &= (p_H x^{n/2} + p_L)(q_H x^{n/2} + q_L) \\ &= p_H q_H x^n + (p_H q_L + p_L q_H) x^{n/2} + p_L q_L\end{aligned}$$

- $p_H q_L + p_L q_H = (p_H + p_L)(q_H + q_L) - p_H q_H - p_L q_L$

Divide-and-Conquer for Polynomial Multiplication

Divide-and-Conquer for Polynomial Multiplication

$$r_H = \text{multiply}(p_H, q_H)$$

$$r_L = \text{multiply}(p_L, q_L)$$

Divide-and-Conquer for Polynomial Multiplication

$$r_H = \text{multiply}(p_H, q_H)$$

$$r_L = \text{multiply}(p_L, q_L)$$

$$\text{multiply}(p, q) = r_H \times x^n$$

$$\begin{aligned} &+ (\text{multiply}(p_H + p_L, q_H + q_L) - r_H - r_L) \times x^{n/2} \\ &+ r_L \end{aligned}$$

Divide-and-Conquer for Polynomial Multiplication

$$r_H = \text{multiply}(p_H, q_H)$$

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Divide-and-Conquer for Polynomial Multiplication

$$r_H = \text{multiply}(p_H, q_H)$$

$$r_L = \text{multiply}(p_L, q_L)$$

$$\begin{aligned}\text{multiply}(p, q) &= r_H \times x^n \\ &\quad + (\text{multiply}(p_H + p_L, q_H + q_L) - r_H - r_L) \times x^{n/2} \\ &\quad + r_L\end{aligned}$$

- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\log_2 3}) = O(n^{1.585})$

Assumption n is a power of 2. Arrays are 0-indexed.

multiply(A, B, n)

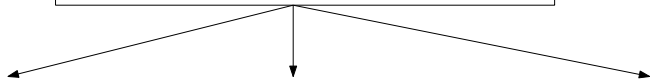
- 1: if $n = 1$ then return ($A[0]B[0]$)
- 2: $A_L \leftarrow A[0 .. n/2 - 1], A_H \leftarrow A[n/2 .. n - 1]$
- 3: $B_L \leftarrow B[0 .. n/2 - 1], B_H \leftarrow B[n/2 .. n - 1]$
- 4: $C_L \leftarrow \text{multiply}(A_L, B_L, n/2)$
- 5: $C_H \leftarrow \text{multiply}(A_H, B_H, n/2)$
- 6: $C_M \leftarrow \text{multiply}(A_L + A_H, B_L + B_H, n/2)$
- 7: $C \leftarrow$ array of $(2n - 1)$ 0's
- 8: **for** $i \leftarrow 0$ to $n - 2$ **do**
- 9: $C[i] \leftarrow C[i] + C_L[i]$
- 10: $C[i + n] \leftarrow C[i + n] + C_H[i]$
- 11: $C[i + n/2] \leftarrow C[i + n/2] + C_M[i] - C_L[i] - C_H[i]$
- 12: **return** C

Example

$$\begin{aligned} & (3 + 2x + 2x^2 + 4x^3 + x^4 + 2x^5 + x^6 + 5x^7) \\ & \times (2 + x - x^2 + 2x^3 - 2x^4 - x^5 + 2x^6 - 2x^7) \end{aligned}$$

Example

$$(3 + 2x + 2x^2 + 4x^3 + x^4 + 2x^5 + x^6 + 5x^7) \\ \times (2 + x - x^2 + 2x^3 - 2x^4 - x^5 + 2x^6 - 2x^7)$$

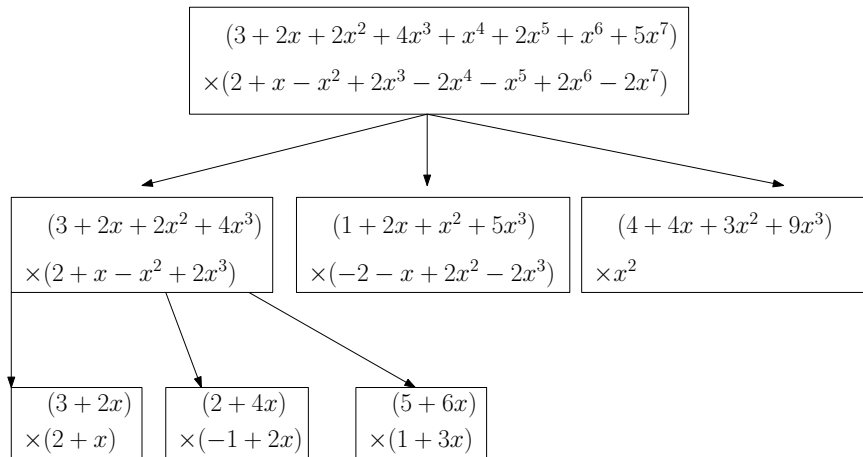


$$(3 + 2x + 2x^2 + 4x^3) \\ \times (2 + x - x^2 + 2x^3)$$

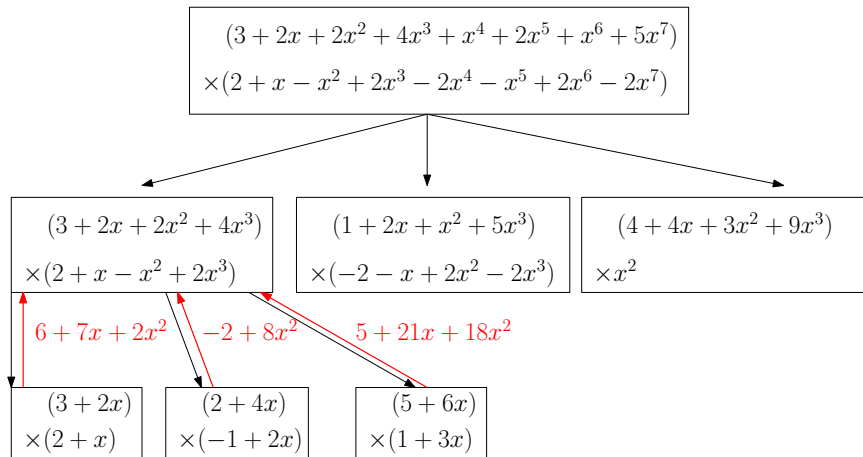
$$(1 + 2x + x^2 + 5x^3) \\ \times (-2 - x + 2x^2 - 2x^3)$$

$$(4 + 4x + 3x^2 + 9x^3) \\ \times x^2$$

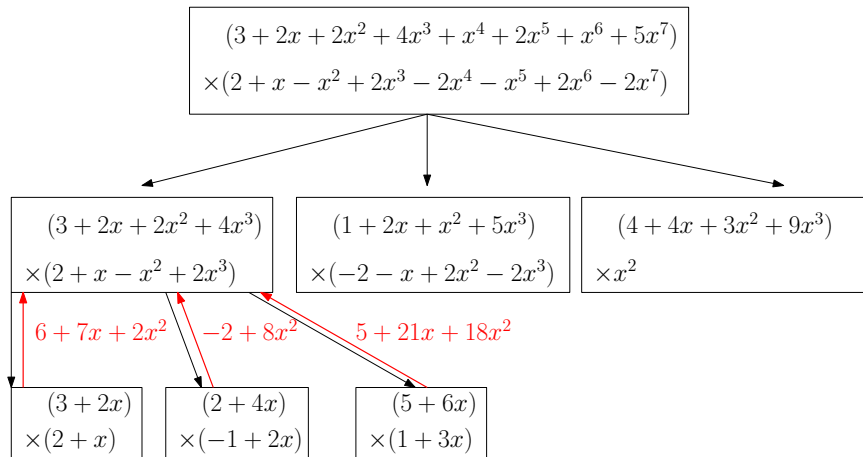
Example



Example

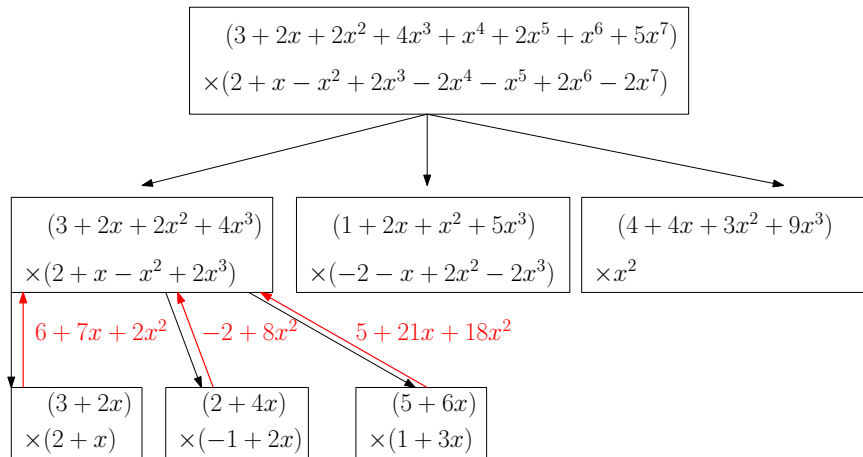


Example



$$(5 + 21x + 18x^2) - (6 + 7x + 2x^2) - (-2 + 8x^2) = 1 + 14x + 8x^2$$

Example

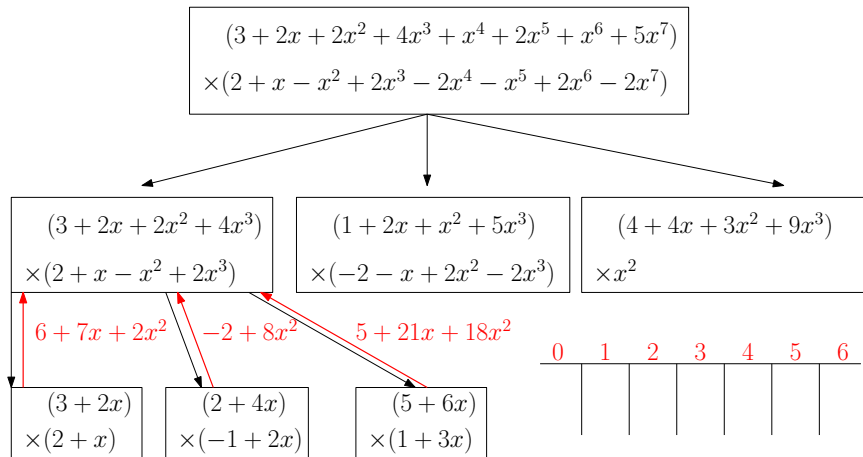


$$(5 + 21x + 18x^2) - (6 + 7x + 2x^2) - (-2 + 8x^2) = 1 + 14x + 8x^2$$

$$(6 + 7x + 2x^2) + (1 + 14x + 8x^2)x^2 + (-2 + 8x^2)x^4$$

$$= 6 + 7x + 3x^2 + 14x^3 + 6x^4 + 8x^6$$

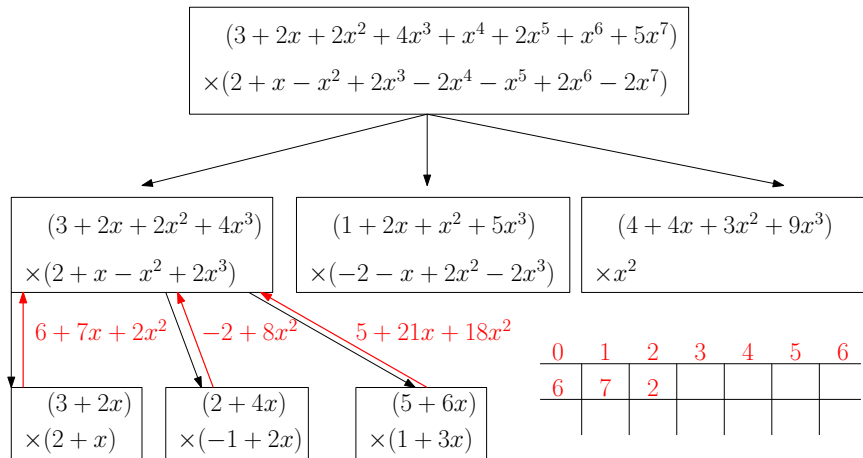
Example



$$(5 + 21x + 18x^2) - (6 + 7x + 2x^2) - (-2 + 8x^2) = 1 + 14x + 8x^2$$

$$(6 + 7x + 2x^2) + (1 + 14x + 8x^2)x^2 + (-2 + 8x^2)x^4 \\ = 6 + 7x + 3x^2 + 14x^3 + 6x^4 + 8x^6$$

Example

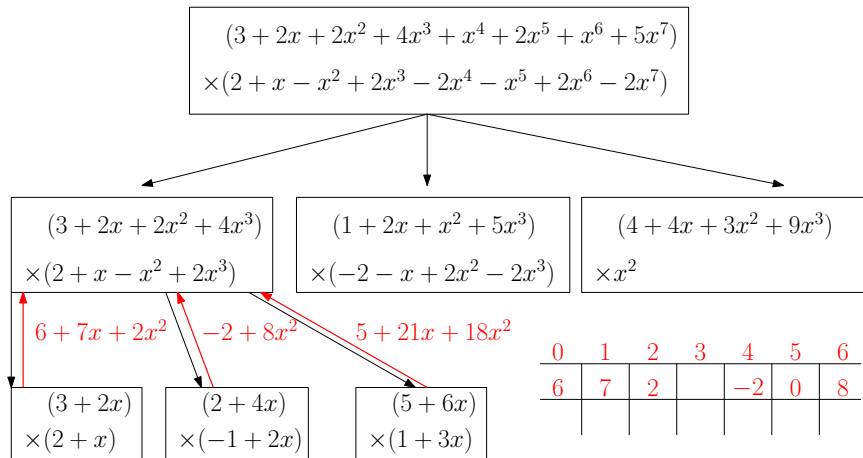


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Example

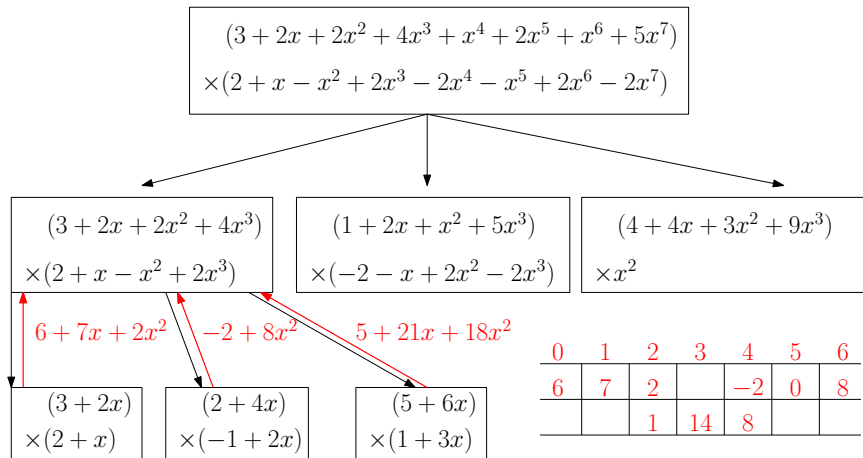


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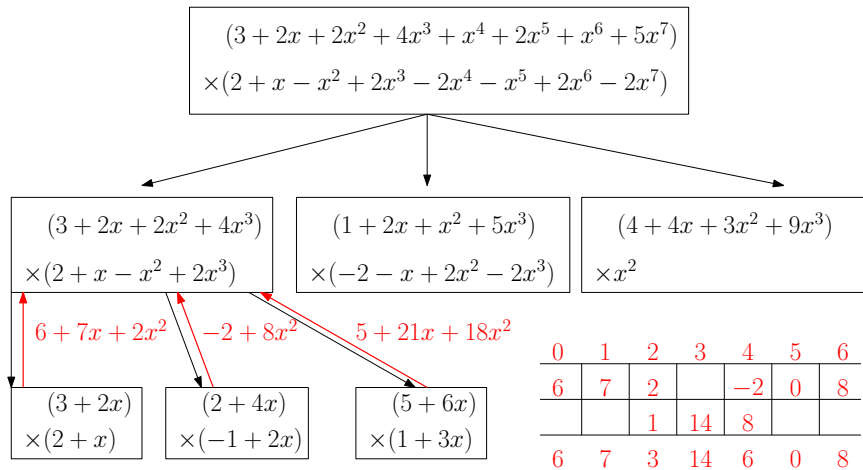


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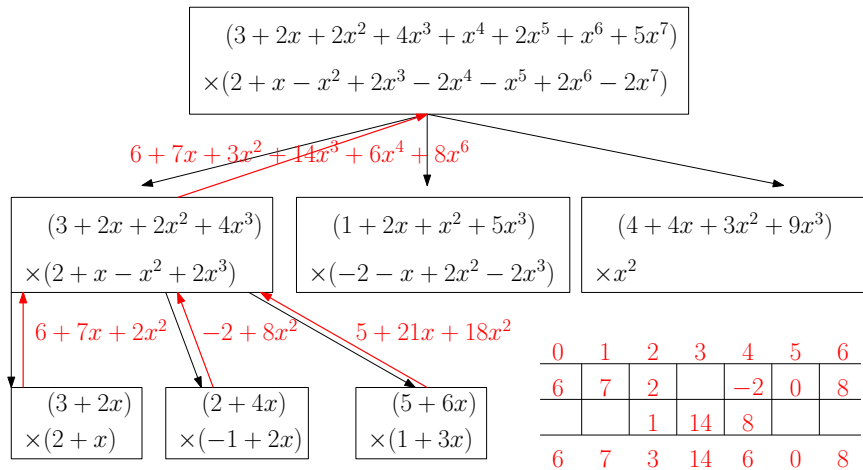


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Input: two $n \times n$ matrices A and B

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1: for  $i \leftarrow 1$  to  $n$  do  
2:   for  $j \leftarrow 1$  to  $n$  do  
3:      $C[i, j] \leftarrow 0$   
4:     for  $k \leftarrow 1$  to  $n$  do  
5:        $C[i, j] \leftarrow C[i, j] + A[i, k] \times B[k, j]$   
6: return  $C$ 
```

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- running time = $O(n^3)$

Try to Use Divide-and-Conquer

$$A = \begin{array}{|c|c|} \hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array} \quad \begin{array}{c} \overbrace{\hspace{1cm}}^{n/2} \\ \left. \hspace{1cm} \right\} n/2 \end{array} \quad B = \begin{array}{|c|c|} \hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array} \quad \begin{array}{c} \overbrace{\hspace{1cm}}^{n/2} \\ \left. \hspace{1cm} \right\} n/2 \end{array}$$

- $C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$
- `matrix_multiplication(A, B)` recursively calls
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...

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- Solving Recurrence $T(n) = O(n^{\log_2 7}) = O(n^{2.808})$

Strassen's Algorithm

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- $M_1 \leftarrow (A_{11} + A_{22}) \times (B_{11} + B_{22})$
- $M_2 \leftarrow (A_{21} + A_{22}) \times B_{11}$
- $M_3 \leftarrow A_{11} \times (B_{12} - B_{22})$
- $M_4 \leftarrow A_{22} \times (B_{21} - B_{11})$
- $M_5 \leftarrow (A_{11} + A_{12}) \times B_{22}$
- $M_6 \leftarrow (A_{21} - A_{11}) \times (B_{11} + B_{12})$
- $M_7 \leftarrow (A_{12} - A_{22}) \times (B_{21} + B_{22})$
- $C_{11} \leftarrow M_1 + M_4 - M_5 + M_7$
- $C_{12} \leftarrow M_3 + M_5$
- $C_{21} \leftarrow M_2 + M_4$
- $C_{22} \leftarrow M_1 - M_2 + M_3 + M_6$

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Interpolation of Polynomials

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- $p(x) = 1 - x + 2x^2 : p(0) = 1, p(1) = 2, p(2) = 7.$

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Using Interpolation for Polynomial Multiplication

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Using Interpolation for Polynomial Multiplication

- $p(x) = 1 - x + 2x^2$, $q(x) = 3 - x^2$
- Interpolation on 5 points $\{0, 1, 2, 3, 4\}$:

$$\begin{aligned} \text{interpolation for } p : & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 7 \\ 16 \\ 29 \end{pmatrix} \\ \text{interpolation for } q : & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \\ -6 \\ -13 \end{pmatrix} \end{aligned}$$

Interpolation of pq :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -7 \\ -102 \\ -377 \end{pmatrix}$$

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$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{25}{12} & 4 & -3 & \frac{4}{3} & -\frac{1}{4} \\ \frac{35}{24} & -\frac{13}{3} & \frac{19}{4} & -\frac{7}{3} & \frac{11}{24} \\ -\frac{5}{12} & \frac{3}{2} & -2 & \frac{7}{6} & -\frac{1}{4} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -7 \\ -96 \\ -377 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 5 \\ 1 \\ -2 \end{pmatrix}$$

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$$pq = (1 - x + 2x^2)(3 - x^2) = 3 - 3x + 5x^2 + x^3 - 2x^4$$

Multiplication of two polynomials of degree $n - 1$

- Choose $2n - 1$ distinct values $x_0, x_1, x_2, \dots, x_{m-1}$ carefully, $m = 2n - 1$
- Compute the interpolation of p and q :

$$M := \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{m-1} & x_{m-1}^2 & x_{m-1}^3 & \cdots & x_{m-1}^{n-1} \end{pmatrix}$$
$$M \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{m-1} \end{pmatrix} \quad M \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{m-1} \end{pmatrix}$$

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$$\begin{aligned} & (a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}) \\ & \times (b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}) \\ & = (c_0 + c_1 x + c_2 x^2 + \cdots + c_{2n-2} x^{2n-2}) \end{aligned}$$

Q: How should we set $x_0, x_1, \dots, x_{n-1}$ so that we can compute Ma and $M^{-1}y$ fast (for any $a, y \in \mathbb{R}^{\{0,1,\dots,n-1\}}$)?

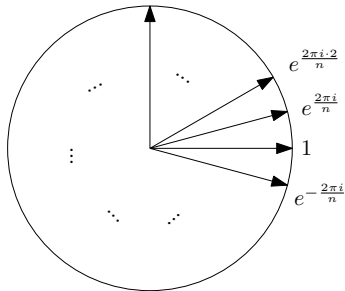
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A: Use the n complex roots of the equation $x^n = 1$

- $e^{\frac{2\pi i \cdot k}{n}} = \cos\left(\frac{2\pi \cdot k}{n}\right) + i \cdot \sin\left(\frac{2\pi \cdot k}{n}\right), k \in \{0, 1, \dots, n-1\}$
- $\omega := e^{\frac{2\pi i}{n}}$, n -th roots are $1, \omega, \omega^2, \dots, \omega^{n-1}$



$$F_n := \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \dots & \omega^{-(n-1)} \end{pmatrix}$$

- Interpolation and Inverse-Interpolation:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} = F_n \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} \quad \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} = F_n^{-1} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

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- Interpolation: **Fast Fourier Transform (FFT)**
- Invert-Interpolation: **Inverse Fast Fourier Transform (iFFT)**

Fast Fourier Transform: Divide and Conquer

- Assume n is even.

Breaking polynomial into even and odd parts

- $p_{\text{even}}(x) := a_0 + a_2x + a_4x^2 + \cdots + a_{n-2}x^{n/2-1}$
- $p_{\text{odd}}(x) := a_1 + a_3x + a_5x^2 + \cdots + a_{n-1}x^{n/2-1}$
- $p(x) = p_{\text{even}}(x^2) + p_{\text{odd}}(x^2) \cdot x$

Fast Fourier Transform: Divide and Conquer

- Assume n is even.

Breaking polynomial into even and odd parts

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$$p(\omega^k) = p_{\text{even}}(\omega^{2k}) + p_{\text{odd}}(\omega^{2k}) \cdot \omega^k, \quad k = 0, 1, \dots, \frac{n}{2} - 1$$

$$p(\omega^{n/2+k}) = p_{\text{even}}(\omega^{2k}) - p_{\text{odd}}(\omega^{2k}) \cdot \omega^k, \quad k = 0, 1, \dots, \frac{n}{2} - 1$$

- Assume n is an integer power of 2

FFT($n, a_0, a_1, \dots, a_{n-1}$)

```
1: if  $n = 1$  then return ( $a_0$ )  
2:  $(e_0, e_1, \dots, e_{n/2-1}) \leftarrow \text{FFT}(n/2, a_0, a_2, \dots, a_{n-2})$   
3:  $(o_0, o_1, \dots, o_{n/2-1}) \leftarrow \text{FFT}(n/2, a_1, a_3, \dots, a_{n-1})$   
4: for  $k \leftarrow 0, 1, 2, \dots, n/2 - 1$  do  
5:    $y_k \leftarrow e_k + o_k \cdot \omega^k$   
6:    $y_{n/2+k} \leftarrow e_k - o_k \cdot \omega^k$   
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- Recurrence for running time: $T(n) = 2T(n/2) + O(n)$
- $T(n) = O(n \log n)$

Example for one recursion of FFT

- $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (3, 2, 1, 2, 5, 6, 1, 4)$

$$\begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 6 \\ -2 \end{pmatrix}$$
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- $y_6 = e_2 - o_2\omega^2 = 6 - 2i$ $y_7 = e_3 - o_3\omega^3$

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$$

$$q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1}$$

multiplying p and q , ▷ assuming n is a power of 2

1: $y \leftarrow \text{FFT}(2n, a_0, a_1, \dots, a_{n-1}, 0, 0, \dots, 0)$

2: $z \leftarrow \text{FFT}(2n, b_0, b_1, \dots, b_{n-1}, 0, 0, \dots, 0)$

3: $c \leftarrow \text{iFFT}(2n, y_0z_0, y_1z_1, \dots, y_{2n-1}z_{2n-1})$

4: **return** $(c_0, c_1, \dots, c_{2n-2})$

- $\text{iFFT}(n, y_0, y_1, \dots, y_{n-1})$: inverse FFT procedure: multiplying input vector y by the inverse of F_n , which is

$$\frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \end{pmatrix}$$

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Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$

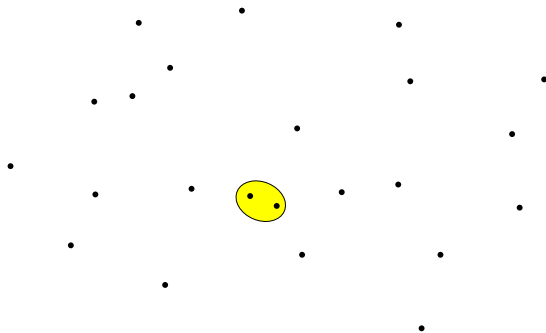
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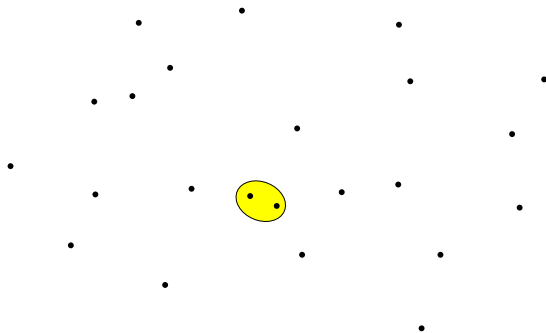
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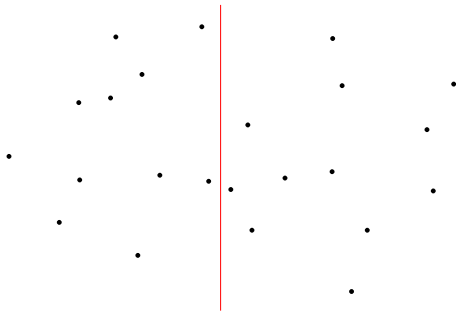
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- Trivial algorithm: $O(n^2)$ running time

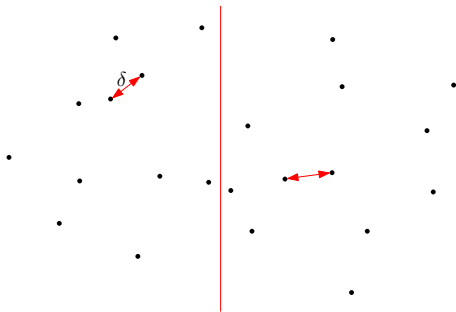
Divide-and-Conquer Algorithm for Closest Pair

- **Divide:** Divide the points into two halves via a vertical line



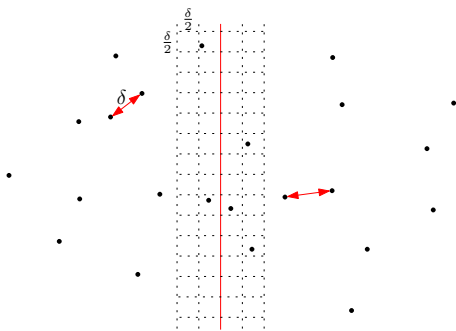
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- **Divide:** Divide the points into two halves via a vertical line
- **Conquer:** Solve two sub-instances recursively

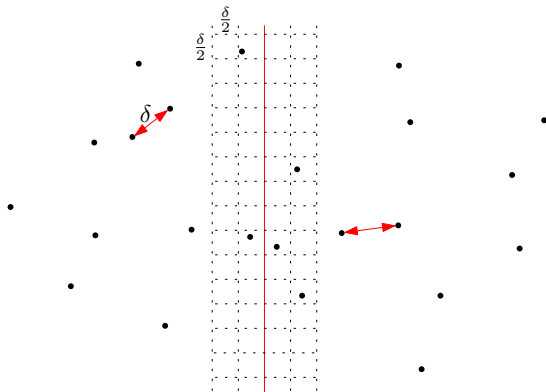


Divide-and-Conquer Algorithm for Closest Pair

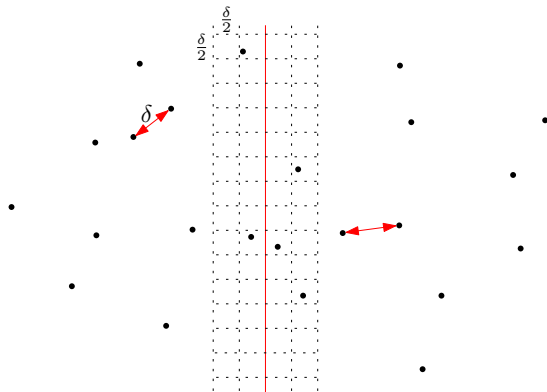
- **Divide:** Divide the points into two halves via a vertical line
- **Conquer:** Solve two sub-instances recursively
- **Combine:** Check if there is a closer pair between left-half and right-half



Divide-and-Conquer Algorithm for Closest Pair

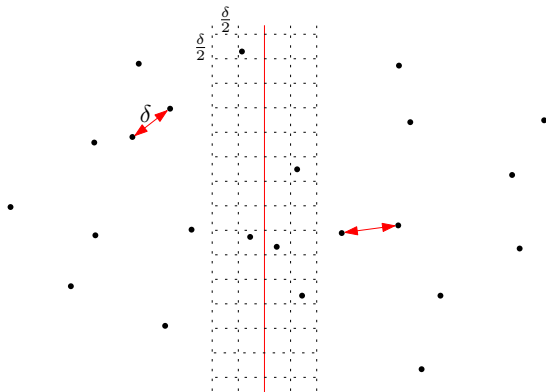


Divide-and-Conquer Algorithm for Closest Pair



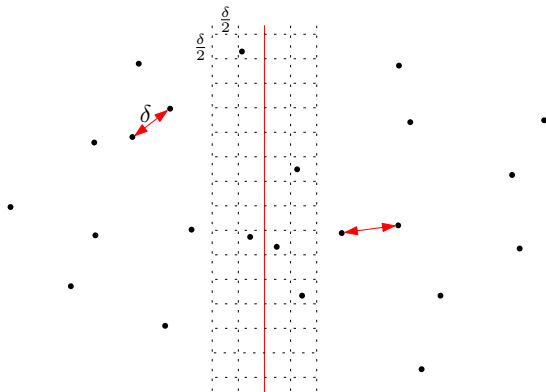
- Each box contains at most one pair

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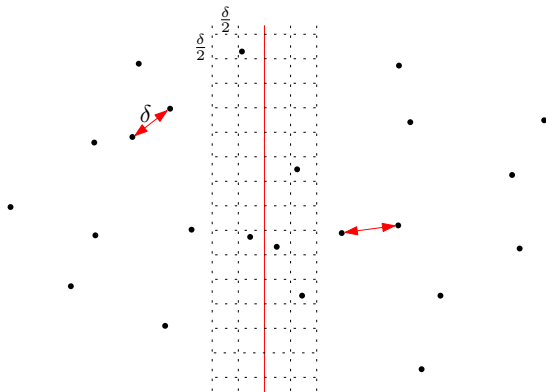
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- For each point, only need to consider $O(1)$ boxes nearby
- Implementation: **Sort** points inside the stripe according to y -coordinates
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- recurrence: $T(n) = 2T(n/2) + O(n \log n)$

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Improve the running time of combine step to $O(n)$

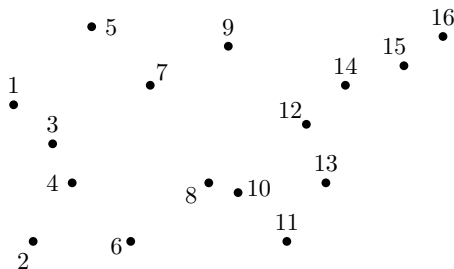
- also sort the points in ascending order of y values at the beginning
- pass the sequence to the root recursion
- constructing two sub-sequences from the sequence, and pass them to the two sub-recursions respectively

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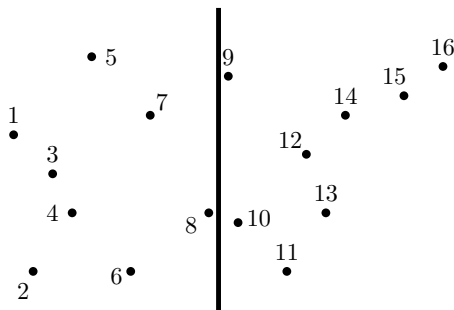
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-
- $T(n) = 2T(n/2) + O(n) \implies T(n) = O(n \log n)$

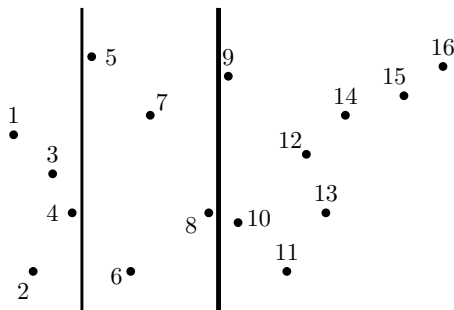
Example for Closest Pair



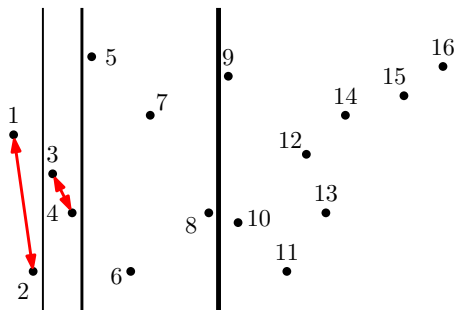
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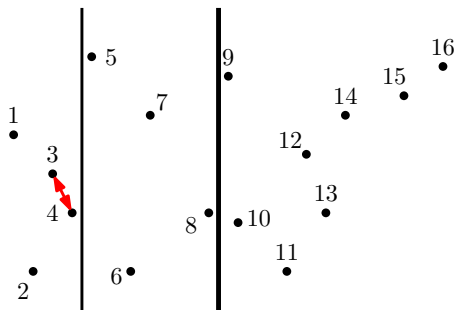
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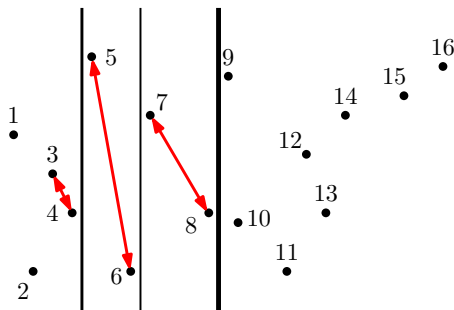
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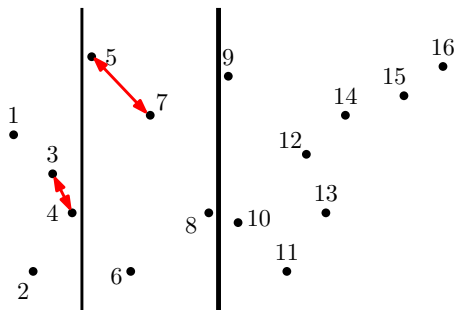
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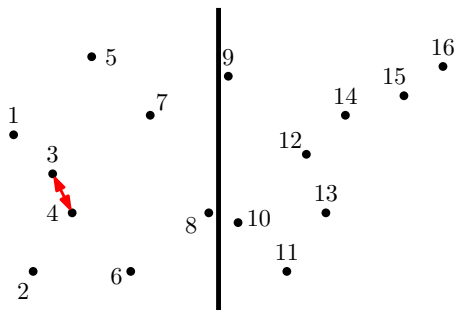
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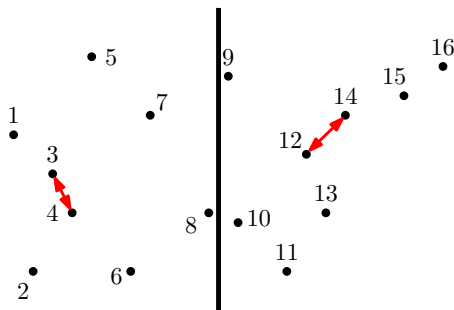
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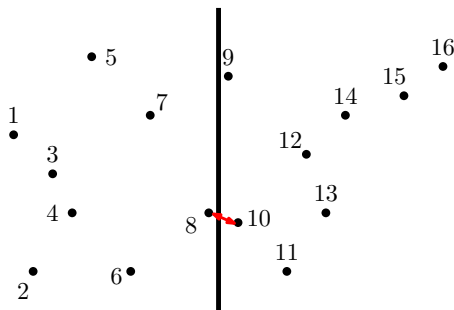
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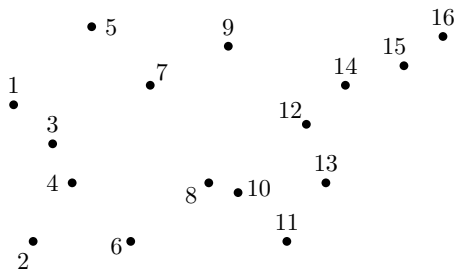
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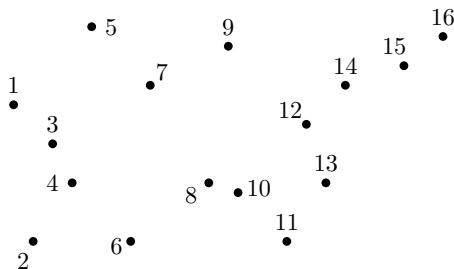
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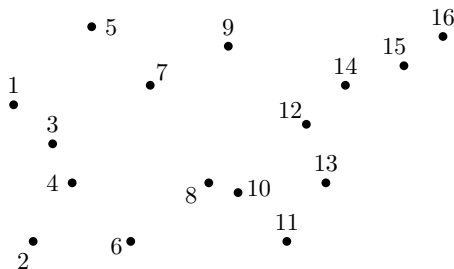


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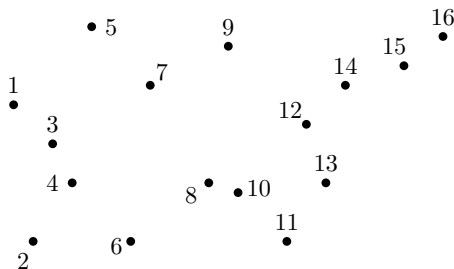
- $CP(1, 16, (5, 16, 9, 15, 7, 14, 1, 12, 3, 4, 8, 13, 10, 11, 2, 6))$

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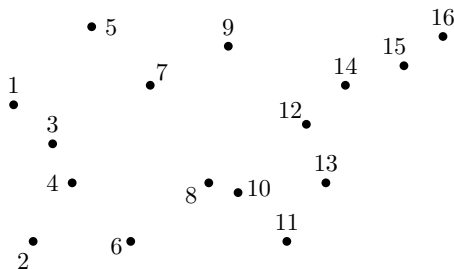
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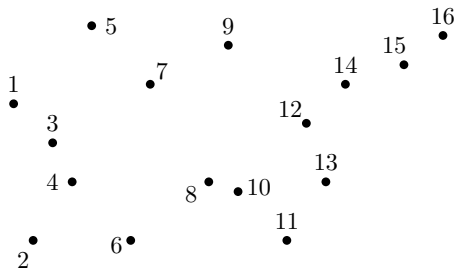
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Fibonacci Numbers

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, $\dots$

n -th Fibonacci Number

Input: integer $n > 0$

Output: F_n

Computing F_n : Stupid Divide-and-Conquer Algorithm

Fib(n)

- 1: if $n = 0$ return 0
- 2: if $n = 1$ return 1
- 3: return $\text{Fib}(n - 1) + \text{Fib}(n - 2)$

Q: Is the running time of the algorithm polynomial or exponential in n ?

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- 3: return $\text{Fib}(n - 1) + \text{Fib}(n - 2)$

Q: Is the running time of the algorithm polynomial or exponential in n ?

A: Exponential

- Running time is at least $\Omega(F_n)$
- F_n is exponential in n

Computing F_n : Reasonable Algorithm

Fib(n)

```
1:  $F[0] \leftarrow 0$   
2:  $F[1] \leftarrow 1$   
3: for  $i \leftarrow 2$  to  $n$  do  
4:    $F[i] \leftarrow F[i - 1] + F[i - 2]$   
5: return  $F[n]$ 
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- Dynamic Programming

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- Dynamic Programming
- Running time = $O(n)$

Computing F_n : Even Better Algorithm

$$\begin{aligned}\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} \\ \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix} \\ &\dots \\ \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}\end{aligned}$$

power(n)

- 1: if $n = 0$ then return $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- 2: $R \leftarrow \text{power}(\lfloor n/2 \rfloor)$
- 3: $R \leftarrow R \times R$
- 4: if n is odd then $R \leftarrow R \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
- 5: **return** R

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Fixing the Problem

To compute F_n , we need $O(\log n)$ basic arithmetic operations on integers

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- **Divide:** Divide instance into many smaller instances
- **Conquer:** Solve each of smaller instances recursively and separately
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- Write down recurrence for running time
- Solve recurrence using master theorem

Summary: Divide-and-Conquer

- Merge sort, quicksort, count-inversions, closest pair, FFT, ...:
 $T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$

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- To improve running time, design better algorithm for “combine” step, or reduce number of recursions, ...