算法设计与分析(2026年春季学期)

Graph Basics

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南京大学计算机学院

Outline

1. Graphs
2. Connectivity and Graph Traversal
    - Testing Bipartiteness
3. Topological Ordering
4. Bridges and 2-Edge-Connected Components
    - $O(n+m)$-Time Algorithm to Find Bridges
    - Related Concept: Cut Vertices
5. Strong Connectivity in Directed Graphs
    - Tarjan's $O(n+m)$-Time Algorithm for Finding SCCes

Examples of Graphs

Figure: Road Networks

Figure: Internet

Figure: Social Networks

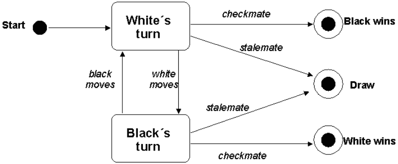
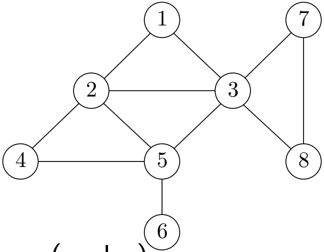


Figure: Transition Graphs

(Undirected) Graph $G = (V, E)$



- V: set of vertices (nodes);
- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2

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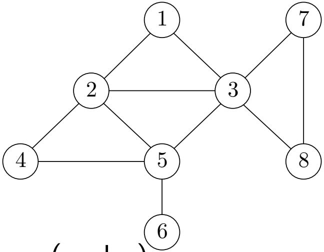
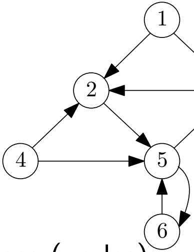


- V: set of vertices (nodes);
  - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2
  - $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\}$

Directed Graph $G = (V, E)$



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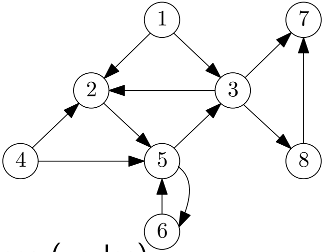

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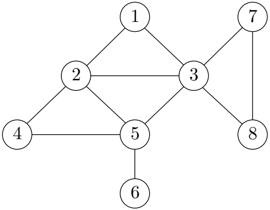
Abuse of Notations

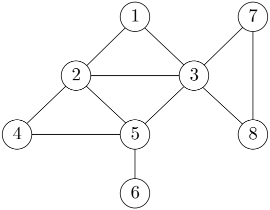
- For (undirected) graphs, we often use (i, j) to denote the set $\{i, j\}$.
- We call (i, j) an unordered pair; in this case $(i, j) = (j, i)$.



- $E = \{(1,2),(1,3),(2,3),(2,4),(2,5),(3,5),(3,7),(3,8),$ $(4,5),(5,6),(7,8)\}$

- Social Network : Undirected
- Transition Graph : Directed
- Road Network : Directed or Undirected
- Internet : Directed or Undirected

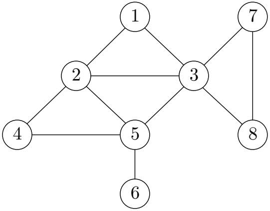
Representation of Graphs

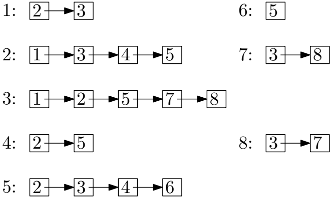


	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

- Adjacency matrix
  - $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
  - A is symmetric if graph is undirected

Representation of Graphs





1: 2 → 3
2: 1 → 3 → 4 → 5
3: 1 → 2 → 5 → 7 → 8
4: 2 → 5
5: 2 → 3 → 4 → 6
6: 5
7: 3 → 8
8: 3 → 7

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- Linked lists
  - For every vertex v, there is a linked list containing all neighbours of v.

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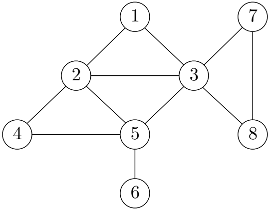

1: 2 3
2: 1 3 4 5
3: 1 2 5 7 8
4: 2 5
5: 2 3 4 6

6: 5
7: 3 8
8: 3 7

$d : (2, 4, 5, 2, 4, 1, 2, 2)$

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  - $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
  - A is symmetric if graph is undirected
- Linked lists
  - For every vertex v, there is a linked list containing all neighbours of v.
- If graph is static: store neighbors of all vertices in a length-$2m$ array, where the neighbors of any vertex are consecutive.

Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- n: number of vertices
- m: number of edges, assuming $n - 1 \le m \le n(n-1)/2$
- d_v: number of neighbors of v

	Matrix	Linked Lists
memory usage		
time to check $(u, v) \in E$		
time to list all neighbours of v		

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memory usage	$O(n^2)$	$O(m)$
time to check $(u,v) \in E$	$O(1)$	
time to list all neighbours of v		

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time to check $(u,v) \in E$	$O(1)$	$O(d_u)$
time to list all neighbours of v		

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time to check $(u,v) \in E$	$O(1)$	$O(d_u)$
time to list all neighbours of v	$O(n)$	

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time to list all neighbours of v	$O(n)$	$O(d_v)$

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Connectivity Problem

Input: graph $G = (V, E)$, (using linked lists)
two vertices $s, t \in V$

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  - Breadth-First Search (BFS)
  - Depth-First Search (DFS)

Breadth-First Search (BFS)

- Build layers $L_0, L_1, L_2, L_3, \cdots$
- $L_0 = \{s\}$
- L_{j+1} contains all nodes that are not in $L_0 \cup L_1 \cup \cdots \cup L_j$ and have an edge to a vertex in L_j

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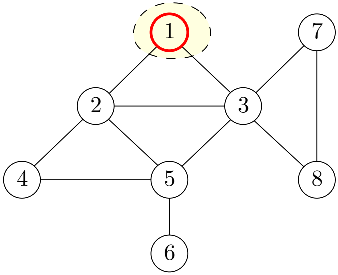

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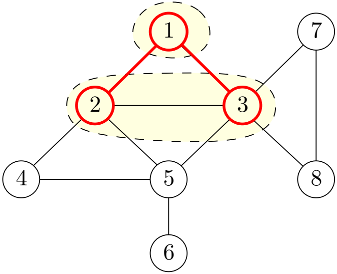

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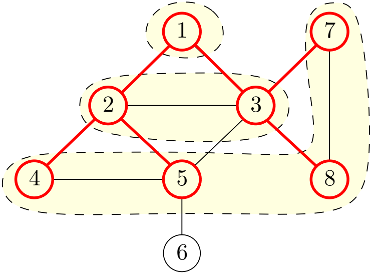

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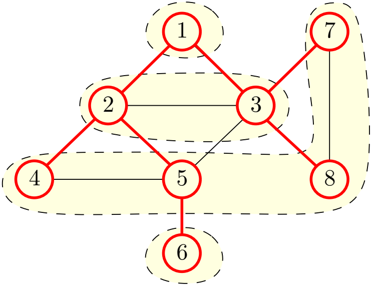


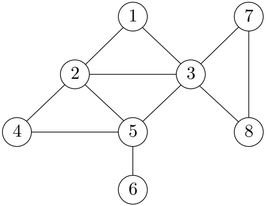
Implementing BFS using a Queue

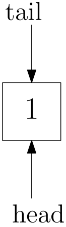
BFS(s)

```
head ← 1, tail ← 1, queue[1] ← s
mark s as "visited" and all other vertices as "unvisited"
while head ≤ tail do
    v ← queue[head], head ← head + 1
    for all neighbours u of v do
        if u is "unvisited" then
            tail ← tail + 1, queue[tail] = u
            mark u as "visited"
```

- Running time: $O(n+m)$.

Example of BFS via Queue





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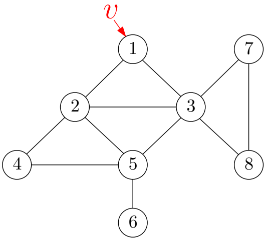

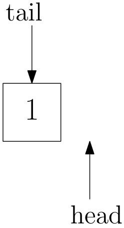

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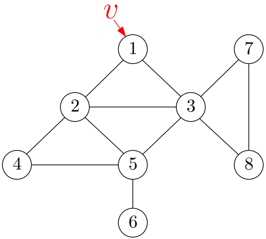

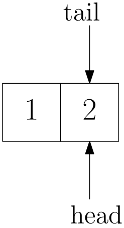

tail

1	2

head

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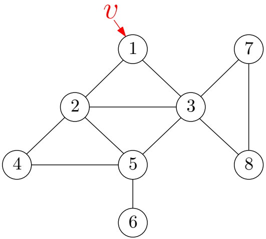

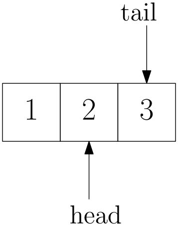

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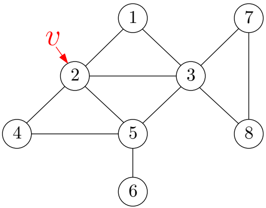

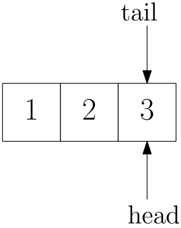

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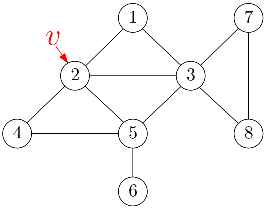

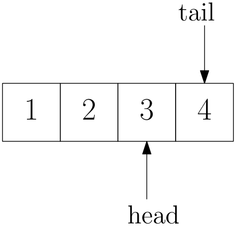

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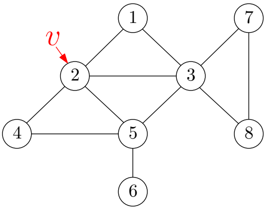

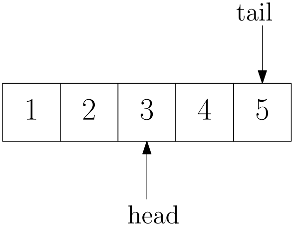

1	2	3	4	5

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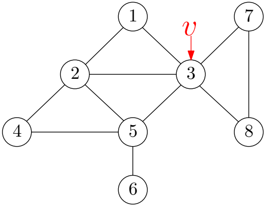

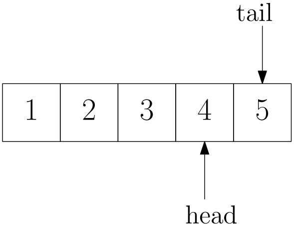

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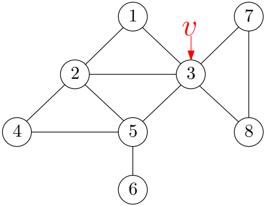

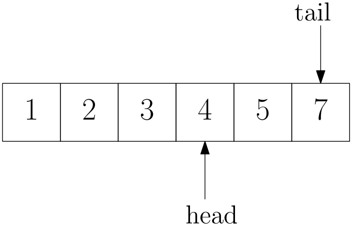

1	2	3	4	5	7

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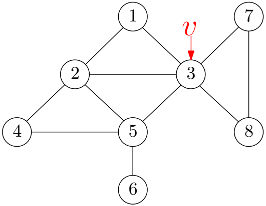

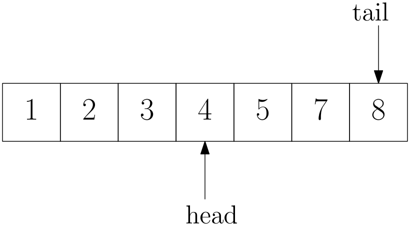

1	2	3	4	5	7	8

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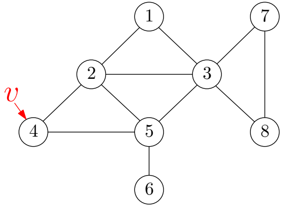

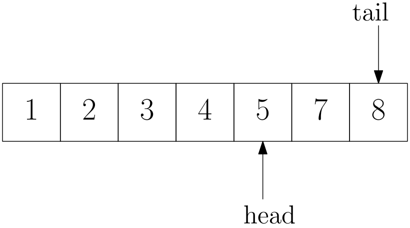

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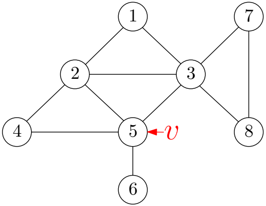

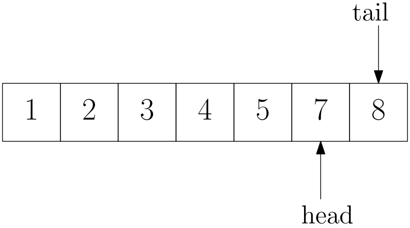

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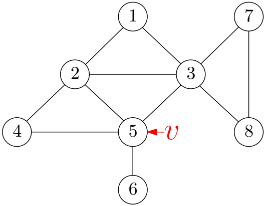

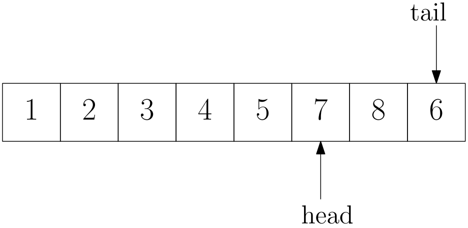

tail

1	2	3	4	5	7	8	6

head

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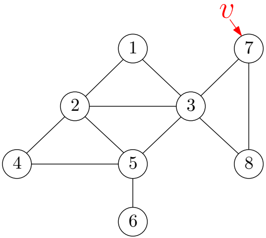

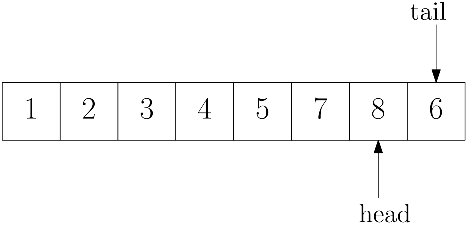

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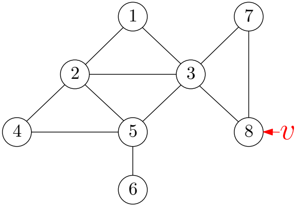
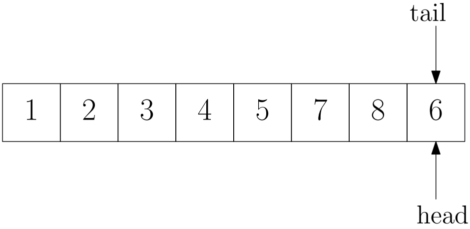


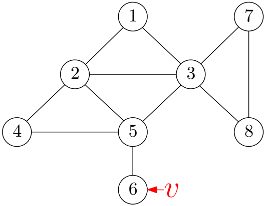


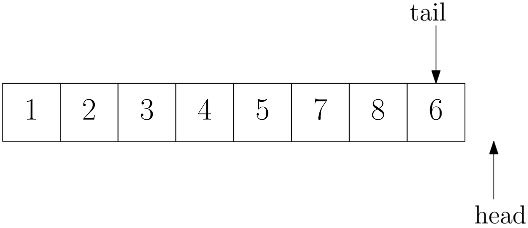
tail

1	2	3	4	5	7	8	6

head

Example of BFS via Queue





tail

1	2	3	4	5	7	8	6

head

Depth-First Search (DFS)

- Starting from s
- Travel through the first edge leading out of the current vertex
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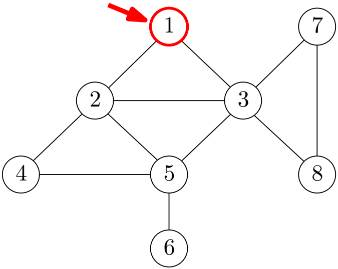

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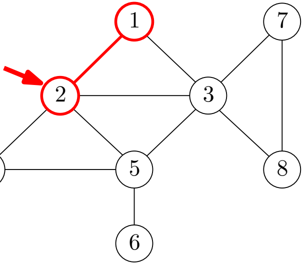

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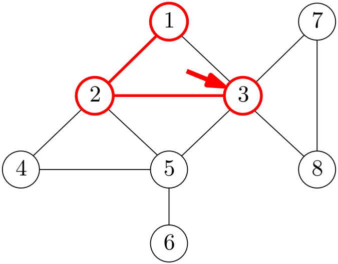

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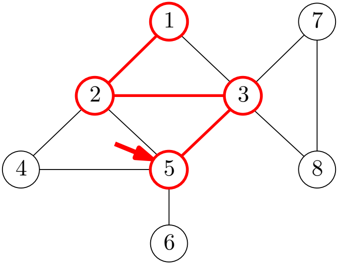

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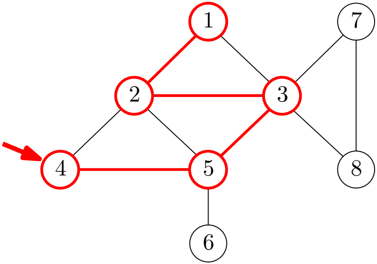

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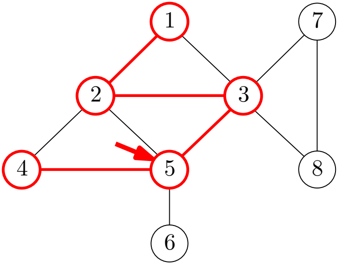

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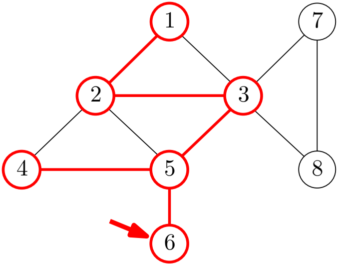

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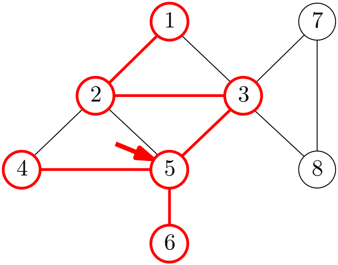

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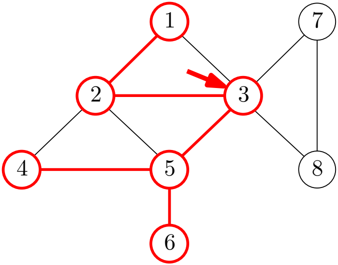

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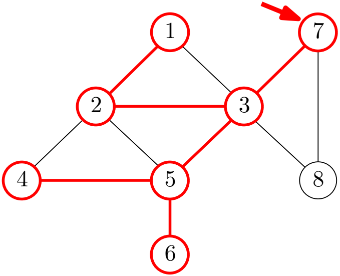

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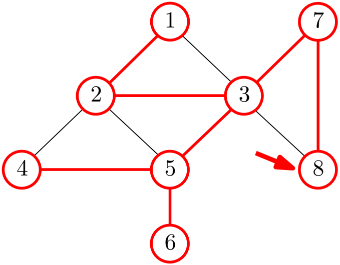


Implementing DFS using Recurrsion

DFS(s)

```
mark all vertices as "unvisited"
recursive-DFS($s$)
```

recursive-DFS(v)

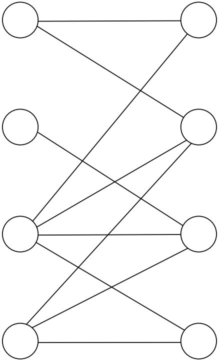
```
mark $v$ as "visited"
for all neighbours $u$ of $v$ do
    if $u$ is unvisited then recursive-DFS($u$)
```

Outline

1. Graphs
2. Connectivity and Graph Traversal
    - Testing Bipartiteness
3. Topological Ordering
4. Bridges and 2-Edge-Connected Components
    - $O(n+m)$-Time Algorithm to Find Bridges
    - Related Concept: Cut Vertices
5. Strong Connectivity in Directed Graphs
    - Tarjan's $O(n+m)$-Time Algorithm for Finding SCCes

Testing Bipartiteness: Applications of BFS

Def. A graph $G = (V, E)$ is a bipartite graph if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, we have either $u \in L, v \in R$ or $v \in L, u \in R$.



Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$

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- $\cdots$
- Report "not a bipartite graph" if contradiction was found

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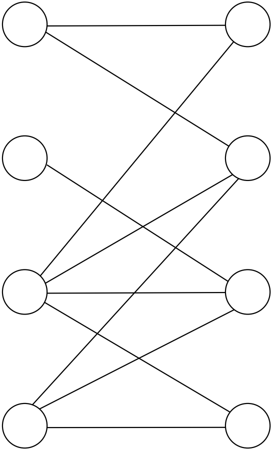
- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
- Neighbors of s must be in R
- Neighbors of neighbors of s must be in L
- $\cdots$
- Report "not a bipartite graph" if contradiction was found
- If G contains multiple connected components, repeat above algorithm for each component

Test Bipartiteness



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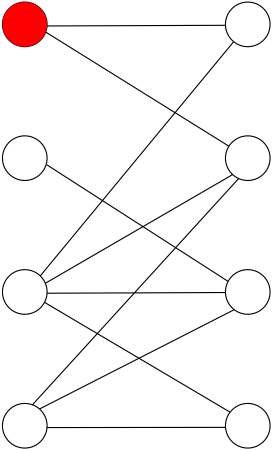

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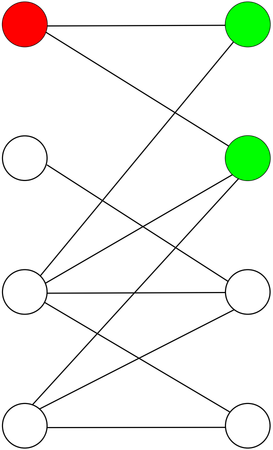

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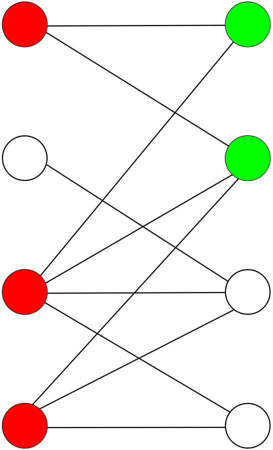

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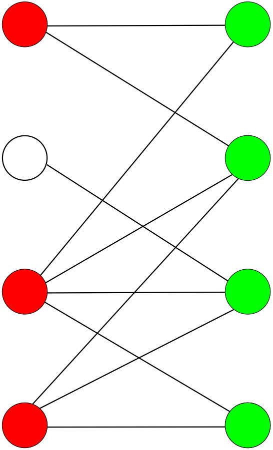

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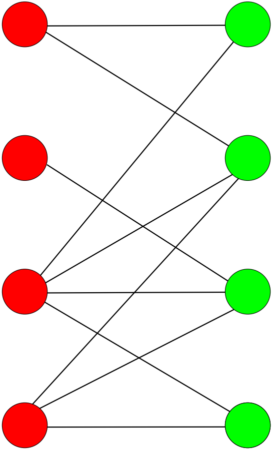

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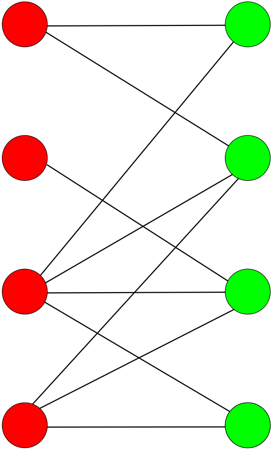

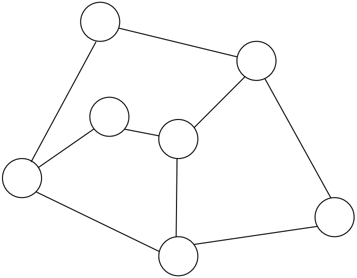

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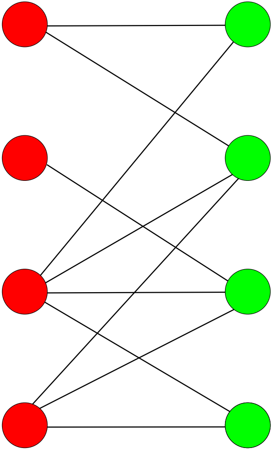

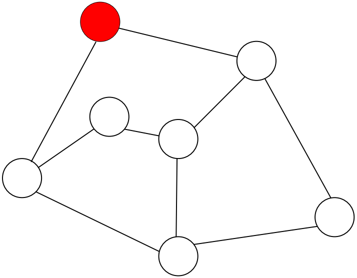

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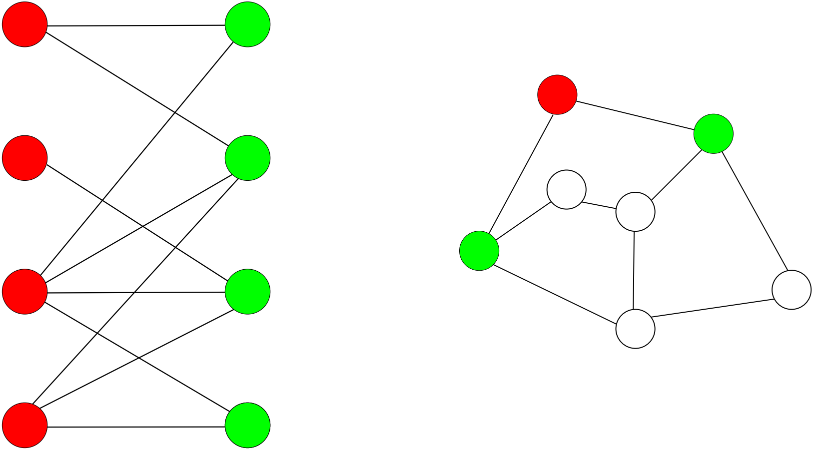

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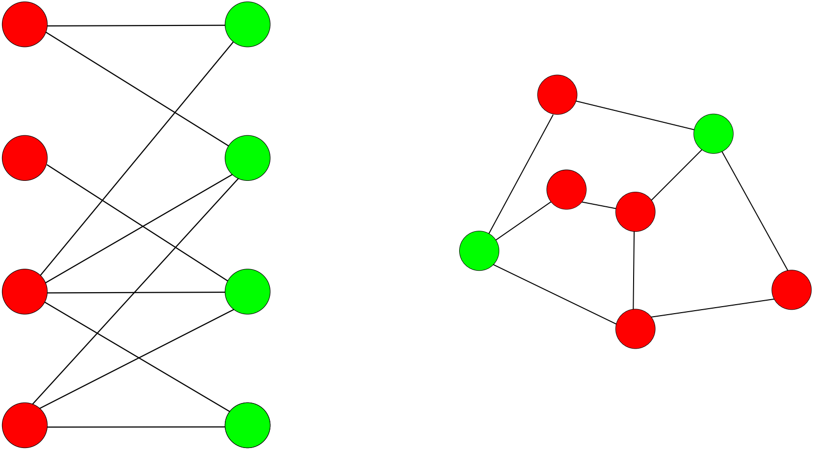

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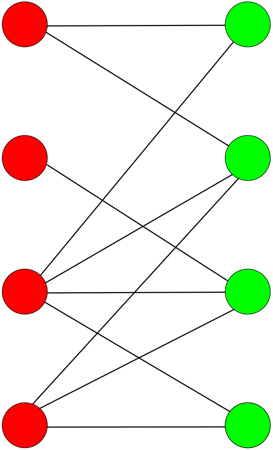

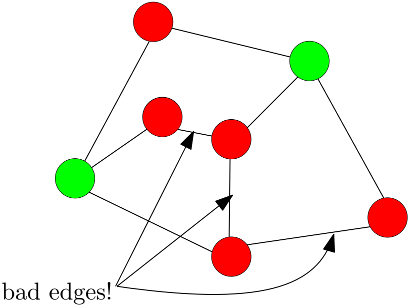

Testing Bipartiteness using BFS

BFS(s)

```
$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
mark $s$ as "visited" and all other vertices as "unvisited"
while $head \le tail$ do
    $v \leftarrow queue[head], head \leftarrow head + 1$
    for all neighbours $u$ of $v$ do
        if $u$ is "unvisited" then
            $tail \leftarrow tail + 1, queue[tail] = u$
            mark $u$ as "visited"
```

Testing Bipartiteness using BFS

test-bipartiteness(s)

```
$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
mark $s$ as "visited" and all other vertices as "unvisited"
$color[s] \leftarrow 0$
while $head \le tail$ do
    $v \leftarrow queue[head], head \leftarrow head + 1$
    for all neighbours $u$ of $v$ do
        if $u$ is "unvisited" then
            $tail \leftarrow tail + 1, queue[tail] = u$
            mark $u$ as "visited"
            $color[u] \leftarrow 1 - color[v]$
        else if $color[u] = color[v]$ then
            print("$G$ is not bipartite") and exit
```

Testing Bipartiteness using BFS

```
mark all vertices as "unvisited"
for each vertex $v \in V$ do
    if $v$ is "unvisited" then
        test-bipartiteness($v$)
print("$G$ is bipartite")
```

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Obs. Running time of algorithm $= O(n+m)$

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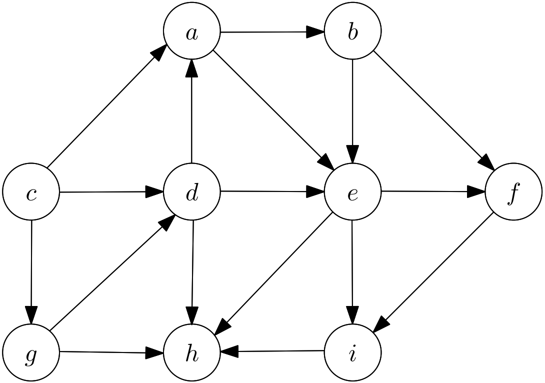
Topological Ordering Problem

Input: a directed acyclic graph (DAG) $G = (V, E)$

Output: 1-to-1 function $\pi : V \to \{1, 2, 3 \cdots, n\}$, so that

- if $(u, v) \in E$ then $\pi(u) < \pi(v)$



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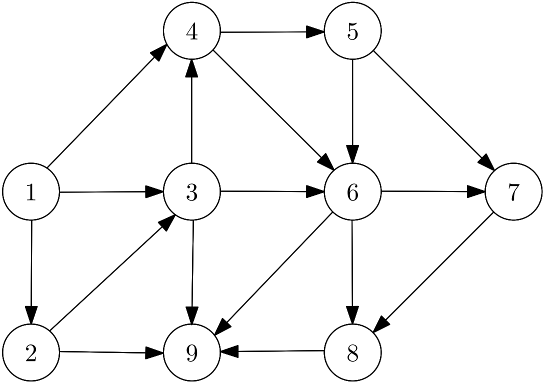
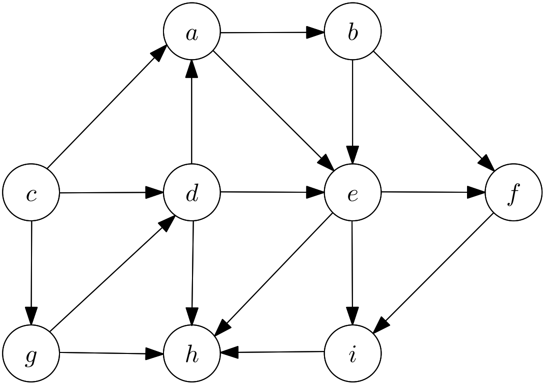


Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.



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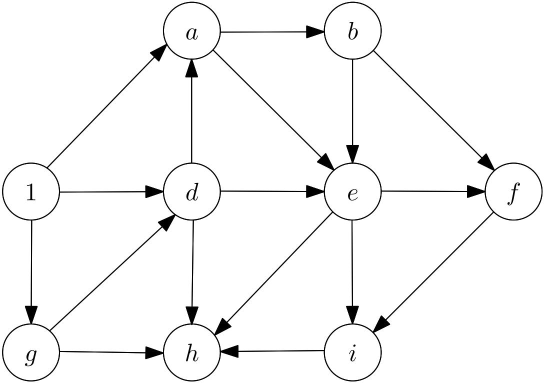

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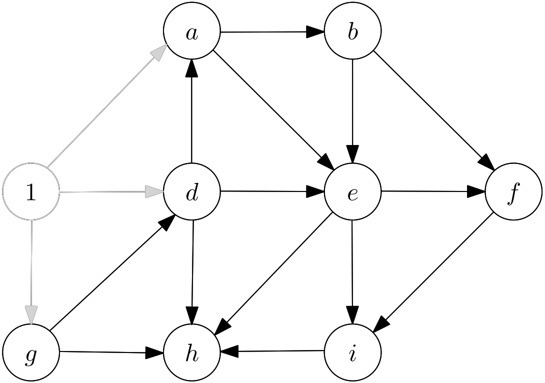

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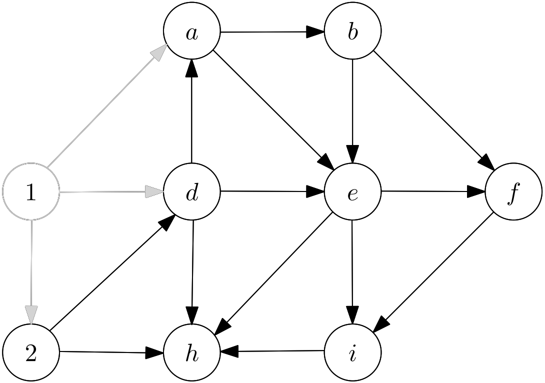

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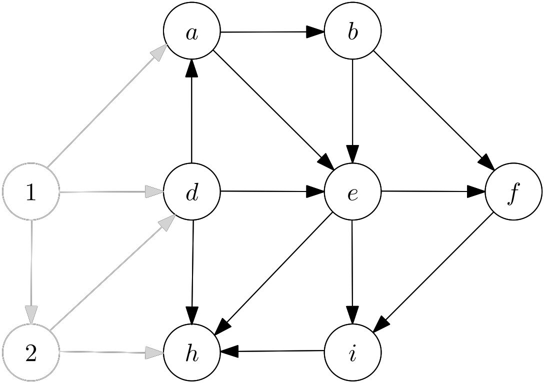

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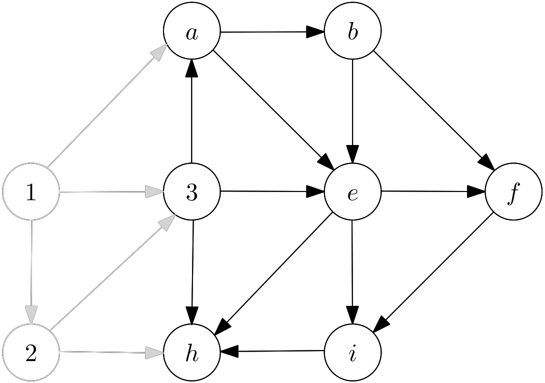

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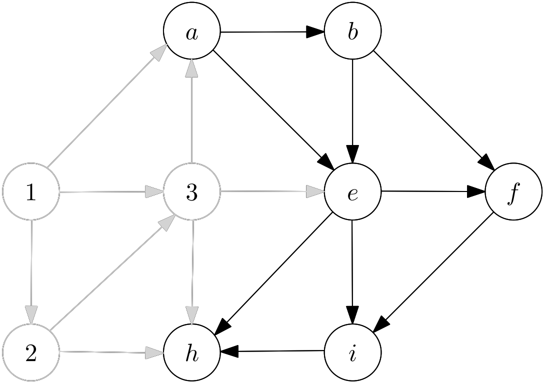

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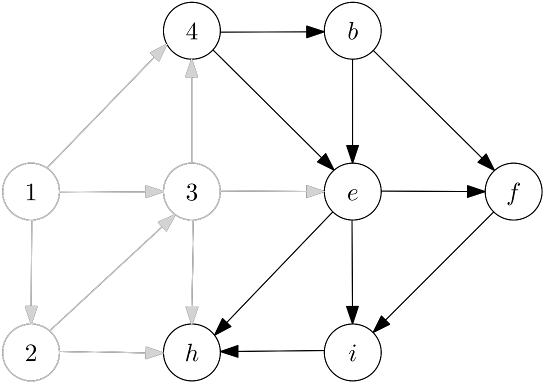

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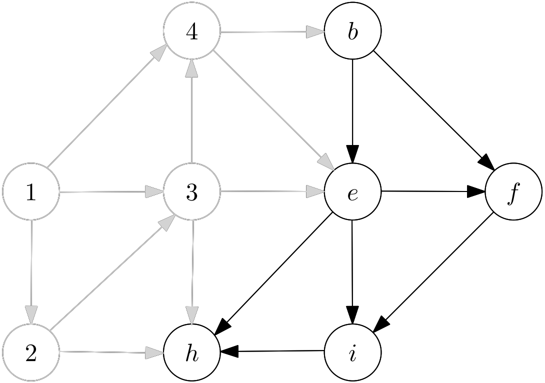

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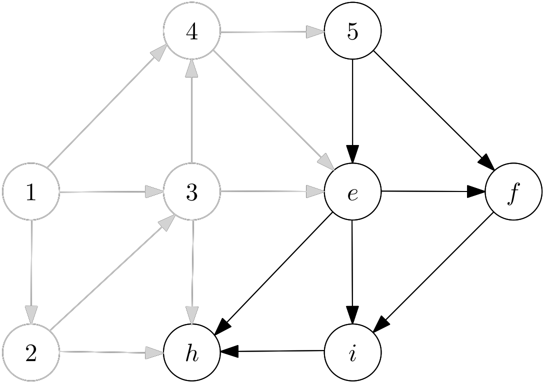

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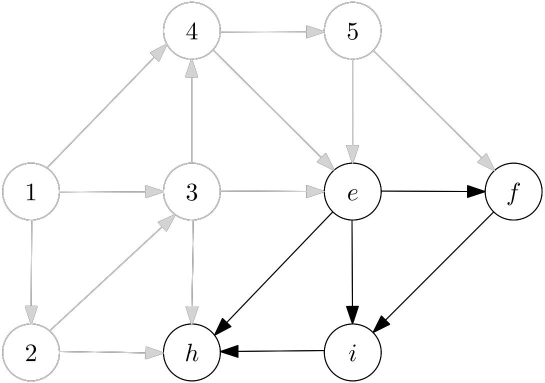

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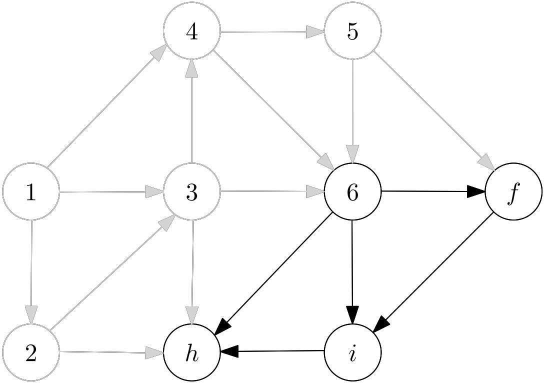

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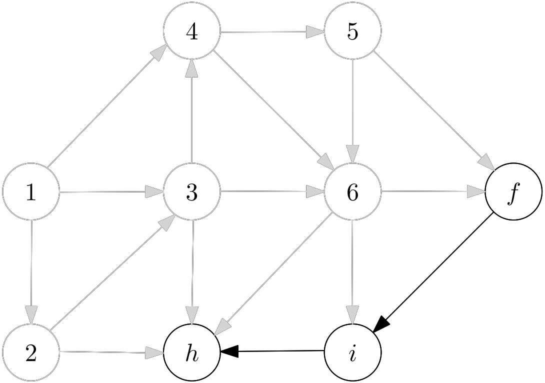

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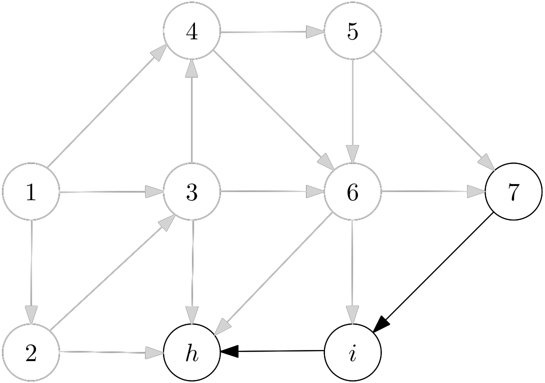

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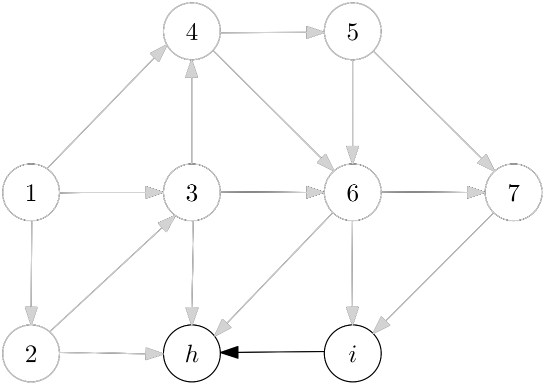

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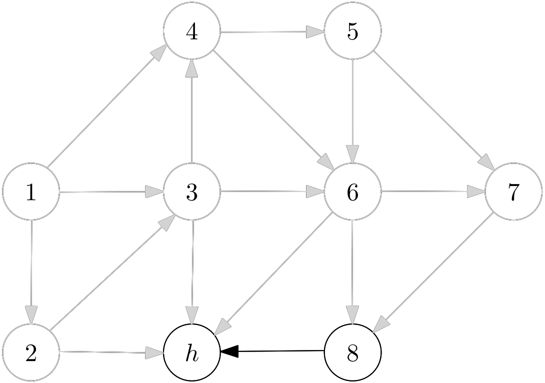

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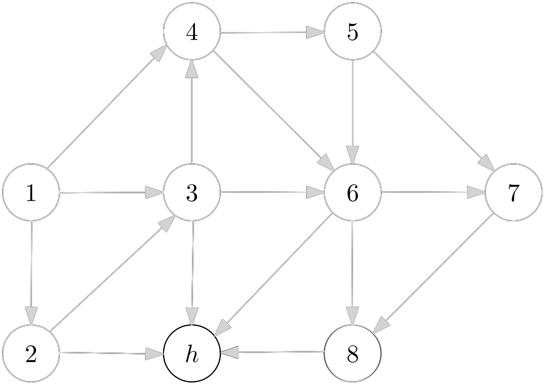

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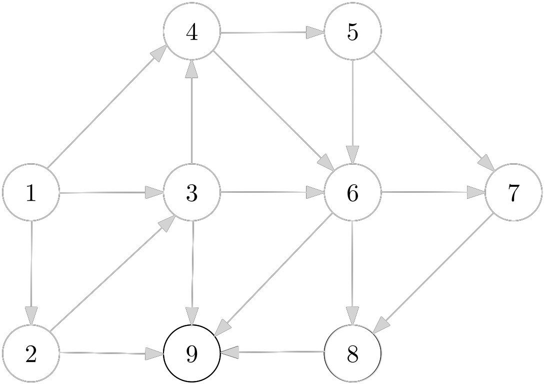
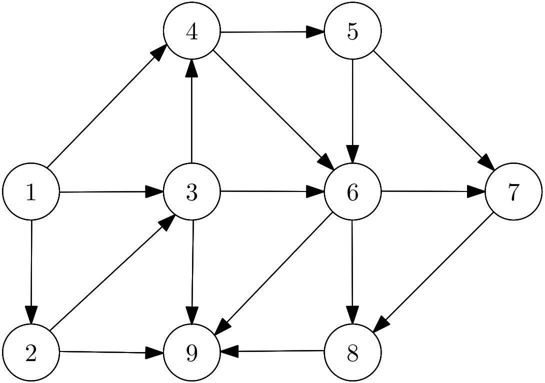


Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.



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Q: How to make the algorithm as efficient as possible?

A:

- Use linked-lists of outgoing edges
- Maintain the in-degree d_v of vertices
- Maintain a queue (or stack) of vertices v with $d_v = 0$

topological-sort(G)

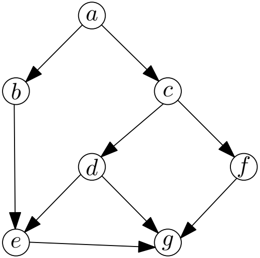
```
let $d_v \leftarrow 0$ for every $v \in V$
for every $v \in V$ do
    for every $u$ such that $(v, u) \in E$ do
        $d_u \leftarrow d_u + 1$
$S \leftarrow \{v : d_v = 0\}, i \leftarrow 0$
while $S \neq \emptyset$ do
    $v \leftarrow$ arbitrary vertex in $S$, $S \leftarrow S \setminus \{v\}$
    $i \leftarrow i + 1$, $\pi(v) \leftarrow i$
    for every $u$ such that $(v, u) \in E$ do
        $d_u \leftarrow d_u - 1$
        if $d_u = 0$ then add $u$ to $S$
if $i < n$ then output "not a DAG"
```

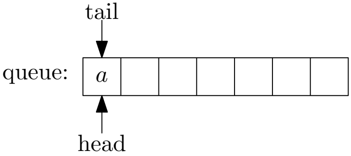
- S can be represented using a queue or a stack
- Running time $= O(n + m)$

S as a Queue or a Stack

DS	Queue	Stack
Initialization	$head \leftarrow 1$, $tail \leftarrow 0$	$top \leftarrow 0$
Non-Empty?	$head \le tail$	$top > 0$
Add(v)	$tail \leftarrow tail + 1$ $S[tail] \leftarrow v$	$top \leftarrow top + 1$ $S[top] \leftarrow v$
Retrieve v	$v \leftarrow S[head]$ $head \leftarrow head + 1$	$v \leftarrow S[top]$ $top \leftarrow top - 1$

Example





	a	b	c	d	e	f	g
degree	0	1	1	1	2	1	3

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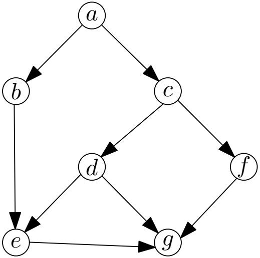
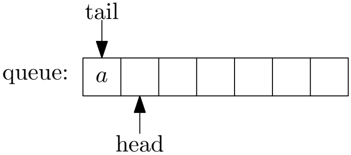


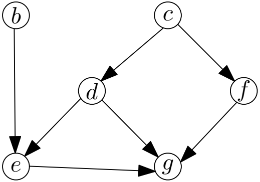

tail

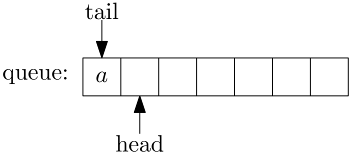
queue: a

head

	a	b	c	d	e	f	g
degree	0	1	1	1	2	1	3

Example





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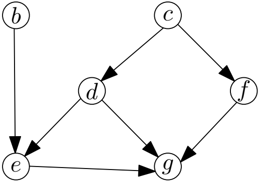
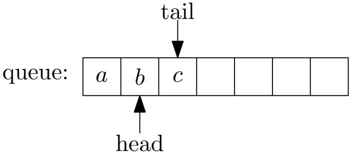



tail

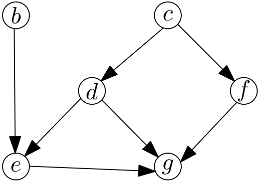
queue:

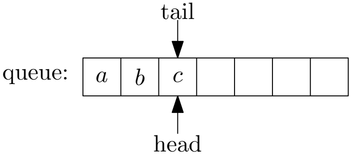
a	b	c				

head

	a	b	c	d	e	f	g
degree	0	0	0	1	2	1	3

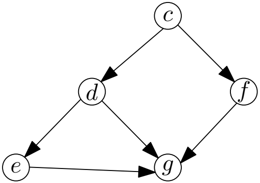
Example

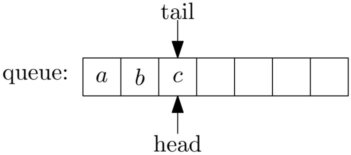




	a	b	c	d	e	f	g
degree	0	0	0	1	2	1	3

Example




tail

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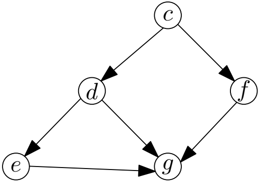

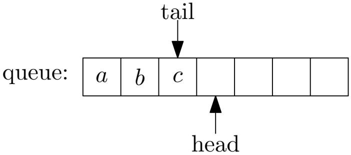
tail

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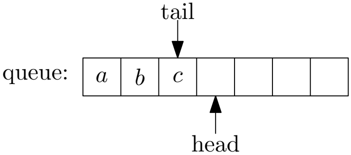
Example



tail

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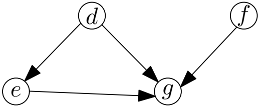
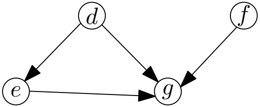
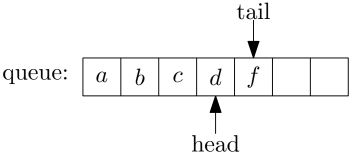


	a	b	c	d	e	f	g
degree	0	0	0	0	1	0	3

Example

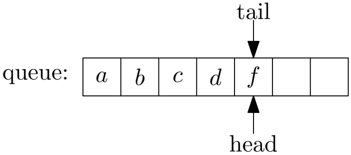




queue: a b c d f

	a	b	c	d	e	f	g
degree	0	0	0	0	1	0	3

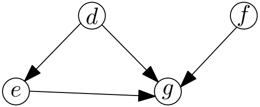
Example



tail

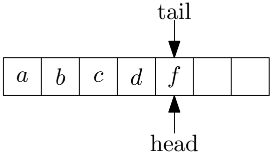
queue:	a	b	c	d	f		

head

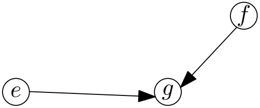


	a	b	c	d	e	f	g
degree	0	0	0	0	1	0	3

Example

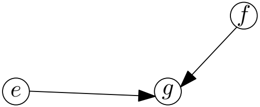


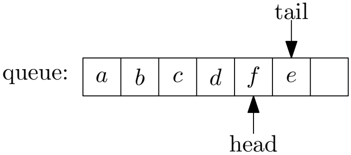
queue:	a	b	c	d	f		



	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	2

Example





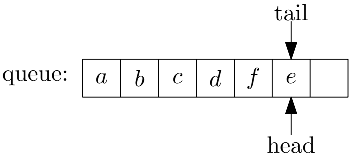
tail

queue: a b c d f e

head

	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	2

Example



tail

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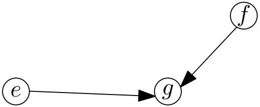
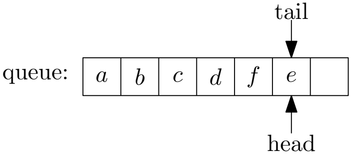


	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	2

Example



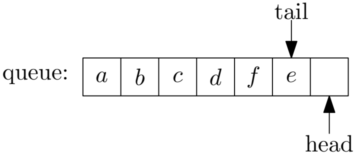
tail

queue:	a	b	c	d	f	e	

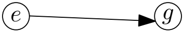
head

	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	1

Example

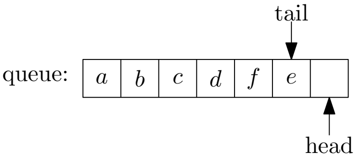


queue:	a	b	c	d	f	e	



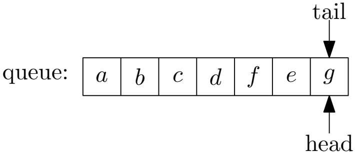
	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	1

Example



	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	0

Example



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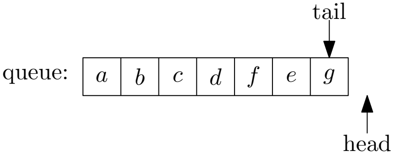

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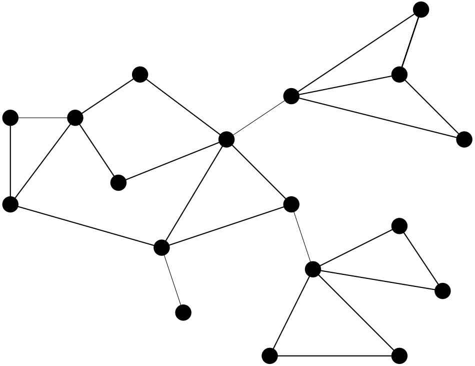
Outline

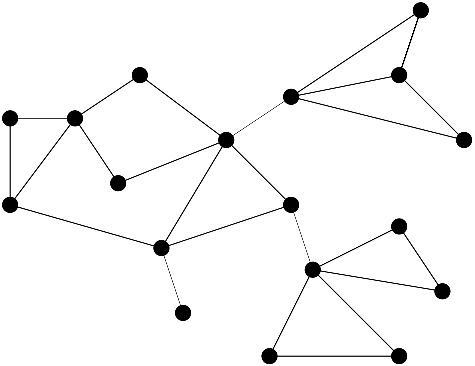
1. Graphs
2. Connectivity and Graph Traversal
    - Testing Bipartiteness
3. Topological Ordering
4. Bridges and 2-Edge-Connected Components
    - $O(n+m)$-Time Algorithm to Find Bridges
    - Related Concept: Cut Vertices
5. Strong Connectivity in Directed Graphs
    - Tarjan's $O(n+m)$-Time Algorithm for Finding SCCes

Def. Given $G = (V, E)$, $e \in E$ is called a bridge if the removal of e from G will increase its number of connected components.



Def. Given $G = (V, E)$, $e \in E$ is called a bridge if the removal of e from G will increase its number of connected components.

- When G is connected, $e \in E$ is a bridge iff its removal will disconnect G.



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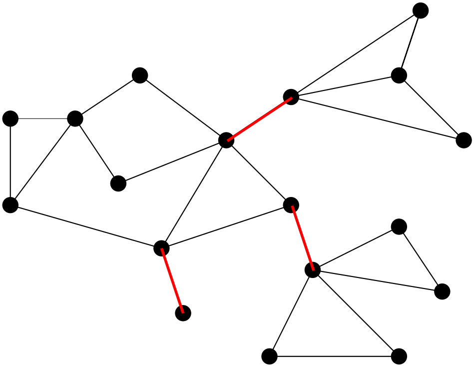

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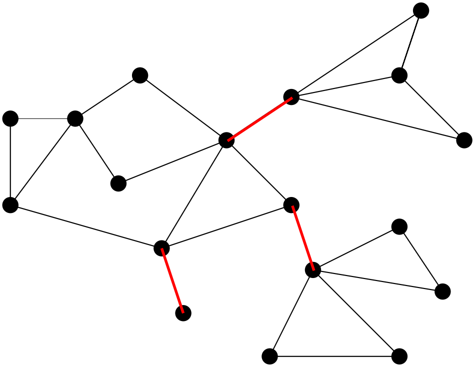

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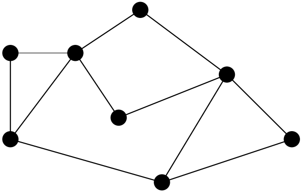

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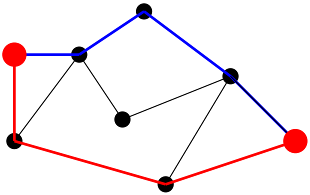

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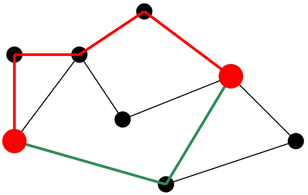

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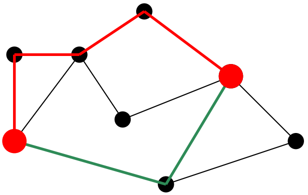

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Lemma Let B be the set of bridges in a graph $G = (V, E)$. Then, every connected component in $(V, E \setminus B)$ is 2-edge-connected. Every such component is called a 2-edge-connected component of G.

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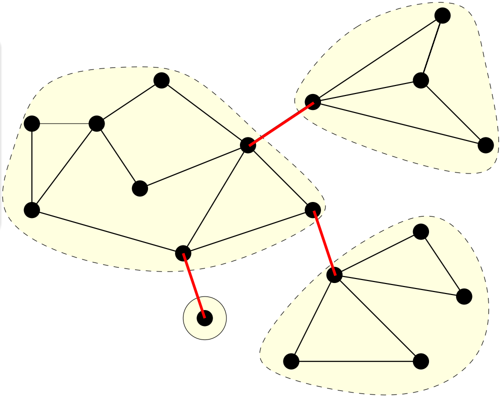

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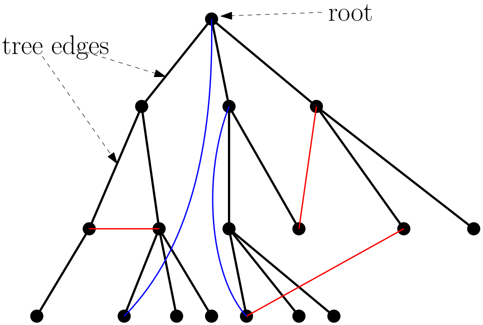
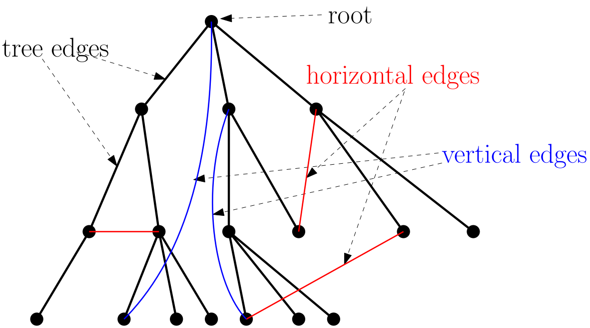
Vertical and Horizontal Edges

- $G = (V, E)$: connected graph
- $T = (V, E_T)$: rooted spanning tree of G

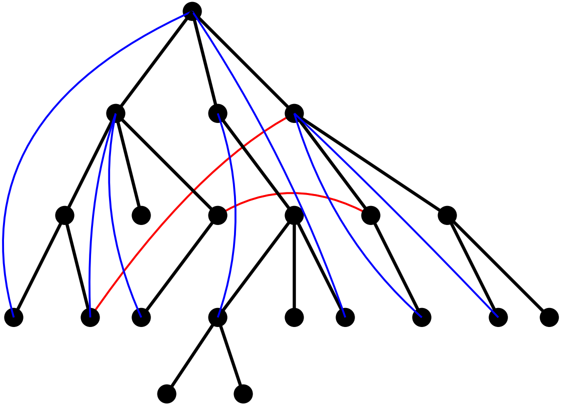


Vertical and Horizontal Edges

- $G = (V, E)$: connected graph
- $T = (V, E_T)$: rooted spanning tree of G
- $(u, v) \in E \setminus E_T$ is
  - vertical if one of u and v is an ancestor of the other in T,
  - horizontal otherwise.



- $G = (V, E)$: connected graph $\qquad$ T: a DFS tree for G



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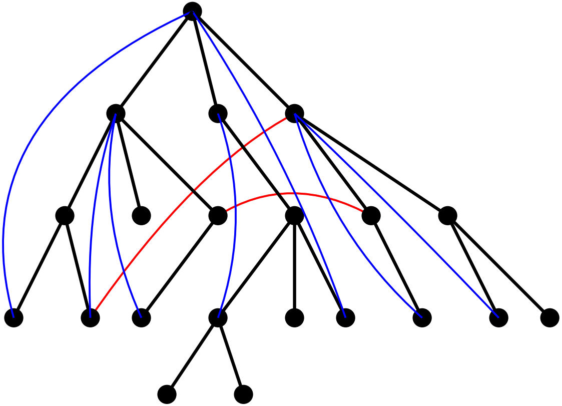

Q: Can there be a horizontal edges (u, v) w.r.t T?

- $G = (V, E)$: connected graph

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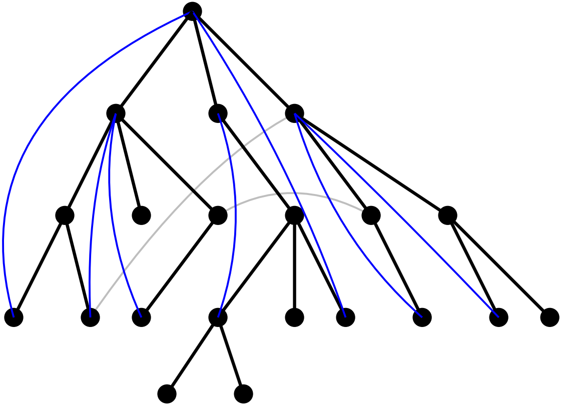
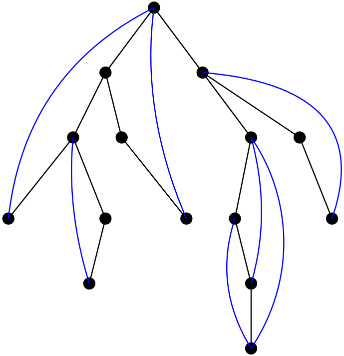


Q: Can there be a horizontal edges (u, v) w.r.t T?

A: No!

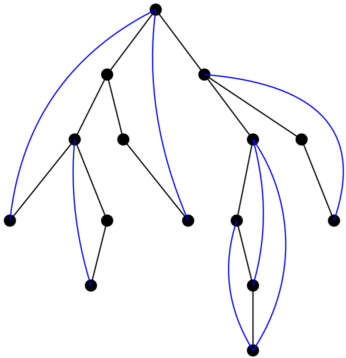
- $G = (V, E)$: connected graph
- T: a DFS tree for G

- G contains only tree and vertical edges



- $G = (V, E)$: connected graph
- T: a DFS tree for G

- G contains only tree and vertical edges
- vertical edges: not bridges



- $G = (V, E)$: connected graph
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- G contains only tree and vertical edges
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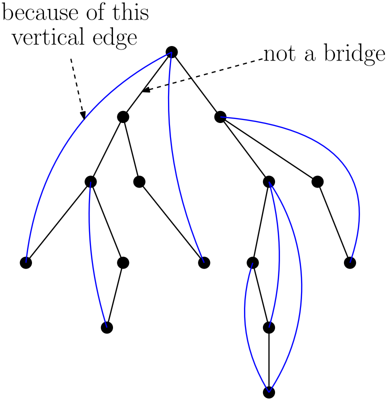
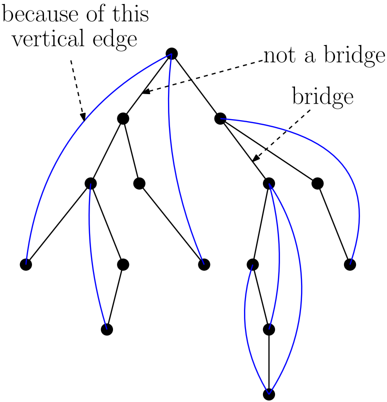


Lemma

- $(u, v) \in T$, u is parent
- (u, v) is not a bridge $\iff \exists$ vertical edge connecting an (inclusive) descendant of v and an (inclusive) ancestor of u

- $G = (V, E)$: connected graph
- T: a DFS tree for G
- G contains only tree and vertical edges
- vertical edges: not bridges



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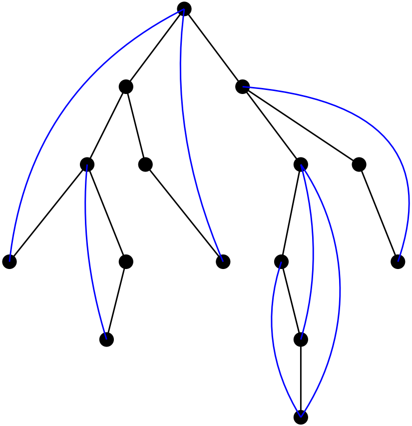
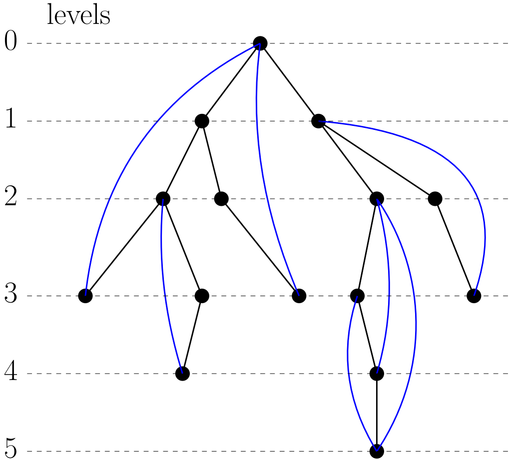
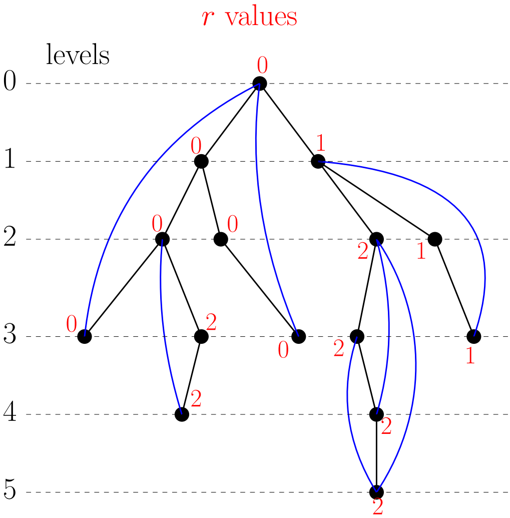


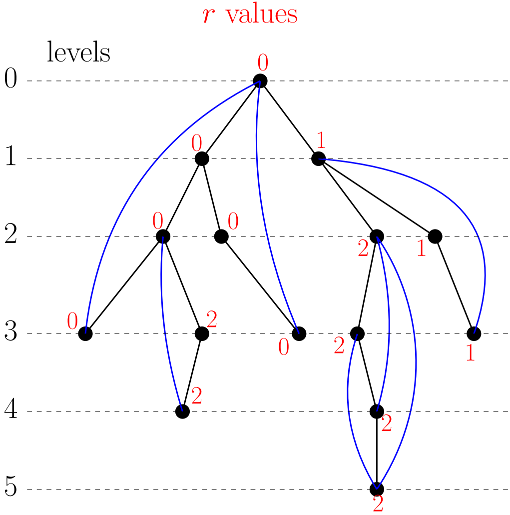
- $v.l$: the level of vertex v in DFS tree



- $v.l$: the level of vertex v in DFS tree
- T_v: subtree rooted at v
- $v.r$: the smallest level that can be reached by a vertical edge from T_v



- $v.l$: the level of vertex v in DFS tree
- T_v: subtree rooted at v
- $v.r$: the smallest level that can be reached by a vertical edge from T_v
- $(parent(u), u)$ is a bridge if and only if $u.r \geq u.l$.



recursive-DFS(v)

```
mark $v$ as "visited"
$v.r \leftarrow \infty$
for all neighbours $u$ of $v$ do
    if $u$ is unvisited then                          ▷ $u$ is a child of $v$
        $u.l \leftarrow v.l + 1$
        recursive-DFS($u$)
        if $u.r \geq u.l$ then claim $(v, u)$ is a bridge
        if $u.r < v.r$ then $v.r \leftarrow u.r$
    else if $u.l < v.l - 1$ then          ▷ $u$ is ancestor but not parent
        if $u.l < v.r$ then $v.r \leftarrow u.l$
```

finding-bridges

```
mark all vertices as "unvisited"
for every $v \in V$ do
    if $v$ is unvisited then
        $v.l \leftarrow 0$
        recursive-DFS($v$)
```

- Running time: $O(n + m)$

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    - $O(n + m)$-Time Algorithm to Find Bridges
    - Related Concept: Cut Vertices

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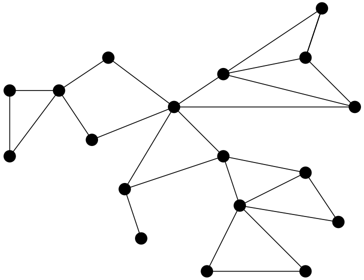
Cut vertices

Def. A vertex is a cut vertex of $G = (V, E)$ if its removal will increase the number of connected components of G.



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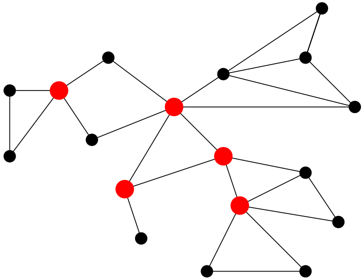

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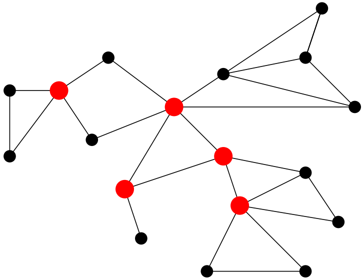
Def. A graph $G = (V, E)$ is 2-(vertex-)connected (or biconnected) if for every $u, v \in V$, there are 2 internally-disjoint paths between u and v.



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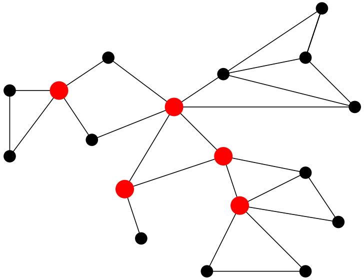

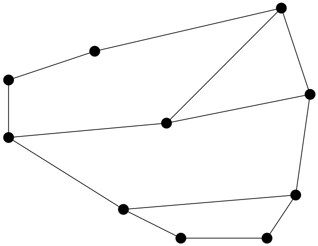

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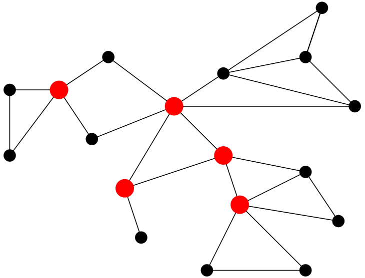

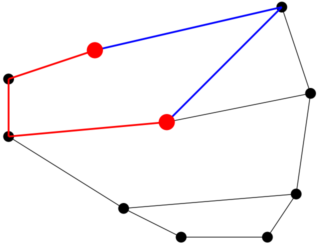

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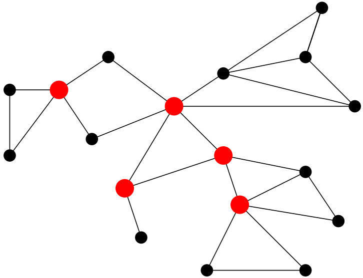

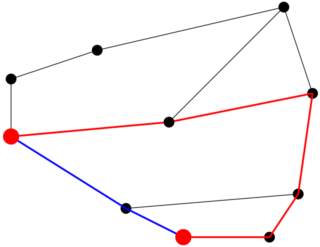

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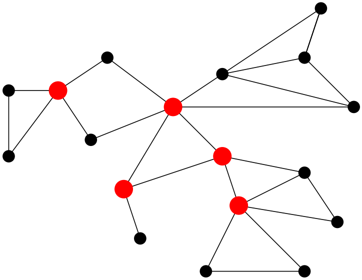
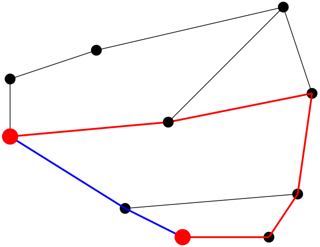
Lemma A graph $G = (V, E)$ with $|V| \geq 3$ does not contain a cut vertex, if and only if it is biconnected.





Q: How can we find the cut vertices?

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A: With a small modification to the algorithm for finding bridges.

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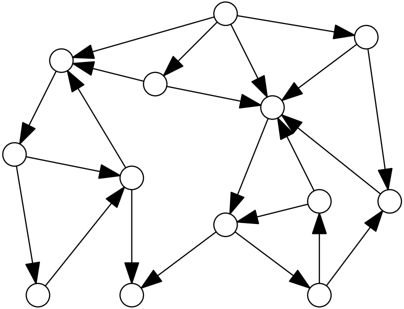

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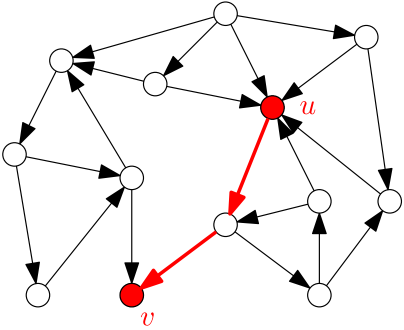

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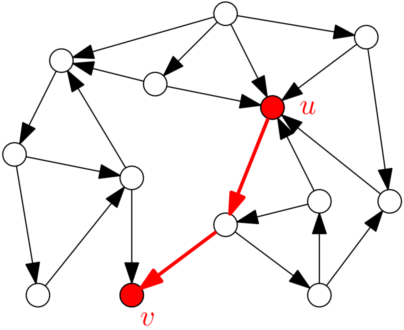

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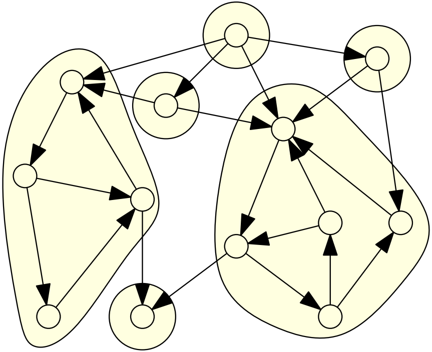

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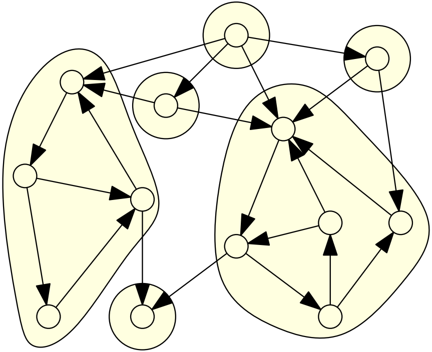

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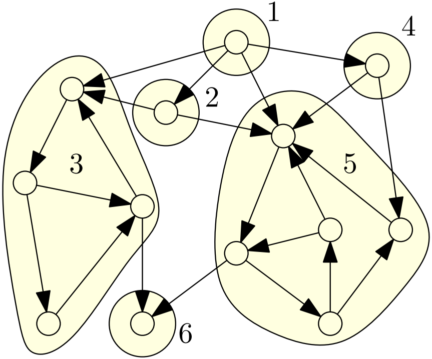

- Define equivalence relation: u and v are related if they are reachable from each other
- equivalence class $\equiv$ SCC
- After contracting each SCC, G becomes a directed-acyclic (multi-)graph (DAG).

Q: How can we check if a directed graph $G = (V, E)$ is strongly-connected?

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A:

- Run a traversal algorithm (either BFS or DFS) from s twice, one on G, one on G with all directions of edges reversed
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Q: How can we find all strongly-connected components (SCCes) of a directed graph G?

A: A much harder problem. Tarjan's $O(n + m)$-time algorithm.

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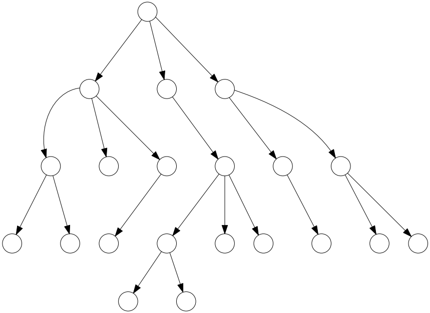
Type of Edges w.r.t a Directed DFS Tree

- directed graph, $G = (V, E)$, a DFS-tree T,
- assuming every vertex is reachable from the root of T



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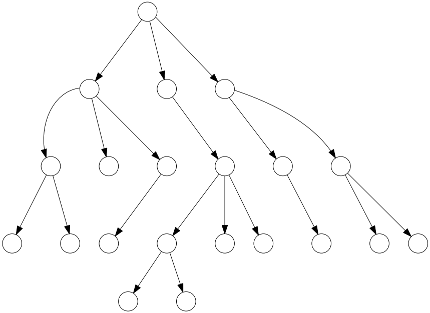

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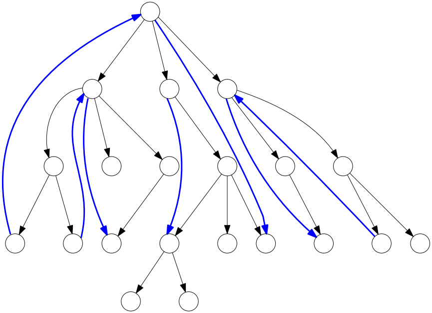

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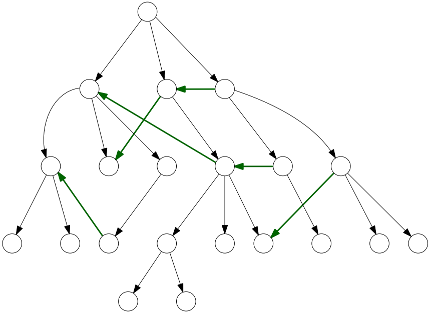

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- upwards (vertical) edges
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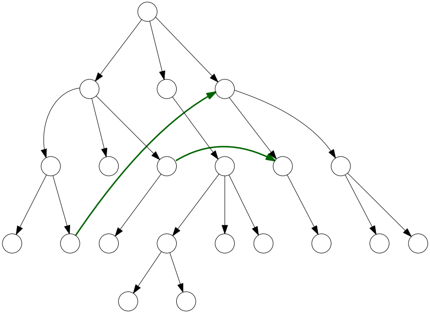

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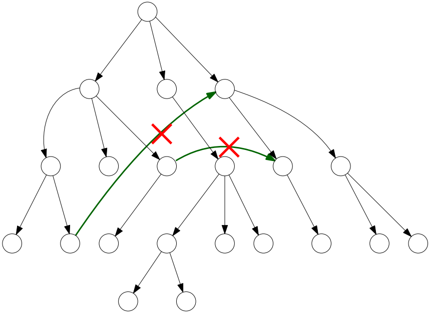

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Q: Can there be rightwards horizontal edges?

A: No!

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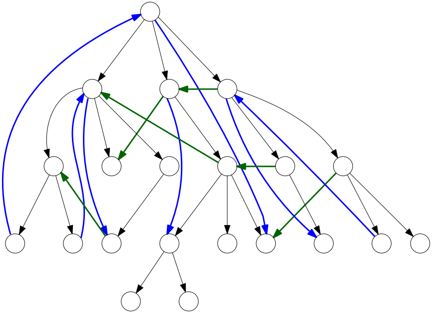

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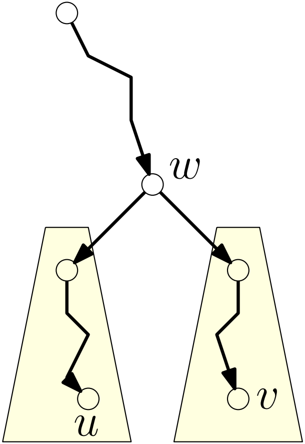
- leftwards horizontal edges

Q: Can there be rightwards horizontal edges?

A: No!

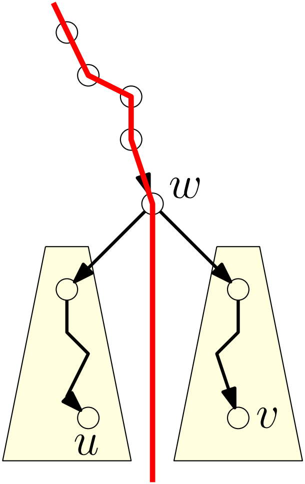
Lemma Suppose u and v are in the same SCC, and w is the lowest common ancestor (LCA) of u and v in T. Then w is the same SCC as u and v.



Lemma Suppose u and v are in the same SCC, and w is the lowest common ancestor (LCA) of u and v in T. Then w is the same SCC as u and v.

Proof.

- Idea: using leftward, upwards and tree edges, u can not reach v without touching w or its ancestors. □



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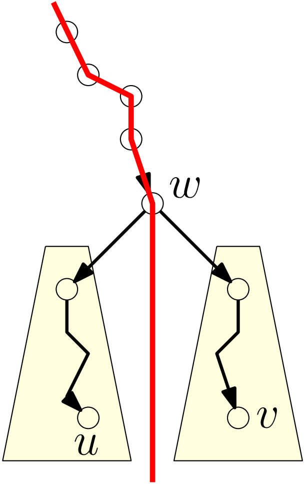
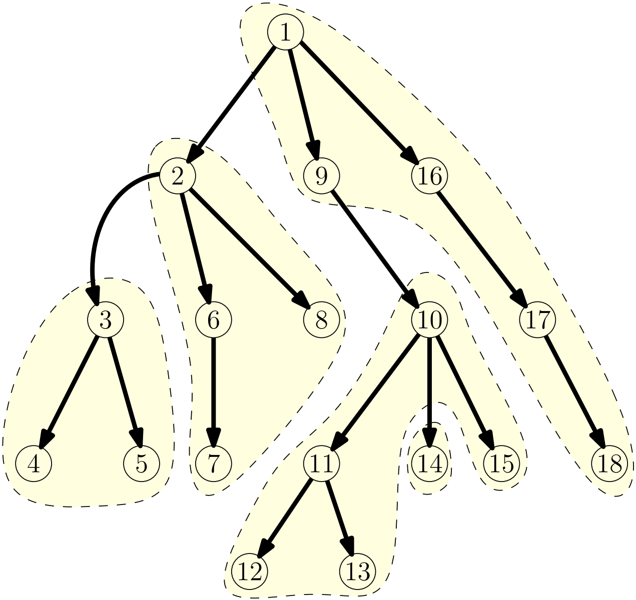


Lemma The vertices in every SCC of G induce a sub-tree in T.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An Intermediate Algorithm to Keep in Mind

```
build the DFS tree $T$
while $T$ is not empty do
    find the first vertex $v$ in the posterior-order-traversal of $T$ satisfying the following property: there are no edges from $T_v$ to outside $T_v$
    claim vertices in $T_v$ as a SCC, remove them from $T$ and all edges incident to them from $T$ and $G$
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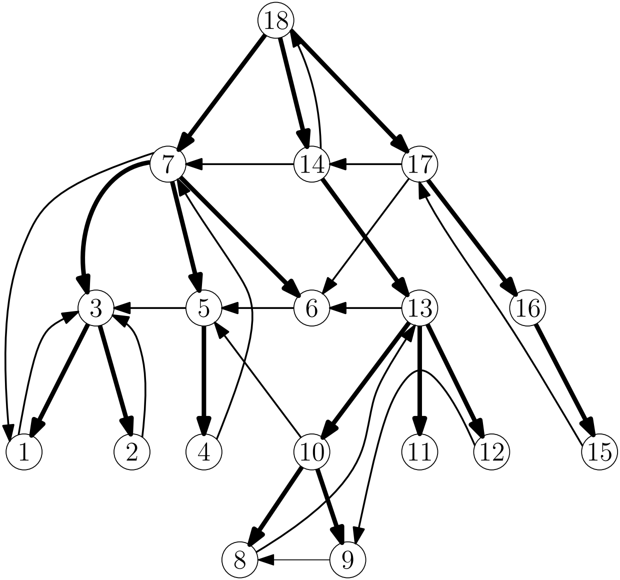
Proof.

- from v, we can reach any vertex in T_v (using tree edges, easy)
- from any vertex in T_v, we can reach v (harder)
- no edges go out of T_v (by our choice, easy) □

Illustration of Intermediate Algorithm



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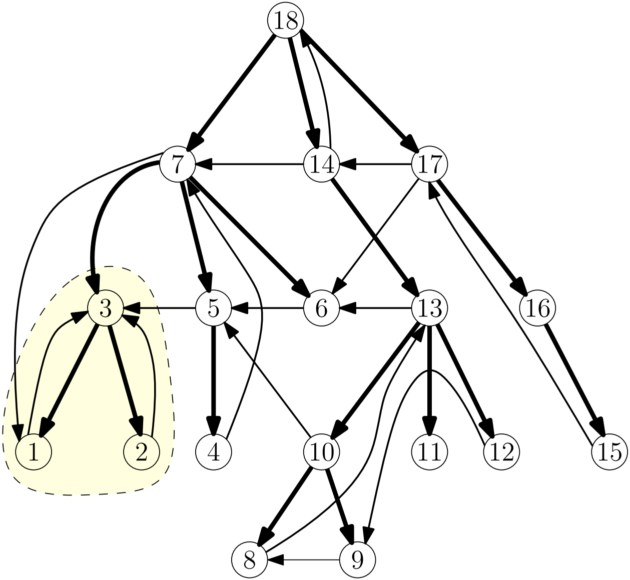

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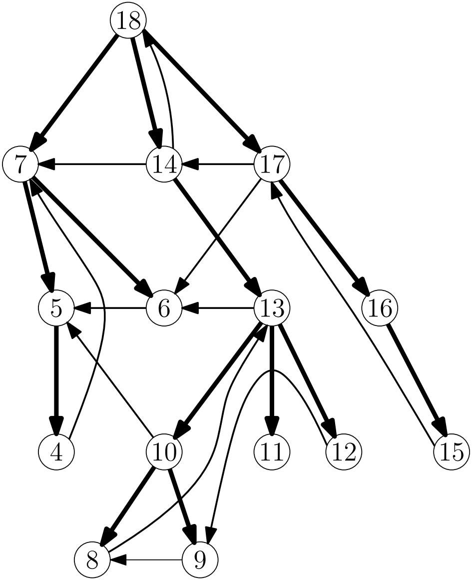

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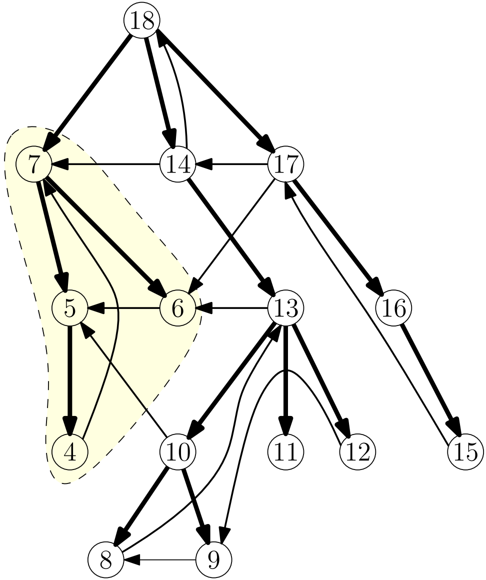

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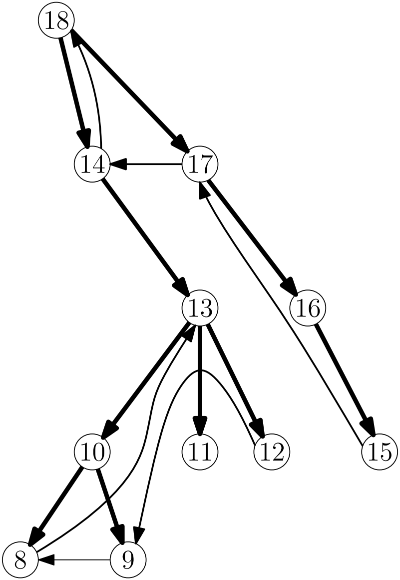

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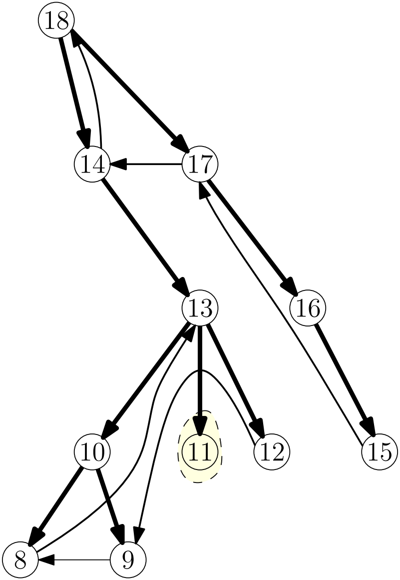
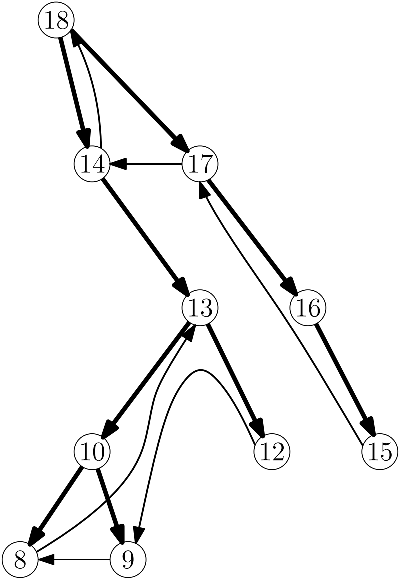


Illustration of Intermediate Algorithm



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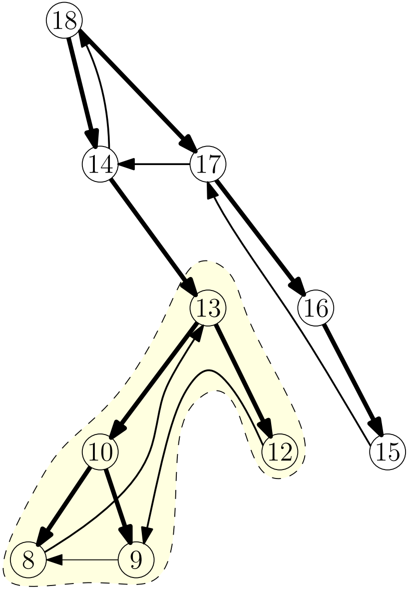

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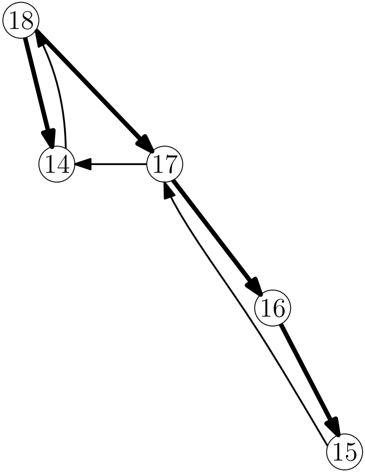

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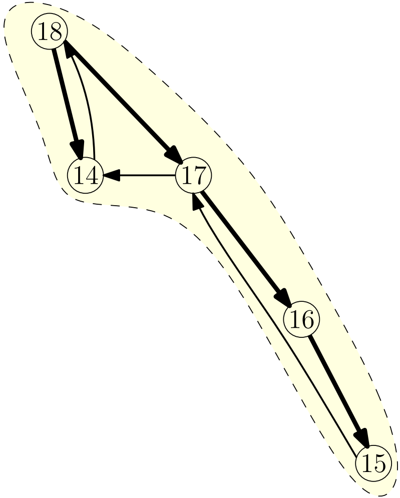

Illustration of Intermediate Algorithm

Tarjan's $O(n + m)$-Time Algorithm

- visited: vertices that are visited by the DFS procedure
  - alive: not in any SCC yet
  - departed: already in some SCC, and removed
- unvisited: not yet visited by the DFS procedure

Tarjan's $O(n+m)$-Time Algorithm

- visited: vertices that are visited by the DFS procedure
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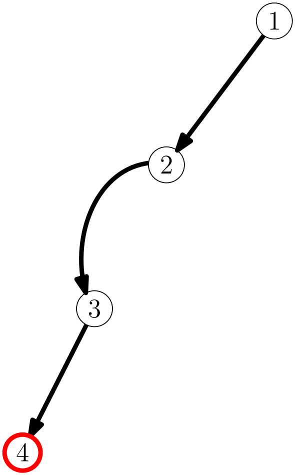
- $stack$: store alive vertices, in visiting order
- $onstack[v]$: whether v is in the stack (i.e, alive)
- $v.i$: the rank of v using the pre-traversal order
- $v.r$, for an alive v: the minimum of $u.i$, over all vertices u that can be reached from v, using alive edges

Illustration of Tarjan's Algorithm



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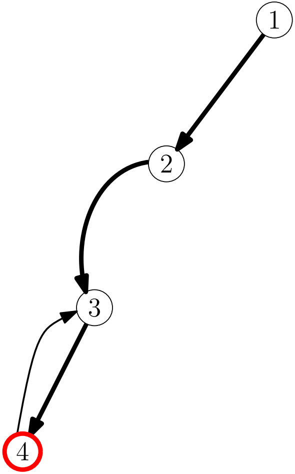

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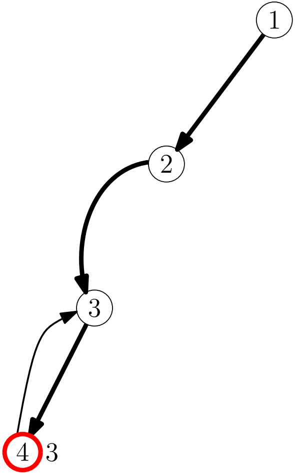

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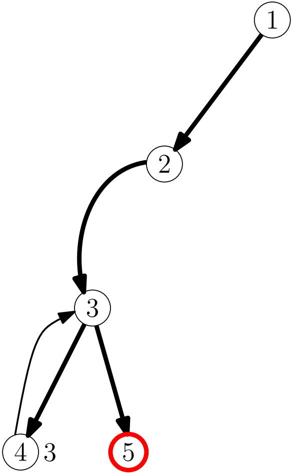

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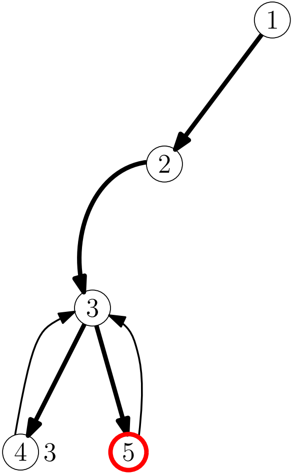

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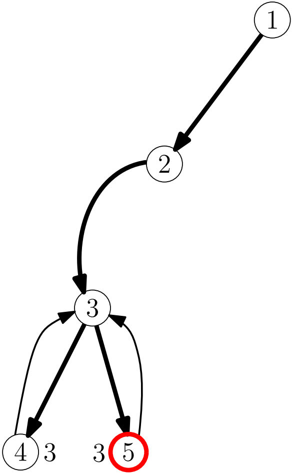

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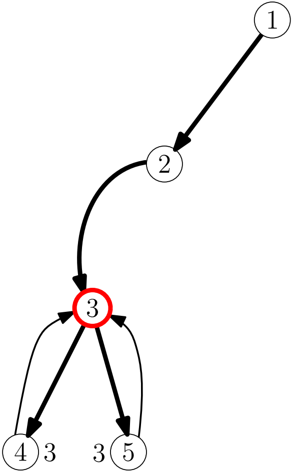

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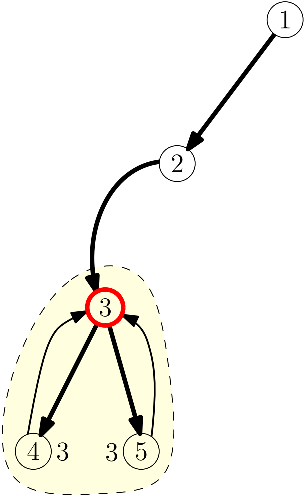

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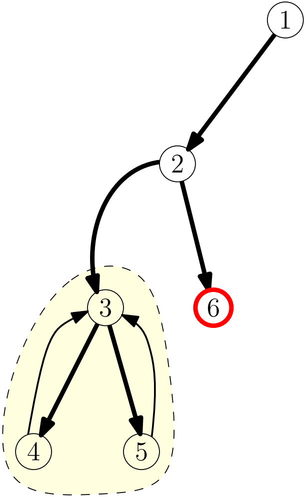

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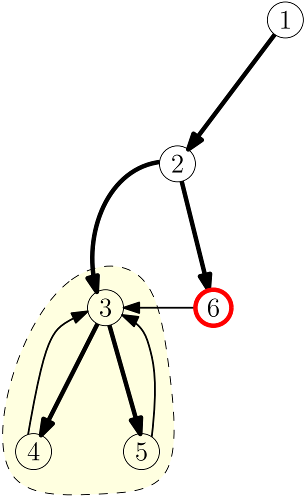

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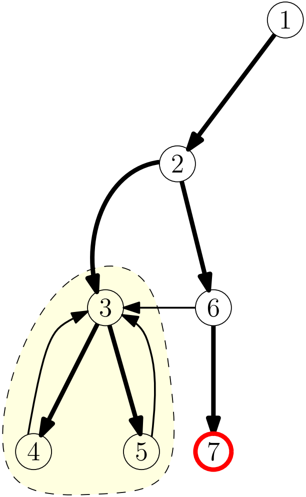

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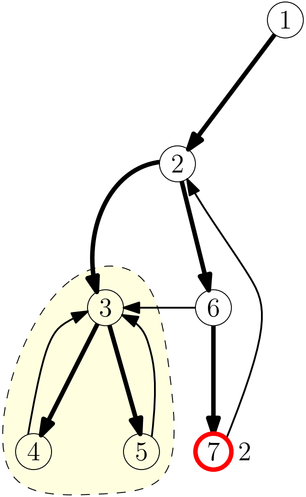

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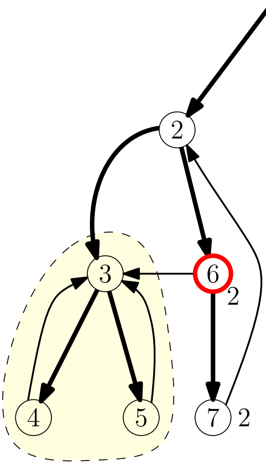

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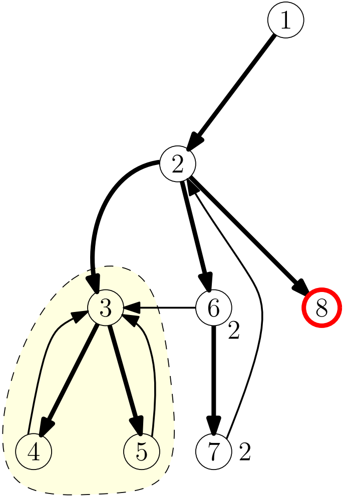

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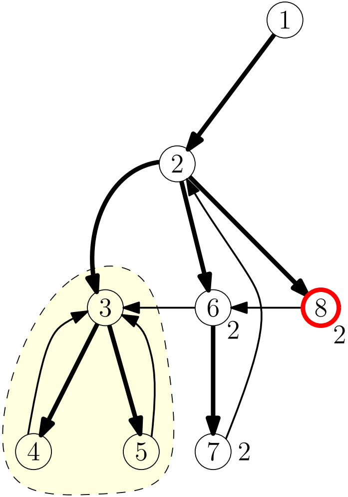

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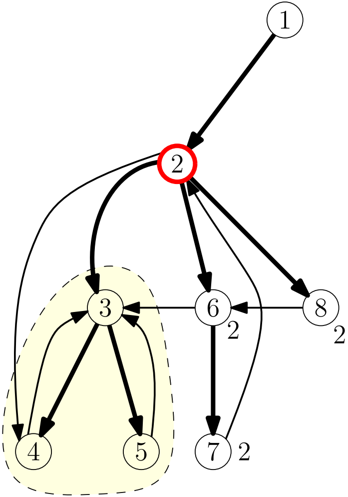

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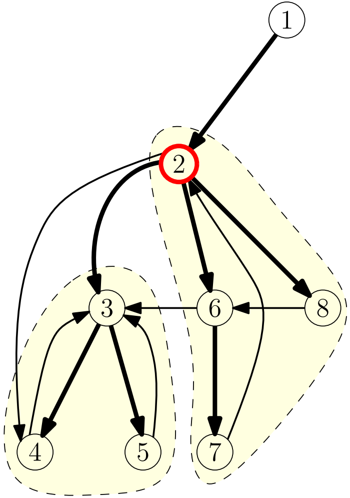

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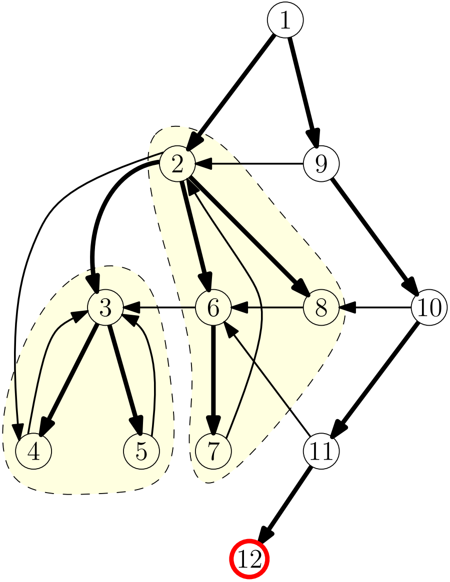

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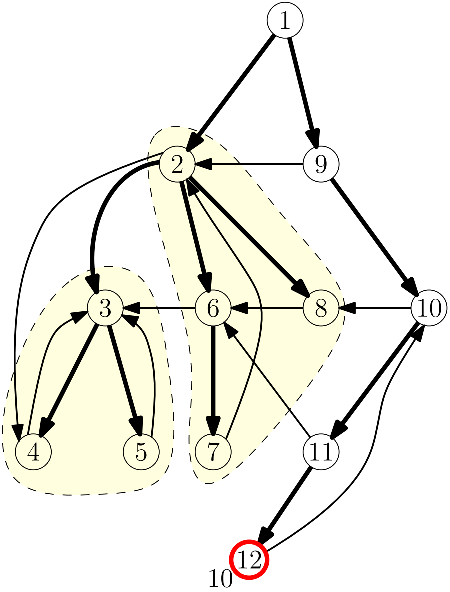

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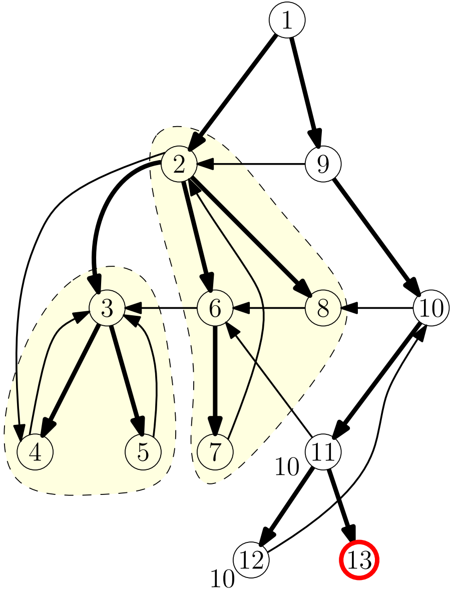

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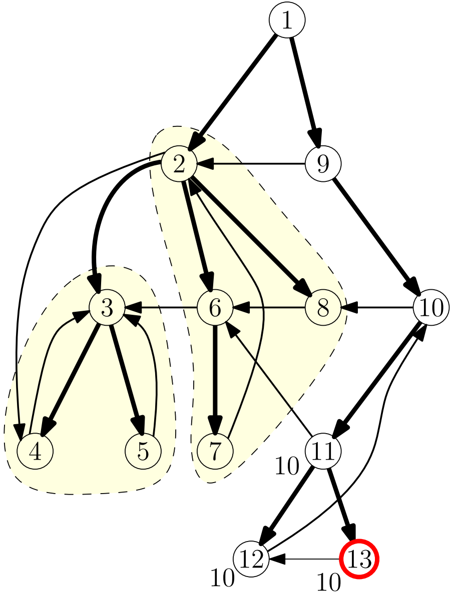

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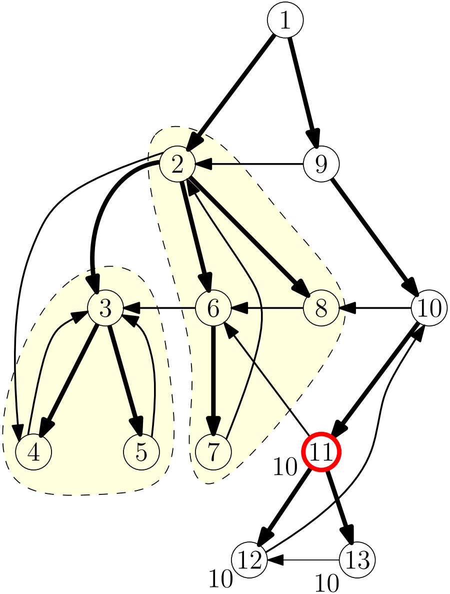

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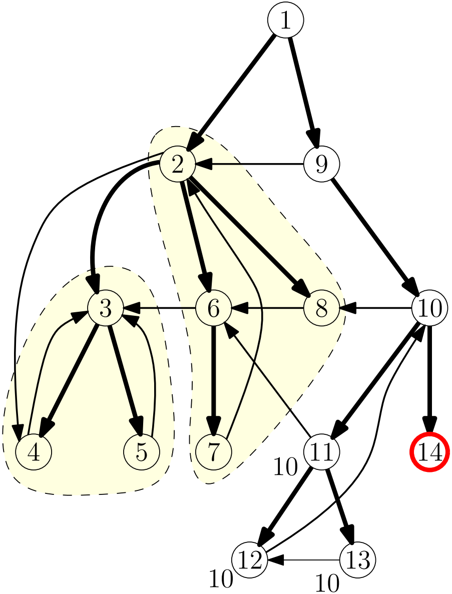

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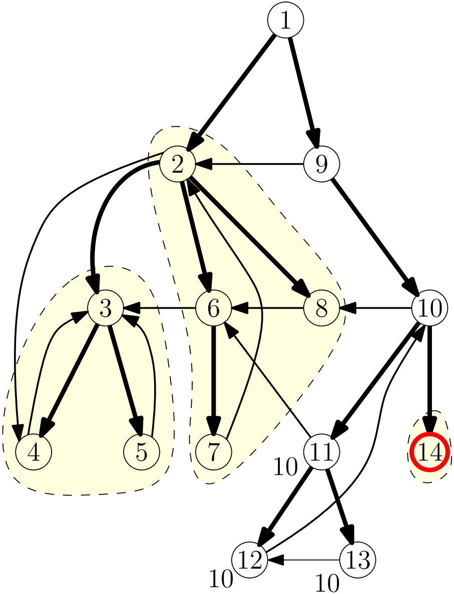

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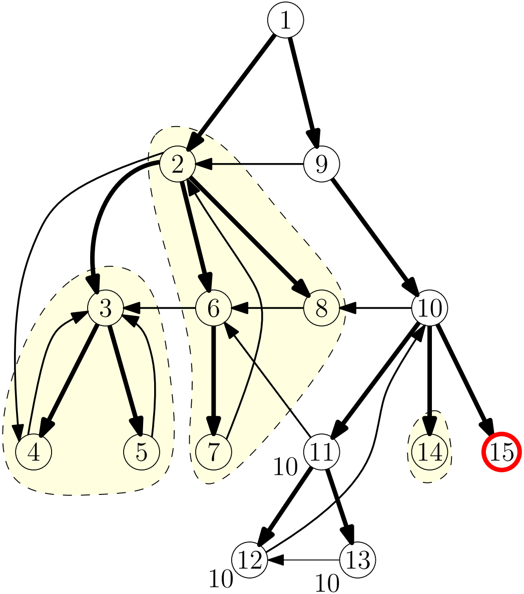

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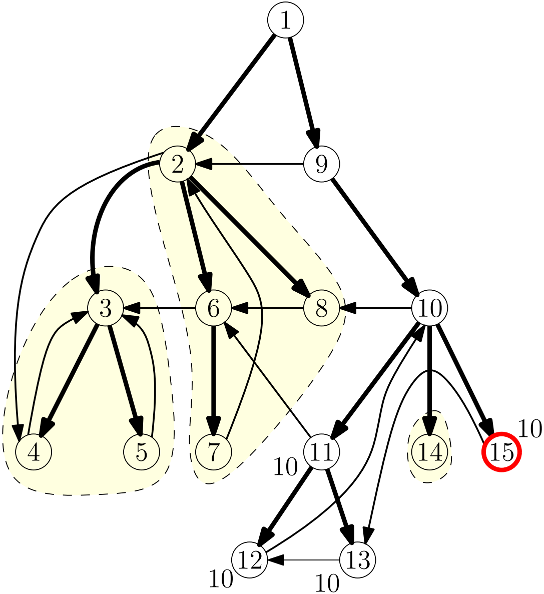

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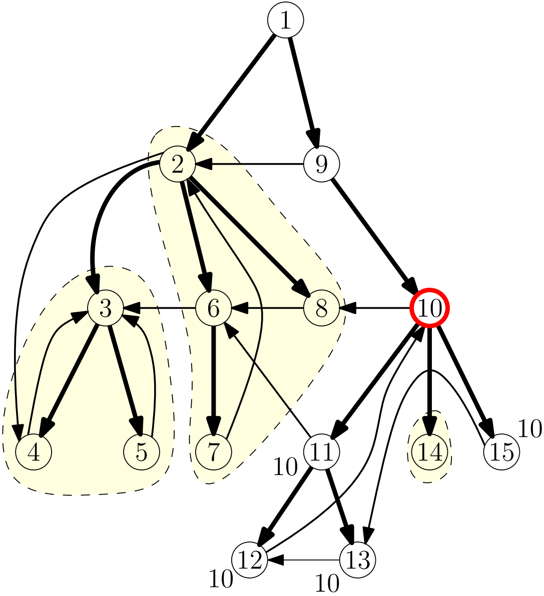

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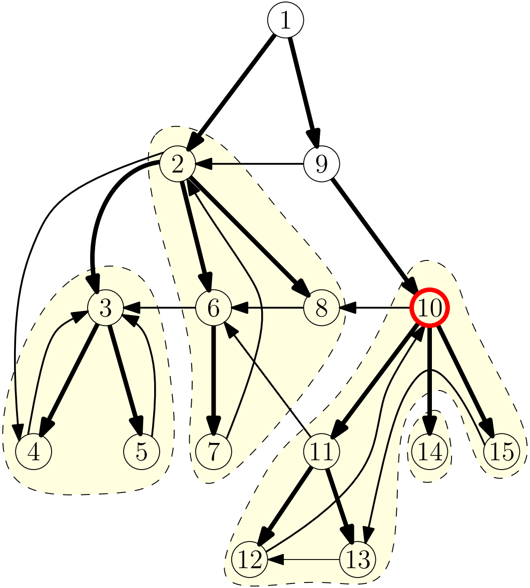

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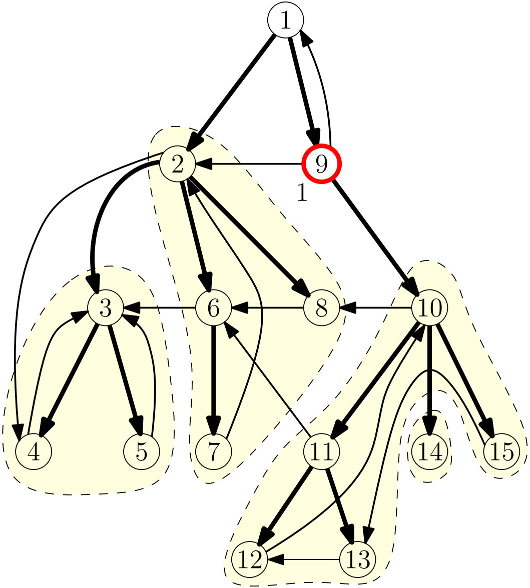

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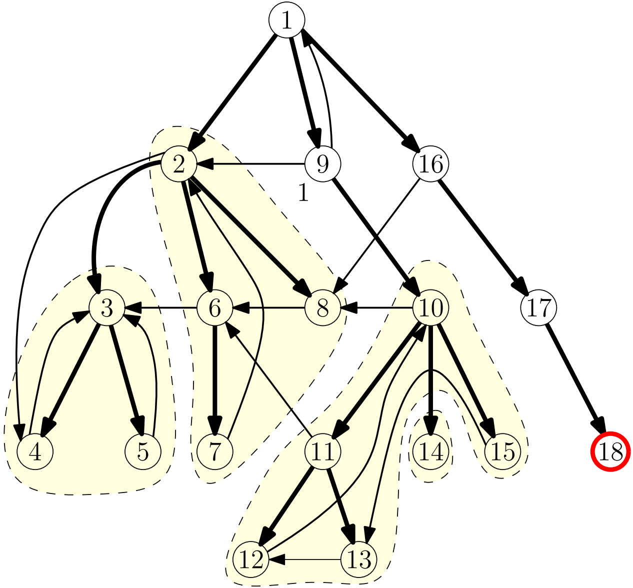

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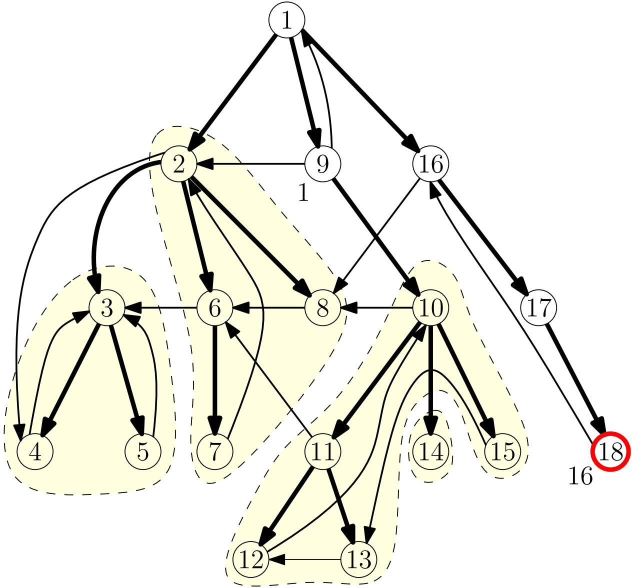

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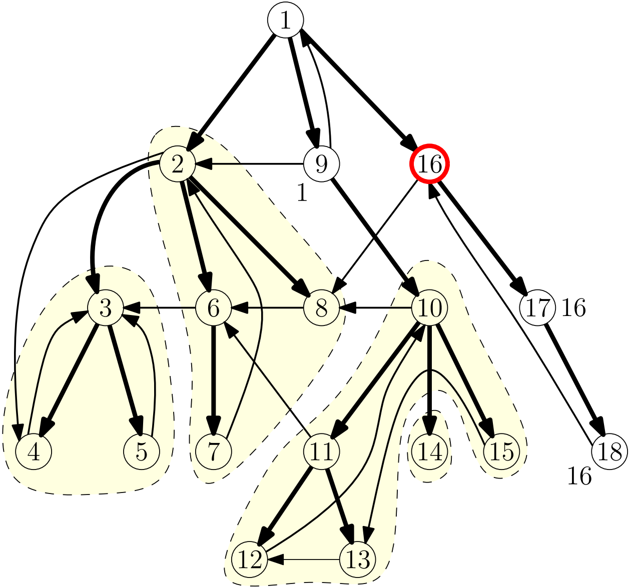

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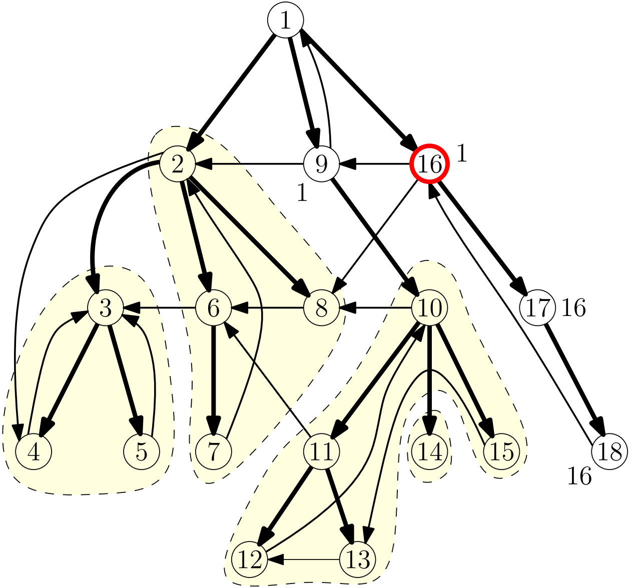

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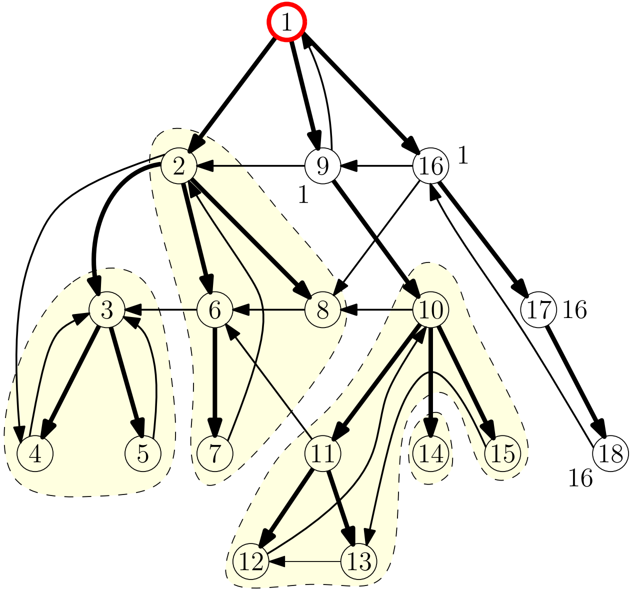

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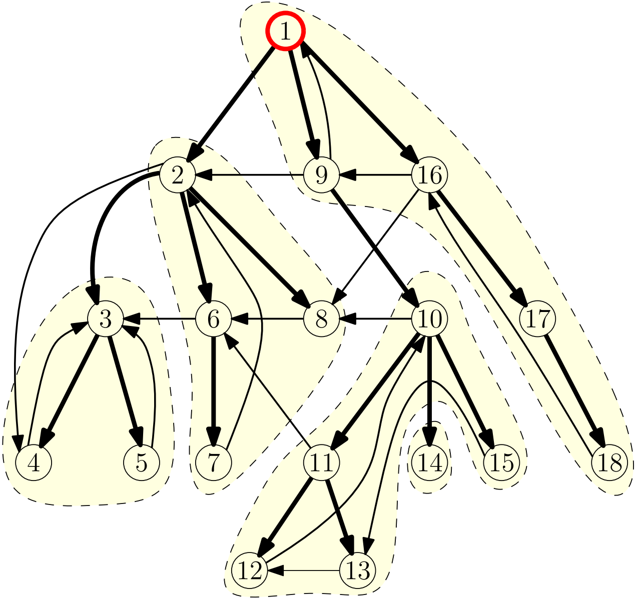

finding strongly connected components

```
$statck \leftarrow$ empty stack, $i \leftarrow 0$
for every $v \in V$ do: $v.i \leftarrow \perp, onstack[i] \leftarrow$ false
for every $v \in V$ do
    if $v.i = \perp$ then recursive-DFS($v$)
```

recursive-DFS(v)

```
$i \leftarrow i + 1, v.i \leftarrow i, v.r \leftarrow i$
$stack.push(v), onstack[v] \leftarrow$ true
for every outgoing edge $(v, u)$ of $v$ do
    if $u.i = \perp$ then recursive-DFS($u$)
    if $onstack[u]$ and $u.r < v.r$ then $v.r \leftarrow u.r$
if $v.r = v.i$ then
    pop all vertices in $stack$ after $v$, including $v$ itself
    set $onstack$ of these vertices to be false
    declare that these vertices form an SCC
```

Running time of the algorithm is $O(n + m)$.