算法设计与分析(2026年春季学期)

Network Flow

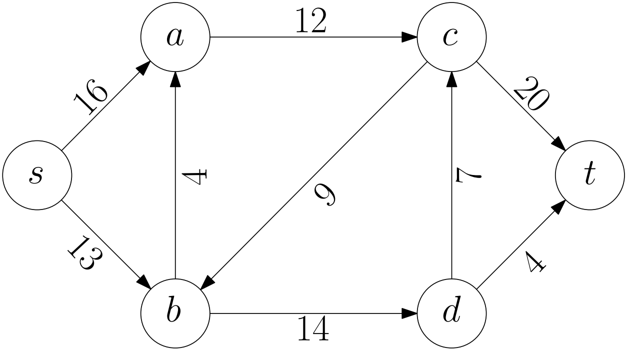
授课老师: 栗师

南京大学计算机学院

Outline

Flow Network

- Abstraction of fluid flowing through edges
- Digraph $G = (V, E)$ with source $s \in V$ and sink $t \in V$
  - No edges enter s
  - No edges leave t
- Edge capacity $c_e \in \mathbb{R}_{>0}$ for every $e \in E$



Def. An s-t flow is a function $f : E \to \mathbb{R}$ such that

- for every $e \in E$: $0 \le f(e) \le c_e$ (capacity conditions)
- for every $v \in V \setminus \{s, t\}$:

$$\sum_{e \in \delta_{\text{in}}(v)} f(e) = \sum_{e \in \delta_{\text{out}}(v)} f(e). \qquad \text{(conservation conditions)}$$

The value of a flow f is

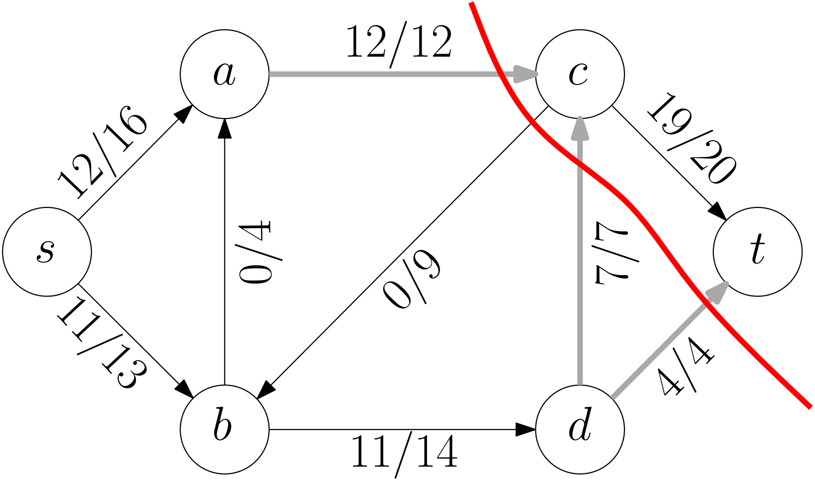
$$\text{val}(f) := \sum_{e \in \delta_{\text{out}}(s)} f(e).$$

Maximum Flow Problem

Input: directed network $G = (V, E)$, capacity function $c : E \to \mathbb{R}_{>0}$, source $s \in V$ and sink $t \in V$

Output: an s-t flow f in G with the maximum $\text{val}(f)$

Maximum Flow Problem: Example

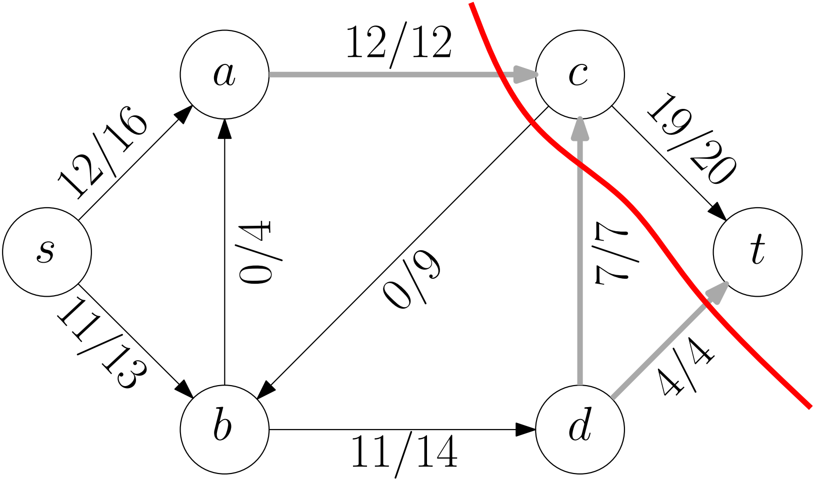


Outline

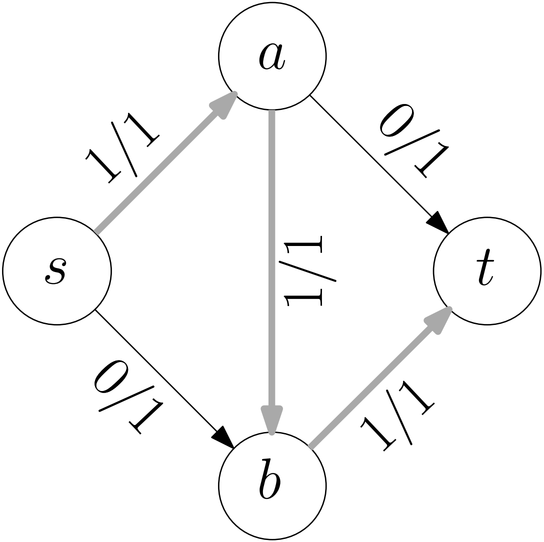
Greedy Algorithm

- Start with empty flow: $f(e) = 0$ for every $e \in E$
- Define the residual capacity of e to be $c_e - f(e)$
- Find an augmenting path: a path from s to t, where all edges have positive residual capacity
- Augment flow along the path as much as possible
- Repeat until we got stuck

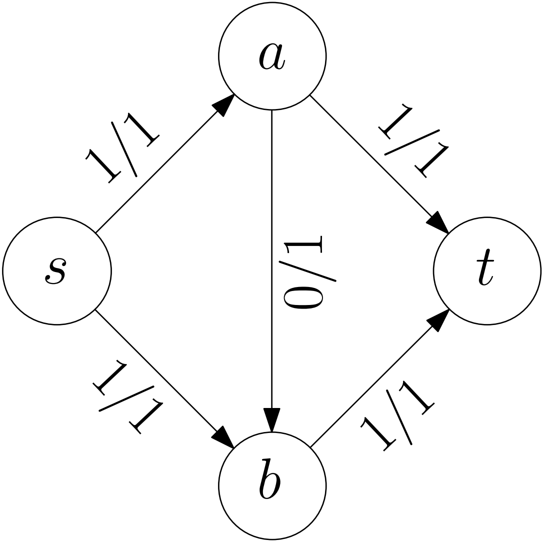
Greedy Algorithm: Example



Greedy Algorithm Does Not Always Give a Optimum Solution



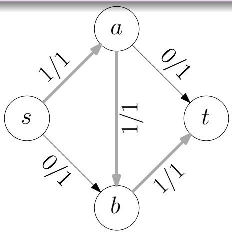
Fix the Issue: Allowing "Undo" Flow Sent



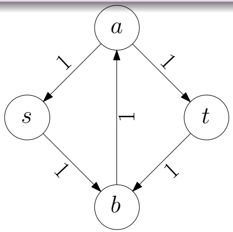
Assumption (u,v) and (v,u) are not both in E

Def. For a s-t flow f, the residual graph G_f of $G=(V,E)$ w.r.t f contains:

- the vertex set V,
- for every $e=(u,v)\in E$ with $f(e)<c_e$, a forward edge $e=(u,v)$, with residual capacity $c_f(e)=c_e-f(e)$,
- for every $e=(u,v)\in E$ with $f(e)>0$, a backward edge $e'=(v,u)$, with residual capacity $c_f(e')=f(e)$.

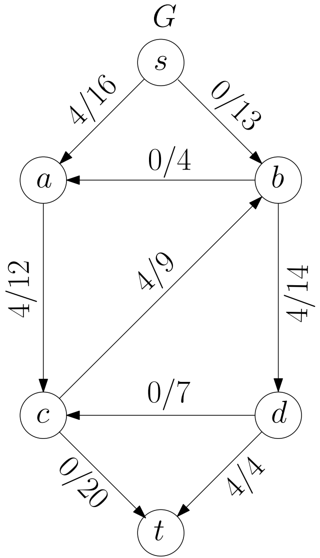


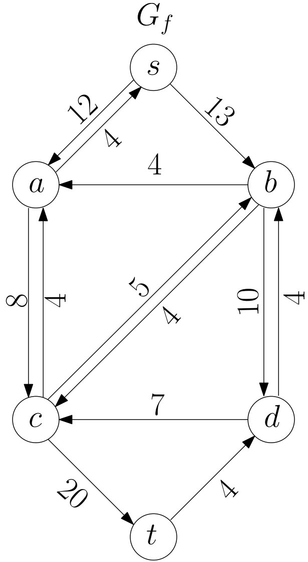
Original graph G and f



Residual Graph G_f

Residual Graph: One More Example





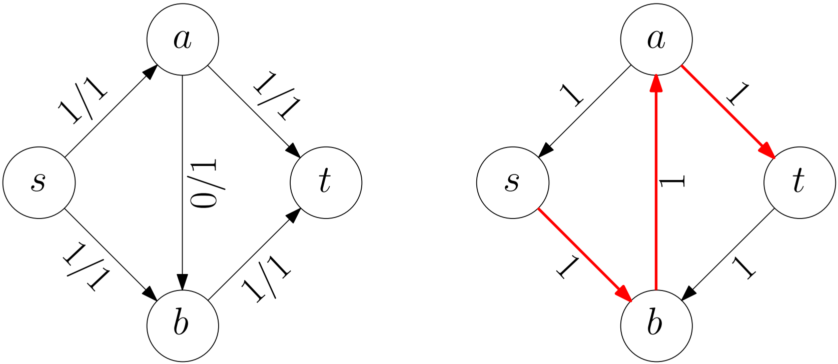
Agumenting Path

Augmenting the flow along a path P from s to t in G_f

Augment(P)

1: $b \leftarrow \min_{e \in P} c_f(e)$
2: **for** every $(u,v) \in P$ **do**
3:     **if** (u,v) is a forward edge **then**
4:         $f(u,v) \leftarrow f(u,v) + b$
5:     **else** $\triangleright$ (u,v) is a backward edge
6:         $f(v,u) \leftarrow f(v,u) - b$
7: **return** f

Example for Augmenting Along a Path

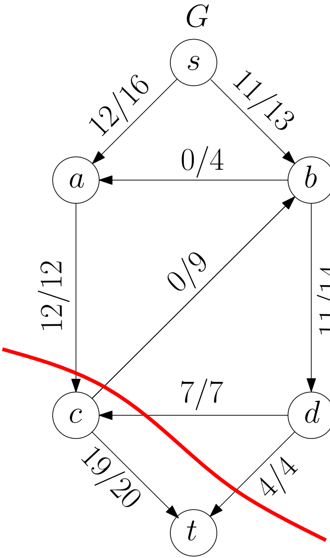


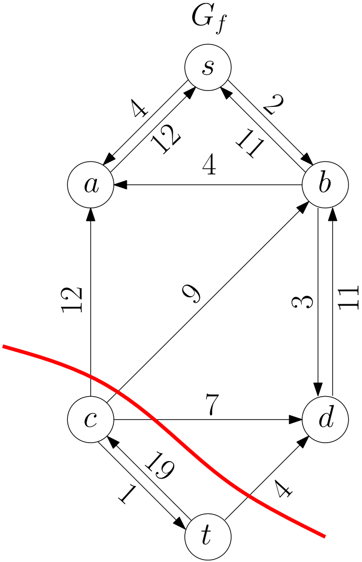
Ford-Fulkerson's Method

Ford-Fulkerson(G, s, t, c)

let $f(e) \leftarrow 0$ for every e in G
while there is a path from s to t in G_f **do**
    let P be any simple path from s to t in G_f
    $f \leftarrow$augment(f, P)
return f

Ford-Fulkerson: Example





Outline

Correctness of Ford-Fulkerson's Method

1. The procedure augment(f, P) maintains the two conditions:
   - for every $e \in E$: $0 \leq f(e) \leq c_e$ (capacity conditions)
   - for every $v \in V \setminus \{s, t\}$:

$$\sum_{e \in \delta_{\mathsf{in}}(v)} f(e) = \sum_{e \in \delta_{\mathsf{out}}(v)} f(e). \qquad \text{(conservation conditions)}$$

2. When Ford-Fulkerson's Method terminates, val(f) is maximized
3. Ford-Fulkerson's Method will terminate

Correctness of Ford-Fulkerson's Method

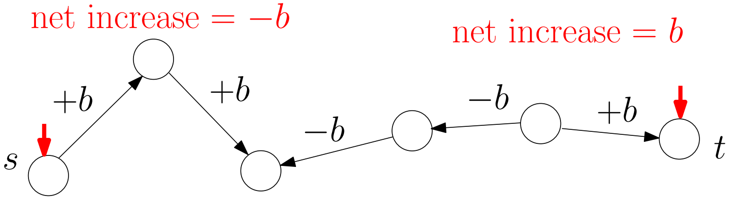
1. The procedure augment(f, P) maintains the two conditions:
   - for every $e \in E$: $0 \leq f(e) \leq c_e$ (capacity conditions)
   - for every $v \in V \setminus \{s, t\}$:

$$\sum_{e \in \delta_{\mathsf{in}}(v)} f(e) = \sum_{e \in \delta_{\mathsf{out}}(v)} f(e). \qquad \text{(conservation conditions)}$$

2. When Ford-Fulkerson's Method terminates, val(f) is maximized
3. Ford-Fulkerson's Method will terminate

- for every $e \in E$: $0 \leq f(e) \leq c_e$ (capacity conditions)
- for every $v \in V \setminus \{s,t\}$:

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e). \qquad \text{(conservation conditions)}$$



- for an edge e correspondent to a forward edge :
  $b \leq c_e - f(e) \implies f(e) + b \leq c_e$
- for an edge e correspondent to a backward edge :
  $b \leq f(e) \implies f(e) - b \geq 0$

Correctness of Ford-Fulkerson's Method

1. The procedure augment(f, P) maintains the two conditions:
   - for every $e \in E$: $0 \leq f(e) \leq c_e$ (capacity conditions)
   - for every $v \in V \setminus \{s, t\}$:

$$\sum_{e \in \delta_{\text{in}}(v)} f(e) = \sum_{e \in \delta_{\text{out}}(v)} f(e). \qquad \text{(conservation conditions)}$$

2. When Ford-Fulkerson's Method terminates, val(f) is maximized
3. Ford-Fulkerson's Method will terminate

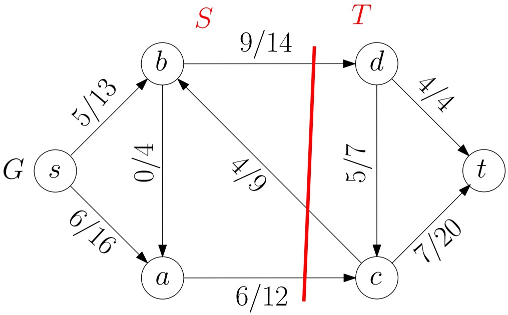
Def. An s-t cut of $G = (V, E)$ is a pair $(S \subseteq V, T = V \setminus S)$ such that $s \in S$ and $t \in T$.

Def. The cut value of an s-t cut is

$$c(S, T) := \sum_{e=(u,v)\in E: u\in S, v\in T} c_e.$$

Def. Given an s-t flow f and an s-t cut (S, T), the net flow sent from S to T is

$$f(S, T) := \sum_{e=(u,v)\in E: u\in S, v\in T} f(e) - \sum_{e=(u,v)\in E: u\in T, v\in S} f(e).$$



$c(S,T) = 14 + 12 = 26$

$f(S,T) = 9 + 6 - 4 = 11$

Obs. $f(S,T) \leq c(S,T)$ s-t cut (S,T).

Obs. $f(S,T) = \text{val}(f)$ for any s-t flow f and any s-t cut (S,T).

Coro.

$$\text{val}(f) \leq \min_{s\text{-}t \text{ cut } (S,T)} c(S,T) \text{ for every } s\text{-}t \text{ flow } f.$$

Coro. $$\mathrm{val}(f) \le \min_{s\text{-}t \text{ cut } (S,T)} c(S,T) \text{ for every } s\text{-}t \text{ flow} f.$$

We will prove

Main Lemma The flow f found by the Ford-Fulkerson's Method satisfies

$$\mathrm{val}(f) = c(S,T) \text{ for some } s\text{-}t \text{ cut } (S,T).$$

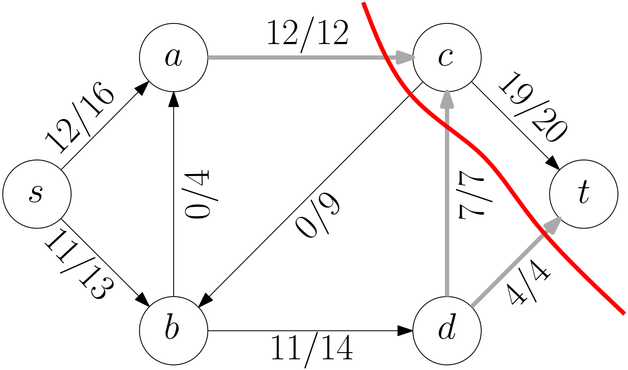
Corollary and Main Lemma implies

Maximum Flow Minimum Cut Theorem

$$\sup_{s\text{-}t \text{ flow } f} \mathrm{val}(f) = \min_{s\text{-}t \text{ cut } (S,T)} c(S,T).$$

Maximum Flow Minimum Cut Theorem

$$\sup_{s\text{-}t \text{ flow } f} \mathsf{val}(f) = \min_{s\text{-}t \text{ cut } (S,T)} c(S,T).$$



Main Lemma The flow f found by the Ford-Fulkerson's Method satisfies

$$\text{val}(f) = c(S,T) \text{ for some } s\text{-}t \text{ cut } (S,T).$$

Proof of Main Lemma.

- When algorithm terminates, no path from s to t in G_f,
- What can we say about G_f?
- There is a s-t cut (S,T), such that there are no edges from S to T
- For every $e = (u,v) \in E, u \in S, v \in T$, we have $f(e) = c_e$
- For every $e = (u,v) \in E, u \in T, v \in S$, we have $f(e) = 0$
- Thus,

$$\text{val}(f) = f(S,T) = \sum_{e=(u,v)\in E, u\in S, v\in T} f(e) - \sum_{e=(u,v)\in E, u\in T, v\in S} f(e) = \sum_{e=(u,v)\in E, u\in S, v\in T} c_e = c(S,T).$$

□

Correctness of Ford-Fulkerson's Method

1. The procedure augment(f, P) maintains the two conditions:
   - for every $e \in E$: $0 \leq f(e) \leq c_e$ (capacity conditions)
   - for every $v \in V \setminus \{s, t\}$:

$$\sum_{e \in \delta_{\text{in}}(v)} f(e) = \sum_{e \in \delta_{\text{out}}(v)} f(e). \qquad \text{(conservation conditions)}$$

2. When Ford-Fulkerson's Method terminates, val(f) is maximized
3. Ford-Fulkerson's Method will terminate

Ford-Fulkerson's Method will Terminate

Intuition:

- In every iteration, we increase the flow value by some amount
- There is a maximum flow value
- So the algorithm will finally reach the maximum value

However, the algorithm may not terminate if some capacities are irrational numbers. ("Pathological cases")

Lemma Ford-Fulkerson's Method will terminate if all capacities are integers.

Proof.

- The maximum flow value is finite (not ∞).
- In every iteration, we increase the flow value by at least 1.
- So the algorithm will terminate. $\square$

- Integers can be replaced by rational numbers.

Correctness of Ford-Fulkerson's Method

1. The procedure augment(f, P) maintains the two conditions:
   - for every $e \in E$: $0 \leq f(e) \leq c_e$ (capacity conditions)
   - for every $v \in V \setminus \{s, t\}$:

$$\sum_{e \in \delta_{\text{in}}(v)} f(e) = \sum_{e \in \delta_{\text{out}}(v)} f(e). \qquad \text{(conservation conditions)}$$

2. When Ford-Fulkerson's Method terminates, val(f) is maximized
3. Ford-Fulkerson's Method will terminate

Outline

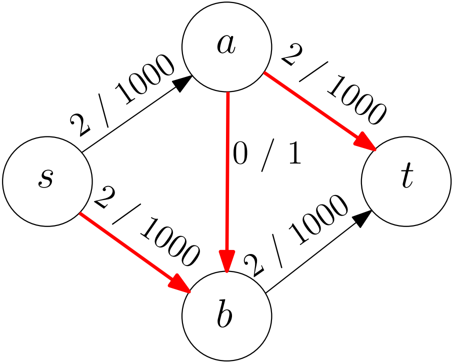
Running time of the Generic Ford-Fulkerson's Algorithm

Ford-Fulkerson(G, s, t, c)

1: let $f(e) \leftarrow 0$ for every e in G
2: **while** there is a path from s to t in G_f **do**
3: let P be any simple path from s to t in G_f
4: $f \leftarrow$augment(f, P)
5: **return** f

- $O(m)$-time for Steps 3 and 4 in each iteration
- Total time $= O(m) \times$ number of iterations
- Assume all capacities are integers, then algorithm may run up to val(f^*) iterations, where f^* is the optimum flow
- Total time $= O(m \cdot \text{val}(f^*))$
- Running time is "Pseudo-polynomial"

The Upper Bound on Running Time Is Tight!



Better choices for choosing augmentation paths:

- Choose the shortest augmentation path
- Choose the augmentation path with the largest bottleneck capacity

Outline

Shortest Augmenting Path

shortest-augmenting-path(G, s, t, c)

```
let f(e) ← 0 for every e in G
while there is a path from s to t in G_f do
    P ← breadth-first-search(G_f, s, t)
    f ← augment(f, P)
return f
```

Due to [Dinitz 1970] and [Edmonds-Karp, 1970]

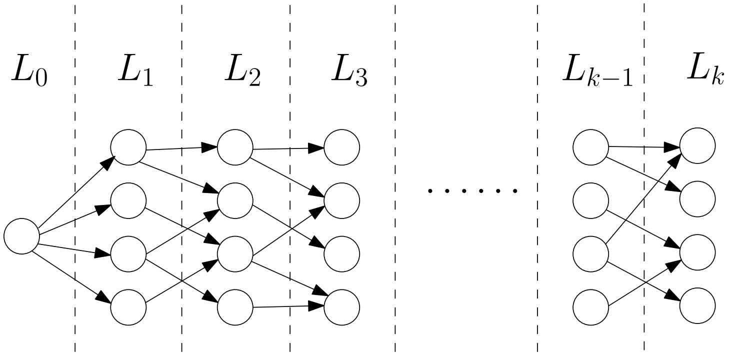
Running Time of Shortest Augmenting Path Algorithm

Lemma 1. Throughout the algorithm, length of shortest path from s to t in G_f never decreases.
2. After at most m shortest path augmentations, the length of the shortest path from s to t in G_f strictly increases.

- Length of shortest path is between 1 and $n-1$
- Algorithm takes at most $O(mn)$ iterations
- Shortest path from s to t can be found in $O(m)$ time using BFS

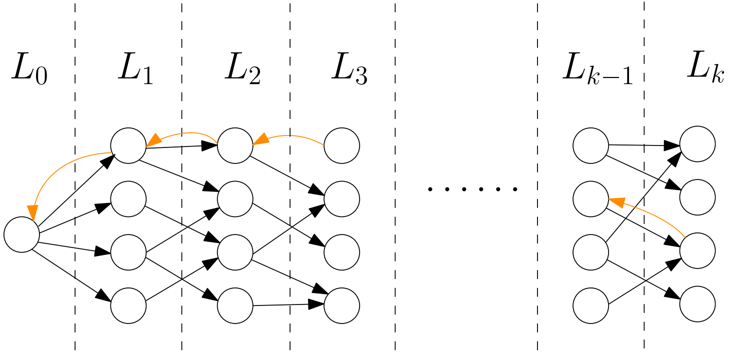
Theorem The shortest-augmenting-path algorithm runs in time $O(m^2n)$.

Proof of Lemma: Focus on G_f



- Divide V into levels: L_i contains the set of vertices v such that the length of shortest path from s to v in G_f is i
- Forth edges : edges from L_i to L_{i+1} for some i
- Side edges : edges from L_i to L_i for some i
- Back edges: edges from L_i to L_j for some $i > j$
- No jump edges: edges from L_i to L_j for $j \geq i + 2$

Proof of Lemma: Focus on G_f



- Assuming $t \in L_k$, shortest $s \to t$ path uses k forth edges
- After augmenting along the path, back edges will be added to G_f
- One forth edge will be removed from G_f
- In $O(m)$ iterations, there will be no paths from s to t of length k in G_f.

Improving the $O(m^2n)$ Running Time for Shortest Path Augmentation Algorithm

- For some networks, $O(mn)$-augmentations are necessary
- Idea for improved running time: reduce running time for each iteration
- Simple idea $\Rightarrow O(mn^2)$ [Dinic 1970]
- Dynamic Trees $\Rightarrow O(mn \log n)$ [Sleator-Tarjan 1983]

Outline

Capacity-Scaling Algorithm

- Idea: find the augment path from s to t with the largest bottleneck capacity
- Assumption: Capacities are integers between 1 and C

capacity-scaling(G, s, t, c)

let $f(e) \leftarrow 0$ for every e in G
$\Delta \leftarrow$ largest power of 2 which is at most C
while $\Delta \geq 1$ do **do**
    while there exists an augmenting path P with bottleneck capacity at least Δ **do**
        $f \leftarrow$augment(f, P)
    $\Delta \leftarrow \Delta/2$
return f

Obs. The outer while loop repeats $1 + \lfloor \log_2 C \rfloor$ times.

Lemma At the beginning of Δ-scale phase, the value of the max-flow is at most val$(f) + 2m\Delta$.

- Each augmentation increases the flow value by at least Δ
- Thus, there are at most $2m$ augmentations for Δ-scale phase.

Theorem The number of augmentations in the scaling max-flow algorithm is at most $O(m \log C)$. The running time of the algorithm is $O(m^2 \log C)$.

Polynomial Time

Assume all capacities are integers between 1 and C.

Ford-Fulkerson	$O(m^2C)$	pseudo-polynomial
Capacity-scaling:	$O(m^2 \log C)$	weakly-polynomial
Shortest-Path-Augmenting:	$O(m^2 n)$	strongly-polynomial

- Polynomial : weakly-polynomial and strongly-polynomial

Brief History

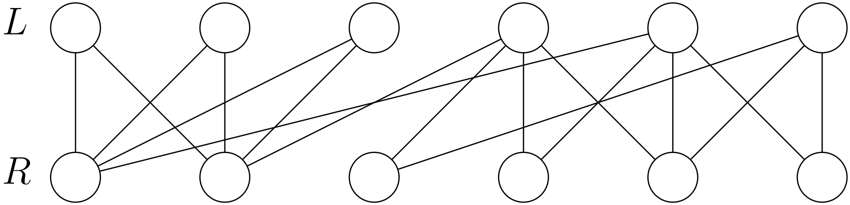
Algorithm	Year	Time	Description
Ford-Fulkerson	1956	$O(mf)$	Ford-Fulkerson Method.
Edmonds-Karp	1972	$O(nm^2)$	Shortest Augmenting Paths
Dinic	1970	$O(n^2m)$	SAP with blocking Flows
Goldberg-Tarjan	1988	$O(n^3)$	Generic Push-Relabel
Goldberg-Tarjan	1988	$O(n^2\sqrt{m})$	PR using highest-label nodes
Chen et al.	2022	$O(m^{1+o(1)})$	LP-solver, dynamic algorithms

- Chen et al. [Chen-Kyng-Liu-Peng-Gutenberg-Sachdeva, 2022].

Outline

Bipartite Graphs

Def. A graph $G = (V, E)$ is bipartite if the vertices V can be partitioned into two subsets L and R such that every edge in E is between a vertex in L and a vertex in R.

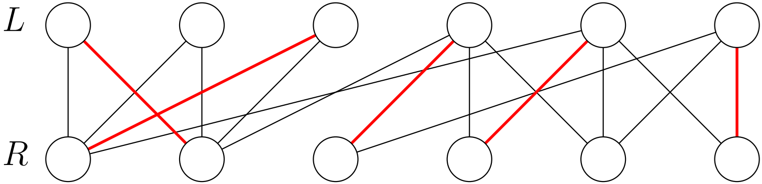


Def. Given a bipartite graph $G = (L \cup R, E)$, a matching in G is a set $M \subseteq E$ of edges such that every vertex in V is an endpoint of at most one edge in M.

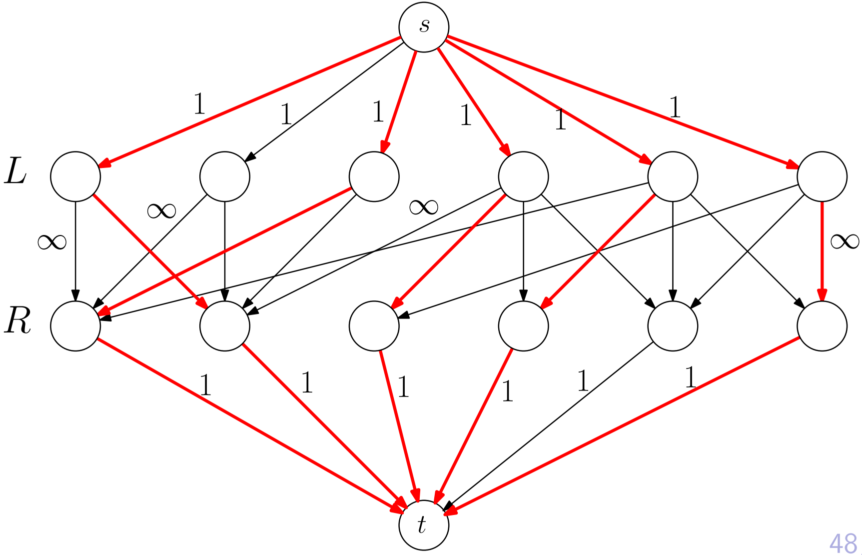
Maximum Bipartite Matching Problem

Input: bipartite graph $G = (L \cup R, E)$

Output: a matching M in G of the maximum size



Reduce Maximum Bipartite Matching to Maximum Flow Problem



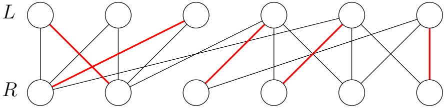
Reduce Maximum Bipartite Matching to Maximum Flow Problem

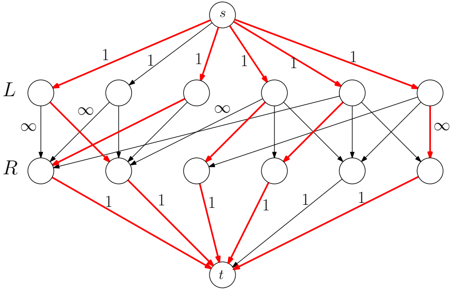
- Create a digraph $G' = (L \cup R \cup \{s, t\}, E')$ with capacity $c : E' \to \mathbb{R}_{\geq 0}$:
  - Add a source s and a sink t
  - Add an edge from s to each vertex $u \in L$ of capacity 1
  - Add an edge from each vertex $v \in R$ to t of capacity 1
  - Direct all edges in E from L to R, and assign ∞ capacity (or capacity 1) to them
- Compute the maximum flow from s to t in G'
- The maximum flow gives a matching
- Running time:
  - Ford-Fulkerson: $O(m \times \text{max flow value}) = O(mn)$.
  - Hopcroft-Karp: $O(mn^{1/2})$ time

Lemma Size of max matching = value of max flow in G'

Proof. $\leq$.

Given a maximum matching $M \subseteq E$, send a flow along each edge $e \in M$ and thus we have a flow of value $|M|$. □

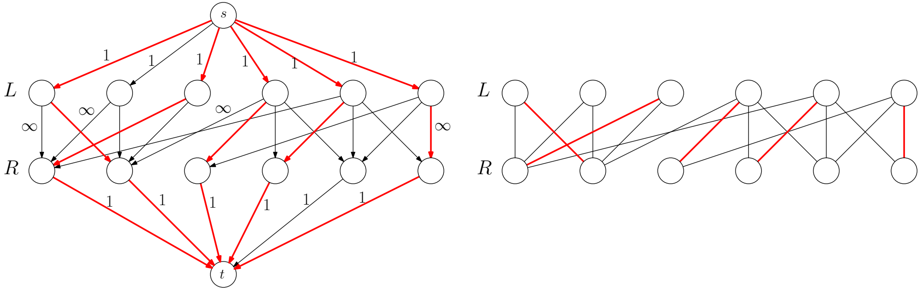




Lemma Size of max matching = value of max flow in G'

Proof. $\geq$.

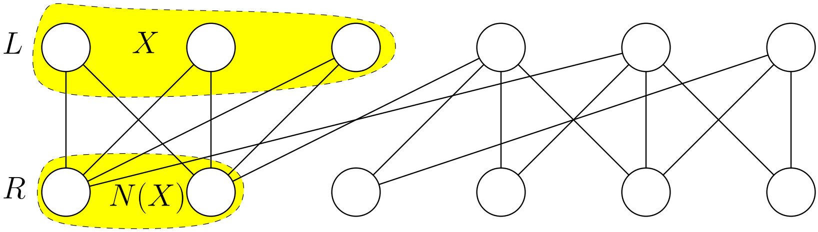
- The maximum flow f in G' is integral since all capacities are integral
- Let M to be the set of edges e from L to R with $f(e) = 1$
- M is a matching of size that equals to the flow value □



Perfect Matching

Def. Given a bipartite graph $G = (L \cup R, E)$ with $|L| = |R|$, a perfect matching M of G is a matching such that every vertex $v \in L \cup R$ participates in exactly one edge in M.

Assuming $|L| = |R| = n$, when does $G = (L \cup R, E)$ not have a perfect matching?



- For $X \subseteq L$, define $N(X) = \{v \in R : \exists u \in X, (u, v) \in E\}$
- $|N(X)| < X$ for some $X \subseteq L \implies$ no perfect matching

Hall's Theorem Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then G has a perfect matching if and only if $|N(X)| \geq |X|$ for every $X \subseteq L$.

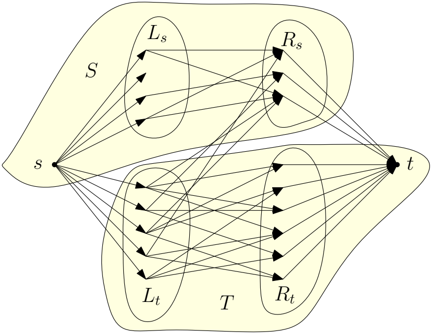
Proof. $\implies$.

If G has a perfect matching, then vertices matched to $X \subseteq N(X)$; thus $|N(X)| \geq |X|$. □

Hall's Theorem Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then G has a perfect matching if and only if $|N(X)| \geq |X|$ for every $X \subseteq L$.

Proof. $\Longleftarrow$.

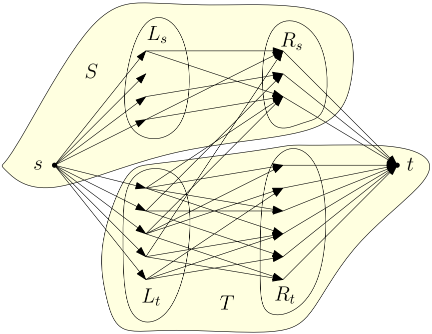
- Contrapositive: if no perfect matching, then $\exists X \subseteq L, |N(X)| < |X|$
- Consider the network flow instance
- There is a s-t cut (S, T) of value at most $n - 1$
- Define L_s, L_t, R_s, R_t as in figure □



Hall's Theorem Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then G has a perfect matching if and only if $|N(X)| \geq |X|$ for every $X \subseteq L$.

Proof. $\Longleftarrow$.

- Contrapositive: if no perfect matching, then $\exists X \subseteq L, |N(X)| < |X|$
- No edges from L_s to R_t, since their capacities are ∞
- $c(S, T) = |L_t| + |R_s| < n$
- $|N(L_s)| \leq |R_s| < n - |L_t| = |L_s|$. □

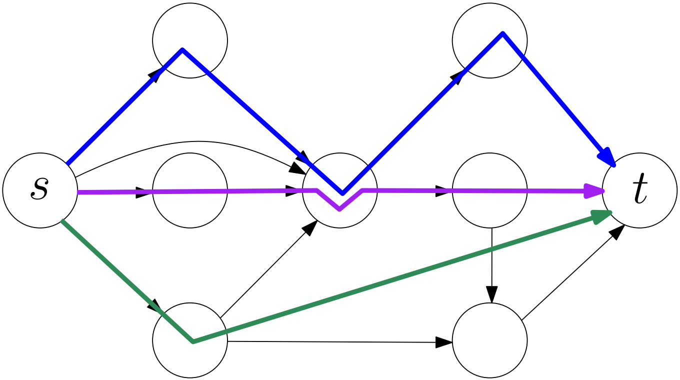


Outline

s-t Edge Disjoint Paths

Input: a directed (or undirected) graph $G = (V, E)$ and $s, t \in V$

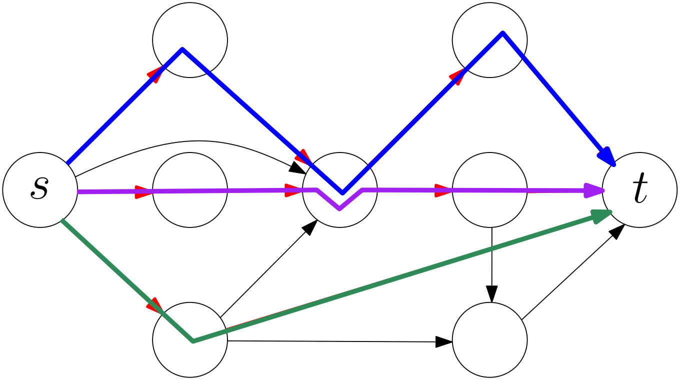
Output: the maximum number of edge-disjoint paths from s to t in G



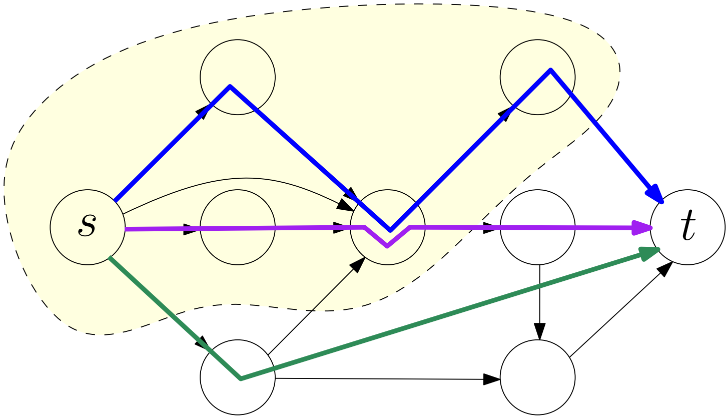
- Solving the maximum flow problem, where all capacities are 1
- All flow values are integral (i.e, either 0 or 1)

From flow to disjoint paths

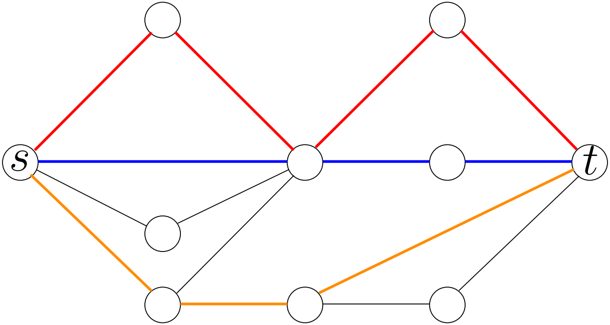
- find an arbitrary $s \to t$ path where all edges have flow value 1
- change the flow values of the path to 0 and repeat



Theorem The maximum number of edge disjoint paths from s to t equals the minimum value of an s-t cut (S, T).



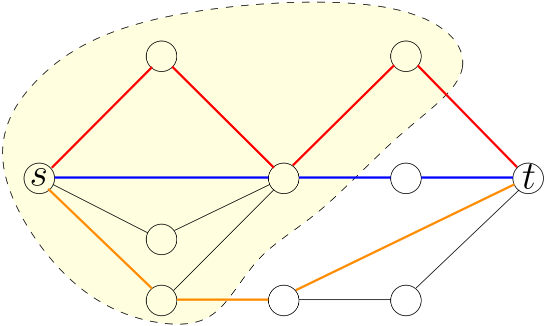
s-t Edge Disjoint Paths in Undirected Graphs



- an undirected edge $\rightarrow$ two anti-parallel directed edges.
- Solving the s-t maximum flow problem in the directed graph
- Convert the flow to paths
- Issue: both $e = (u, v)$ and $e' = (v, u)$ are used
- Fix: if this happens we change $f(e) = f(e') = 0$

Menger's Theorem

Menger's Theorem In an undirected graph, the maximum number of edge-disjoint paths between s to t is equal to the minimum number of edges whose removal disconnects s and t.

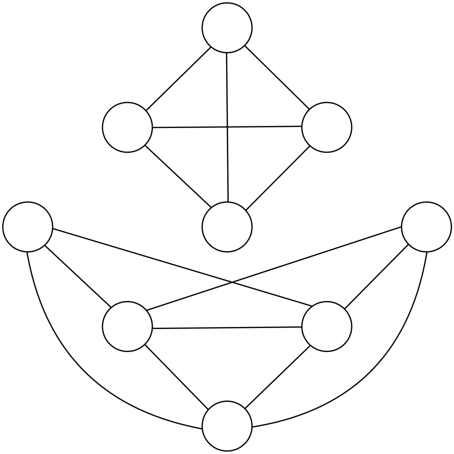


s-t connectivity measures how well s and t are connected.

Global Min-Cut Problem

Input: a connected graph $G = (V, E)$

Output: the minimum number of edges whose removal will disconnect G



Solving Global Min-Cut Using Maximum Flow

let G' be the directed graph obtained from G by replacing every edge with two anti-parallel edges
for every pair $s \neq t$ of vertices **do**
    obtain the minimum cut separating s and t in G, by solving the maximum flow instance with graph G',source s and sink t
output the smallest minimum cut we found

- Need to solve $\Theta(n^2)$ maximum flow instances
- Can we do better?
- Yes. We can fix s. We only need to enumerate t

Outline

Extension of Network Flow: Circulation Problem

Input: A digraph $G = (V, E)$
capacities $c \in \mathbb{Z}_{\geq 0}^E$
supply vector $d \in \mathbb{Z}^V$ with $\sum_{v \in V} d_v = 0$

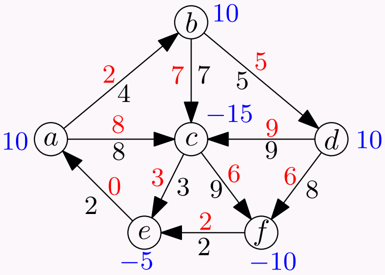
Output: whether there exists $f : E \to \mathbb{Z}_{\geq 0}$ s.t.

$$\sum_{e \in \delta^{\text{out}}(v)} f(e) - \sum_{e \in \delta^{\text{in}}(v)} f(e) = d_v \qquad \forall v \in V$$

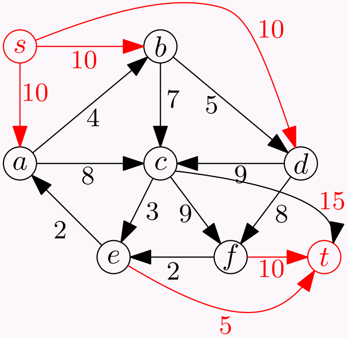
$$0 \leq f(e) \leq c_e \qquad \forall e \in E$$

- d_v denotes the net supply of a good
- $d_v > 0$: there is a supply of d_v at v
- $d_v < 0$: there is a demand of $-d_v$ at v
- problem: whether we can match the supplies and demands without violating capacity constraints

Example



Reduction



Reduction to maximum flow

- add a super-source s and a super-sink t to network
- for every $v \in V$ with $d_v > 0$: add edge (s, v) of capacity d_v
- for every $v \in V$ with $d_v < 0$: add edge (v, t) of capacity $-d_v$
- check if maximum flow has value $\sum_{v: d_v > 0} d_v$

- $d(S) := \sum_{v \in S} d_v, \forall S \subseteq V$.
- $c(S, V \setminus S) := \sum_{(u,v) \in E: u \in S, v \notin S} c_{(u,v)}$.

Lemma The instance is feasible if and only if for every $S \subseteq V$, $d(S) \leq c(S, V \setminus S)$.

Proof of "only if" direction.

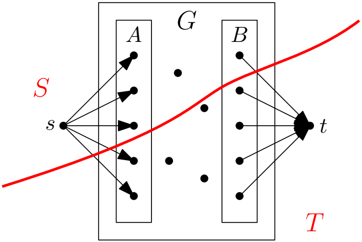
- if for some $S \subseteq V$, $c(S, V \setminus S) < d(S)$, then the demand in S can not be sent out of S. □

- It remains to consider the "if" direction

Proof of "if" Direction

Lemma The instance is feasible if and only if for every $S \subseteq V$, $d(S) \leq c(S, V \setminus S)$.

- assume instance is infeasible: max-flow $< d(A)$
- $A := \{v \in V : d_v > 0\}$
- $B := \{v \in V : d_v < 0\}$
- $(S \ni s, T \ni t)$: min-cut



$$d(T \cap A) + |d(S \cap B)| + c(S \setminus \{s\}, T \setminus \{t\}) < d(A)$$
$$d(T \cap A) - d(S \cap B) + c(S \setminus \{s\}, T \setminus \{t\}) < d(A)$$
$$c(S \setminus \{s\}, T \setminus \{t\}) < d(S \cap A) + d(S \cap B) = d(S \setminus \{s\})$$

- Define $S' = S \setminus \{s\}$: $d(S') > c(S', V \setminus S')$.

Circulation Problem with Capacity Lower Bounds

Input: A digraph $G = (V, E)$

capacities $c \in \mathbb{Z}_{\geq 0}^E$

capacity lower bounds $l \in \mathbb{Z}_{\geq 0}^E$, $0 \leq l_e \leq c_e$

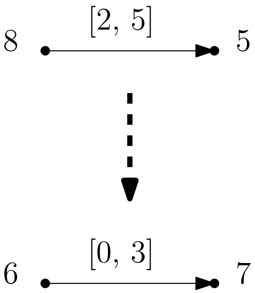
supply vector $d \in \mathbb{Z}^V$ with $\sum_{v \in V} d_v = 0$

Output: whether there exists $f : E \to \mathbb{Z}_{\geq 0}$ s.t.

$$\sum_{e \in \delta^{\text{out}}(v)} f(e) - \sum_{e \in \delta^{\text{in}}(v)} f(e) = d_v \qquad \forall v \in V$$

$$l_e \leq f(e) \leq c_e \qquad \forall e \in E$$

Removing Capacity Lower Bounds



handling $e = (u, v)$ with $l_e > 0$

- $d'_u \leftarrow d_u - l_e$
- $d'_v \leftarrow d_v + l_e$
- $c'_e \leftarrow c_e - l_e$
- $l'_e \leftarrow 0$

- in old instance: flow is $f(e) \in [l_e, c_e] \implies f(e) - l_e \in [0, c_e - l_e]$
- in new instance: flow is $f(e) - l_e \in [0, c'_e]$

Survey Design

Input: integers $n, k \geq 1$ and $E \subseteq [n] \times [k]$

integers $0 \leq c_i \leq c'_i, \forall i \in [n]$

integers $0 \leq p_j \leq p'_j, \forall j \in [k]$

Output: $E' \subseteq E$ s.t.

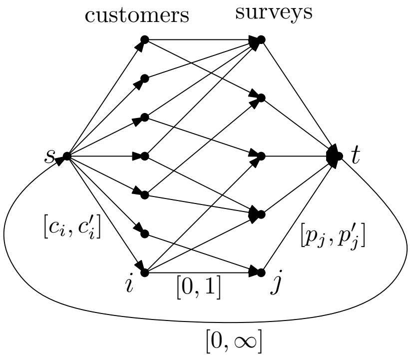
$$c_i \leq |\{j \in [k] : (i,j) \in E'\}| \leq c'_i, \qquad \forall i \in [n]$$
$$p_j \leq |\{i \in [m] : (i,j) \in E'\}| \leq p'_j, \qquad \forall j \in [k]$$

Background

- $[n]$: customers, $[k]$:products
- $ij \in E$: customer i purchased product j and can do a survey
- every customer i needs to do between c_i and c'_i surveys
- every product j needs to collect between p_j and p'_j surveys

Reduction to Circulation

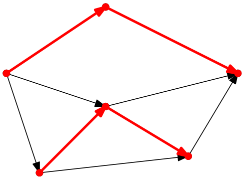
- vertices $\{s, t\} \uplus [n] \uplus [k]$,
- $(i, j) \in E$: (i, j) with bounds $[0, 1]$
- $\forall i$: (s, i) with bounds $[c_i, c_i']$
- $\forall j$: (j, t) with bounds $[p_j, p_i']$
- (t, s) with bounds $[0, \infty]$



Airline Scheduling

Input: a DAG $G = (V, E)$

Output: the minimum number of disjoint paths in G to cover all vertices



Background

- vertex : a flight
- edge (u, v): an aircraft that serves u can serve v immediately
- goal: minimize the number of aircrafts

Reduction to the Circulation Problem

- split v into $(v_{\text{in}}, v_{\text{out}})$
- add s, and $(s, v_{\text{in}}), \forall v$
- add t, and $(v_{\text{out}}, t), \forall v$
- set lower and upper bounds
- add $t \to s$ of capacity k
- find minimum k s.t. instance is feasible

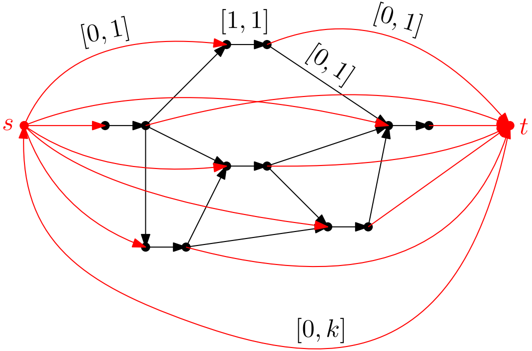


Image Segmentation

Input: A graph $G = (V, E)$, with edge costs $c \in \mathbb{Z}_{\geq 0}^E$
two reward vectors $a, b \in \mathbb{Z}_{\geq 0}^V$

Output: a cut (A, B) of G so as to maximize

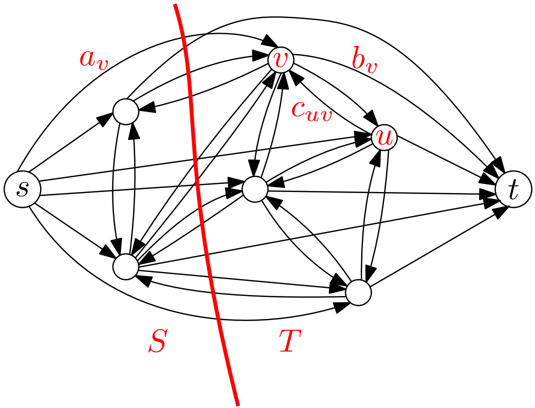
$$\sum_{v \in A} a_v + \sum_{v \in B} b_v - \sum_{(u,v) \in E: |\{u,v\} \cap A| = 1} c_{(u,v)}$$

Background

- a_v: the likelihood of v being a foreground pixel
- b_v: the likelihood of v being a background pixel
- $c_{(u,v)}$: the penalty for separating u and v
- need to maximize total reward - total penalty

Reduction to Network Flow

- replace (u, v) with two anti-parallel arcs
- add source s and arcs $(s, v), \forall v$
- add sink t and arcs $(v, t), \forall v$
- set capacities



- The cut value of $(S = \{s\} \cup A, \{t\} \cup B)$ is

$$\sum_{v \in B} a_v + \sum_{v \in A} b_v + \sum_{(u,v) \in E : |\{u,v\} \cap A| = 1} c_{(u,v)}$$

$$= \sum_{v \in V} (a_v + b_v) - \left(\sum_{v \in A} a_v + \sum_{v \in B} b_v - \sum_{(u,v) \in E : |\{u,v\} \cap A| = 1} c_{(u,v)} \right)$$

- The cut value of $(S = \{s\} \cup A, \{t\} \cup B)$ is

$$\sum_{v \in V}(a_v + b_v) - \Big(\sum_{v \in A} a_v + \sum_{v \in B} b_v - \sum_{(u,v) \in E : |\{u,v\} \cap A| = 1} c_{(u,v)} \Big)$$

$$= \sum_{v \in V}(a_v + b_v) - \big(\text{objective of } (A, B)\big)$$

- So, maximizing the objective of (A, B) is equivalent to minimizing the cut value.

Project Selection

Input: A DAG $G = (V, E)$

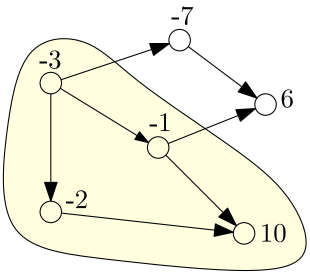
revenue on vertices: $p \in \mathbb{Z}^V$; p_v's could be negative.

Output: A set $B \subseteq V$ satisfying the precedence constraints:

$$v \in B \implies u \in B, \quad \forall (u, v) \in E$$

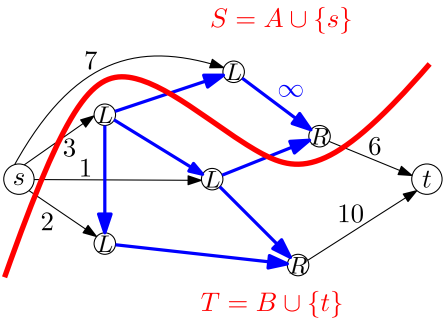
Motivation

- Motivation: $(u, v) \in E$: u is a prerequisite of v, to select v, we must select u
- Goal: maximize the revenue subject to the precedence constraint.



Reduction

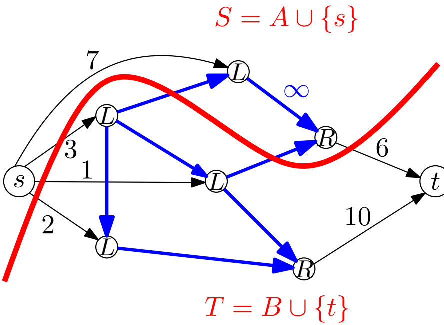
- add source s and sink t
- $p_v < 0$: (s, v) of capacity $-p_v$
- $p_v > 0$: (v, t) of capacity p_v
- $L = \{v : p_v < 0\}$
- $R = \{v : p_v > 0\}$.
- precedence edges: ∞ capacity



- min-cut $(S = \{s\} \cup A, T = \{t\} \cup B)$
- no ∞-capacity edges from A to B
- cut value is

$$\sum_{v \in B \cap L} (-p_v) + \sum_{v \in A \cap R} p_v = -\sum_{v \in B \cap L} p_v - \sum_{v \in B \cap R} p_v + \sum_{v \in R} p_v$$

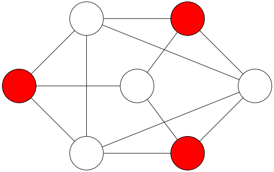
$$= \sum_{v \in R} p_v - \sum_{v \in B} p_v$$



- B is a valid solution $\iff c(S,T) \neq \infty$
- when B is valid, $c(S,T) = \sum_{v\in R} p_v - \sum_{v \in B} p_v$
- so, to maximize $\sum_{v\in B} p_v$, we need to minimize $c(S,T)$.

Maximum Independent Set Problem

Def. An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in I are adjacent in G.



Maximum Independent Set Problem

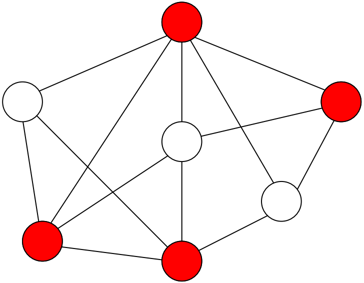
Input: graph $G = (V, E)$

Output: the size of the maximum independent set of G

- Maximum Independent Set is NP-hard

Vertex-Cover

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$.



Vertex-Cover Problem

Input: $G = (V, E)$ and integer k

Output: whether there is a vertex cover of G of size at most k

Q: What is the relationship between Vertex-Cover and Ind-Set?

A: S is a vertex-cover of $G = (V, E)$ if and only if $V \setminus S$ is an independent set of G.

- So, MinVC $= n-$ MaxIS

- MinVC: size of minimum vertex cover
- MaxIS: size of maximum independent set

Lemma In a bipartite graph $G = (L \cup R, E)$, we have

- MaxM $=$ MinVC $= n-$ MaxIS

- MaxM: size of maximum matching

- First, MaxM $\leq$ MinVC, for any graph.
- In bipartite graphs, there is a VC of size MaxM:

- (S, T): s-t cut
- $(u \in L, v \in R) \in M$
  - $v \in S \equiv u \in S$
  - $v \in T \equiv u \in T$
- unmatched vertices are in $(L \cap S) \cup (R \cap T)$
- No edges in E between $L \cap S$ and $R \cap T$
- $(L \cap T) \cup (R \cap S)$ is a VC, whose size is $|M|$