

算法设计与分析(2026年春季学期)

Network Flow

授课老师: 栗师

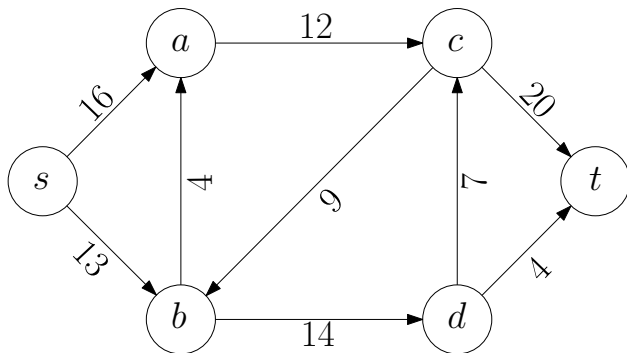
南京大学计算机学院

Outline

- 1 Network Flow
- 2 Ford-Fulkerson Method
- 3 Correctness of Ford-Fulkerson's Method and Maximum Flow Minimum Cut Theorem
- 4 Running Time of Ford-Fulkerson-Type Algorithm
 - Shortest Augmenting Path Algorithm
 - Capacity-Scaling Algorithm
- 5 Bipartite Matching Problem
- 6 s - t Edge-Disjoint Paths Problem
- 7 More Applications

Flow Network

- Abstraction of fluid flowing through edges
- Digraph $G = (V, E)$ with **source** $s \in V$ and **sink** $t \in V$
 - No edges enter s
 - No edges leave t
- Edge **capacity** $c_e \in \mathbb{R}_{>0}$ for every $e \in E$



Def. An *s-t flow* is a function $f : E \rightarrow \mathbb{R}$ such that

- for every $e \in E$: $0 \leq f(e) \leq c_e$ (capacity conditions)
- for every $v \in V \setminus \{s, t\}$:

$$\sum_{e \in \delta_{\text{in}}(v)} f(e) = \sum_{e \in \delta_{\text{out}}(v)} f(e). \quad (\text{conservation conditions})$$

The *value* of a flow f is

$$\text{val}(f) := \sum_{e \in \delta_{\text{out}}(s)} f(e).$$

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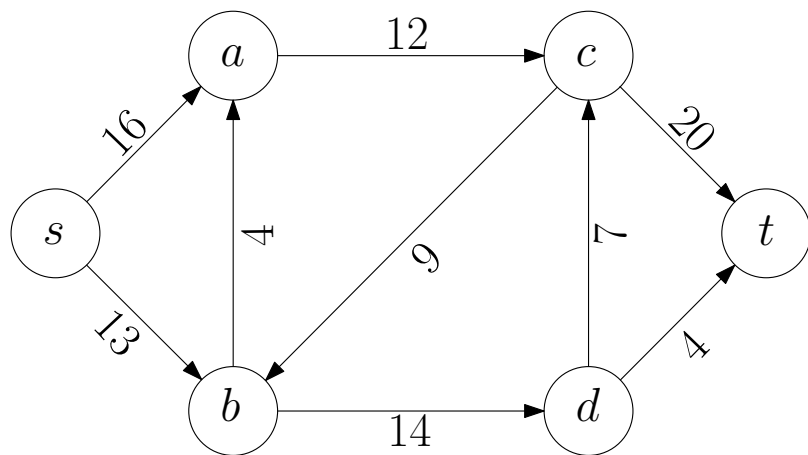
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Maximum Flow Problem

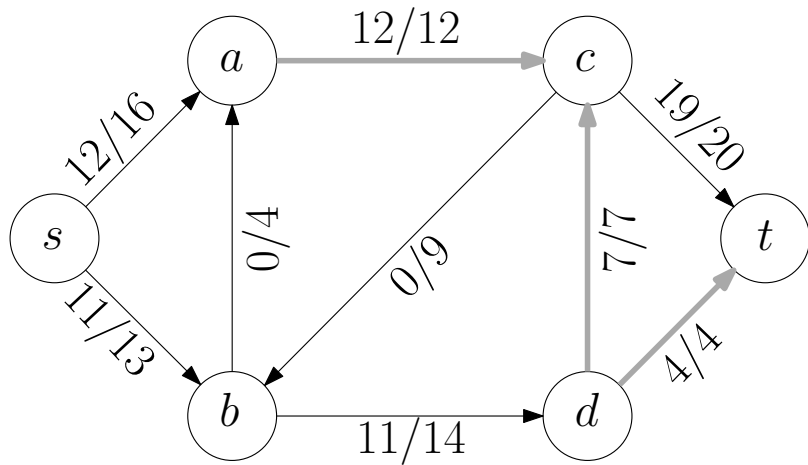
Input: directed network $G = (V, E)$, capacity function $c : E \rightarrow \mathbb{R}_{>0}$, source $s \in V$ and sink $t \in V$

Output: an *s-t* flow f in G with the maximum $\text{val}(f)$

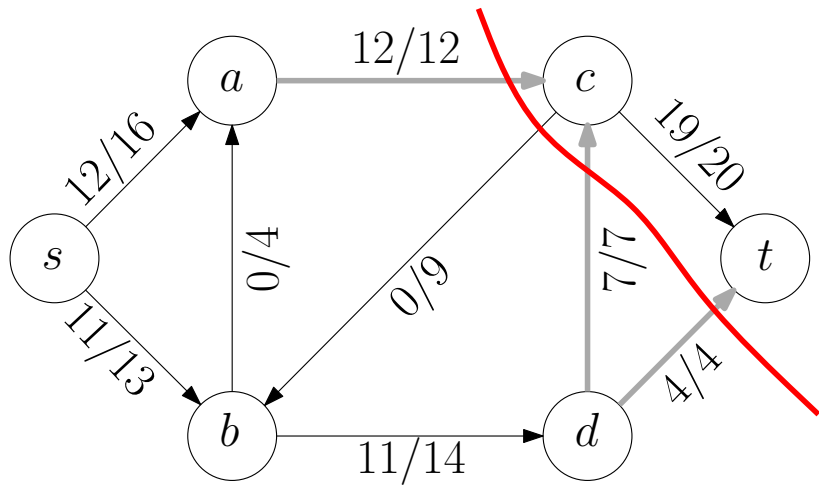
Maximum Flow Problem: Example



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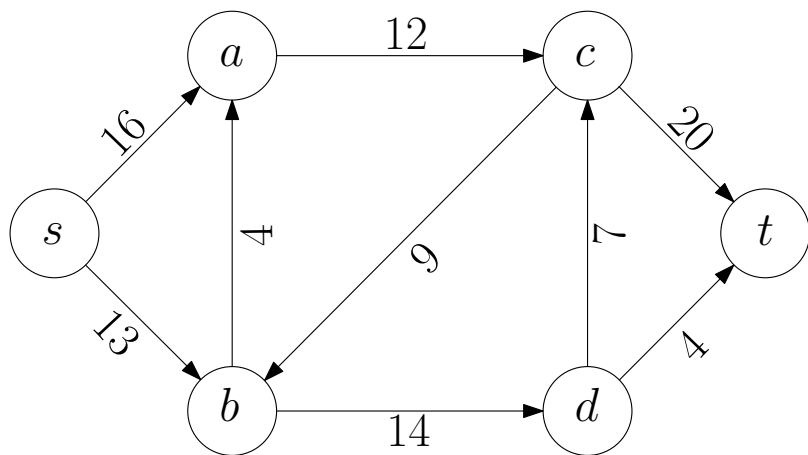
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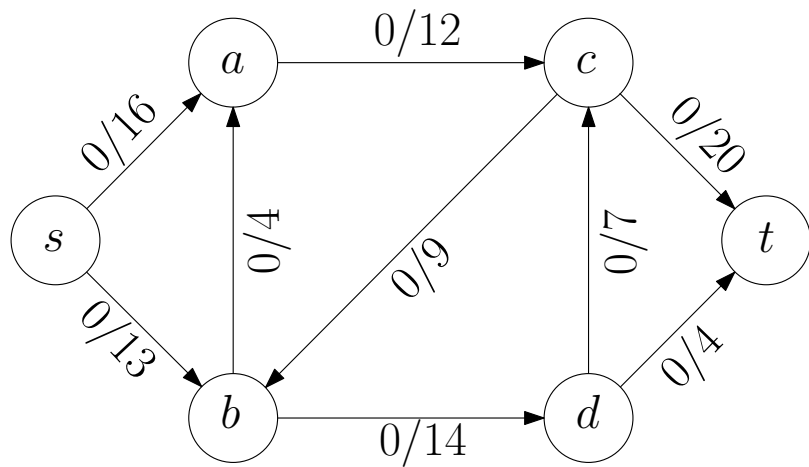
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- Repeat until we got stuck

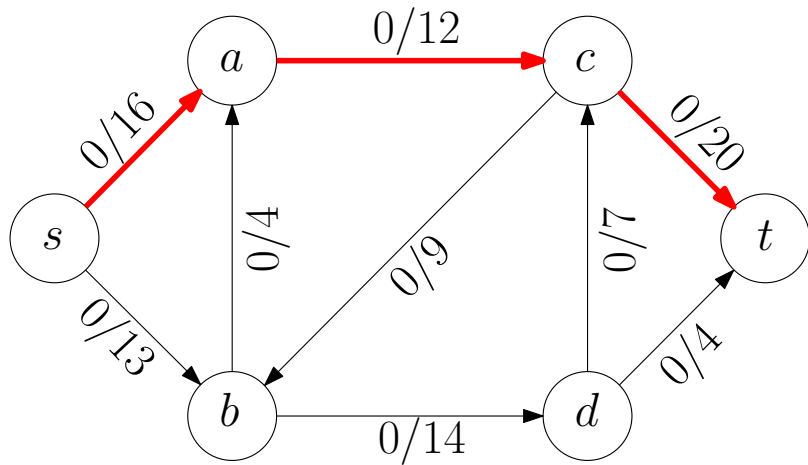
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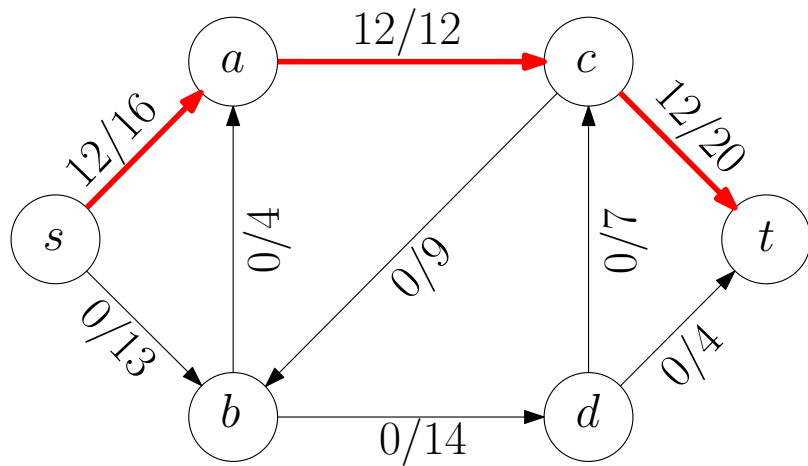
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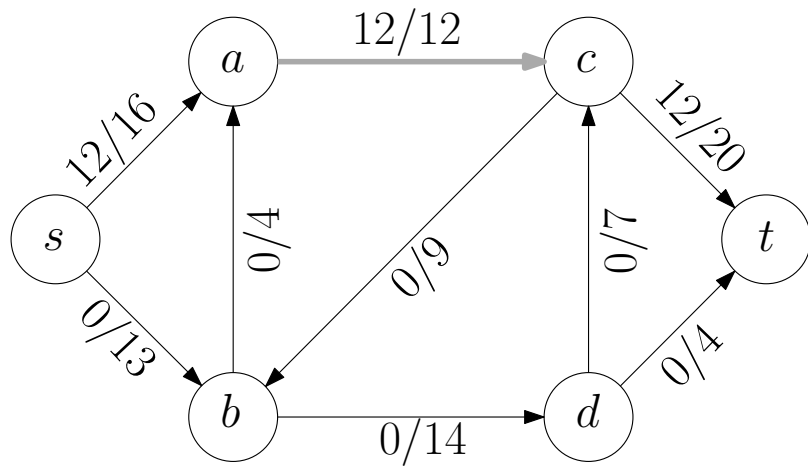
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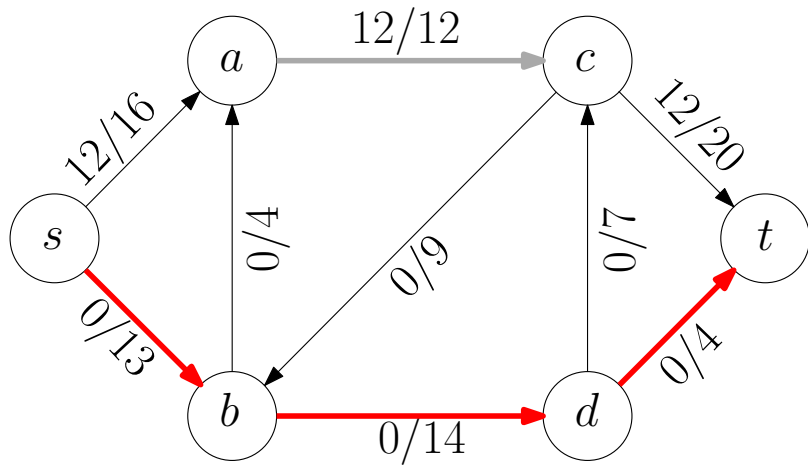
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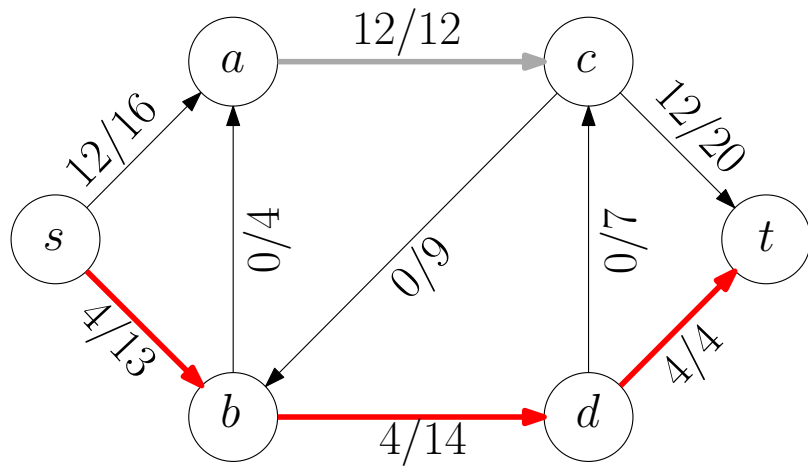
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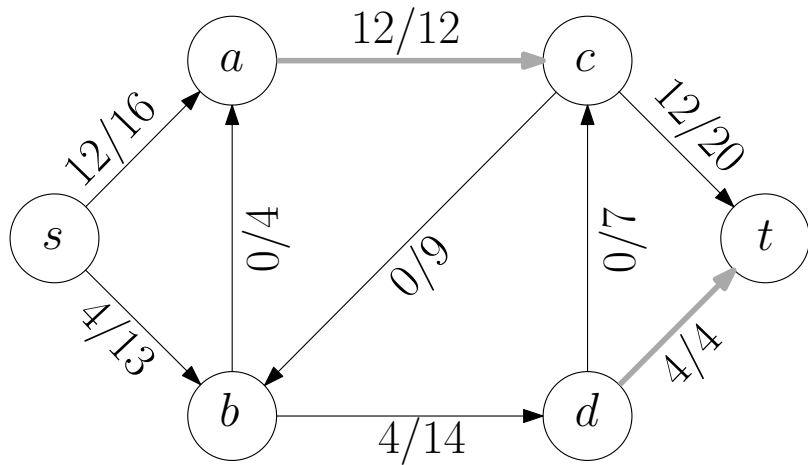
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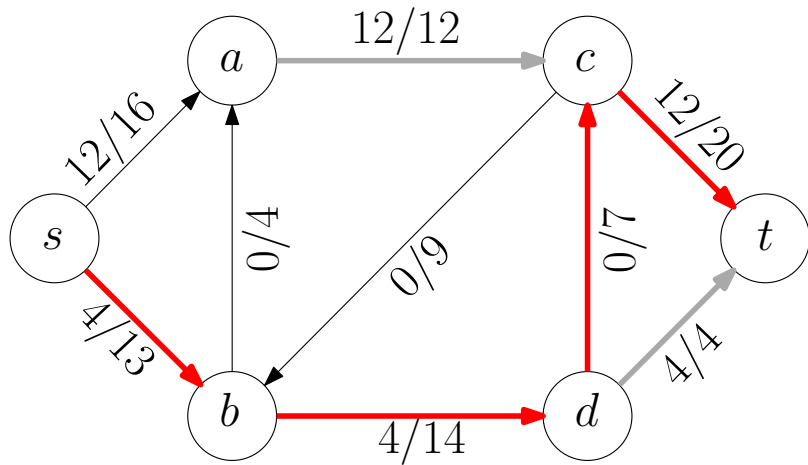
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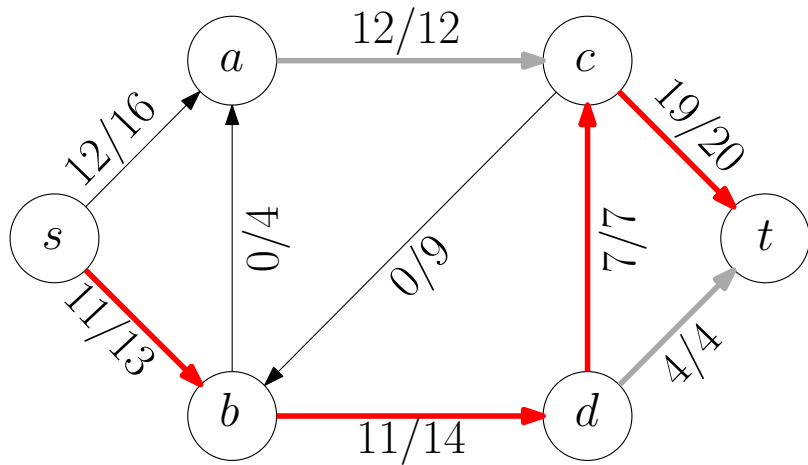
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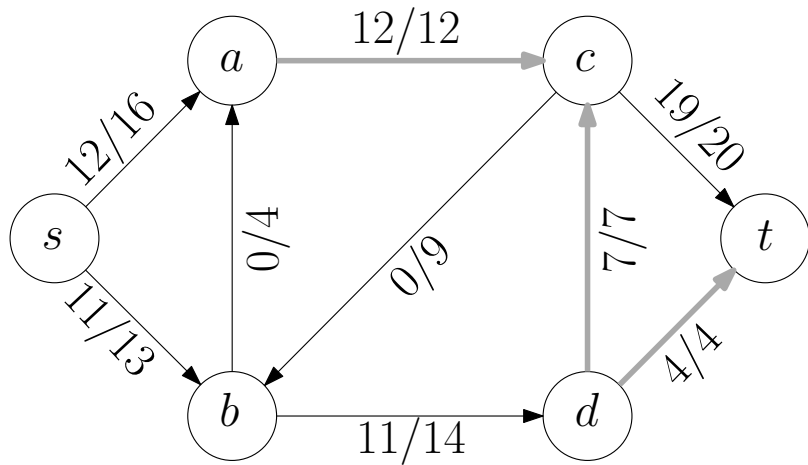
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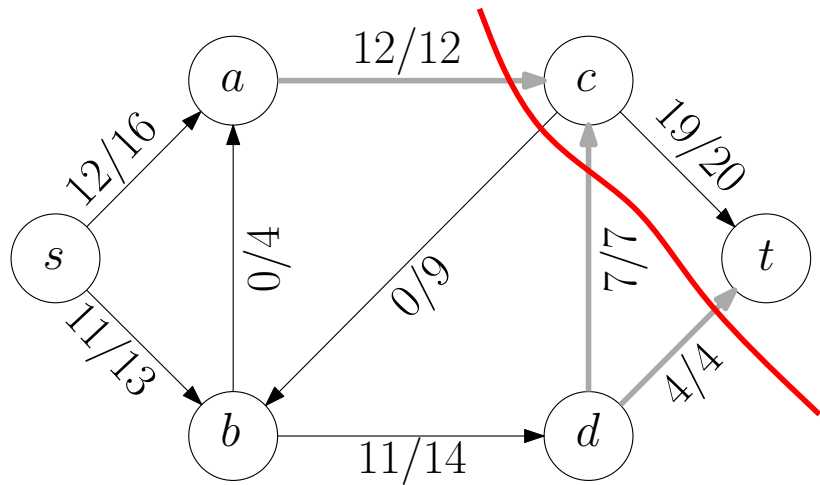
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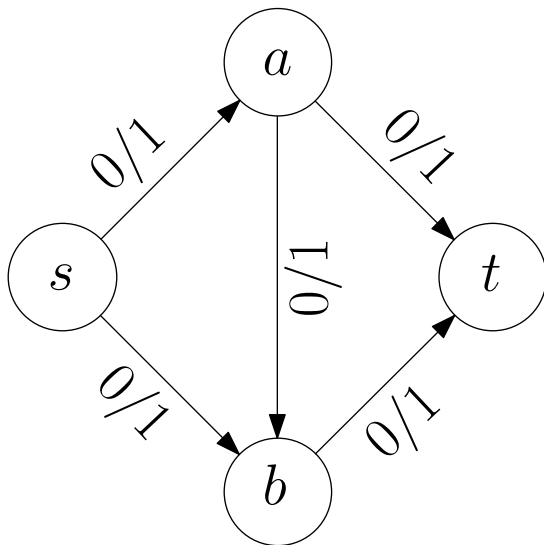
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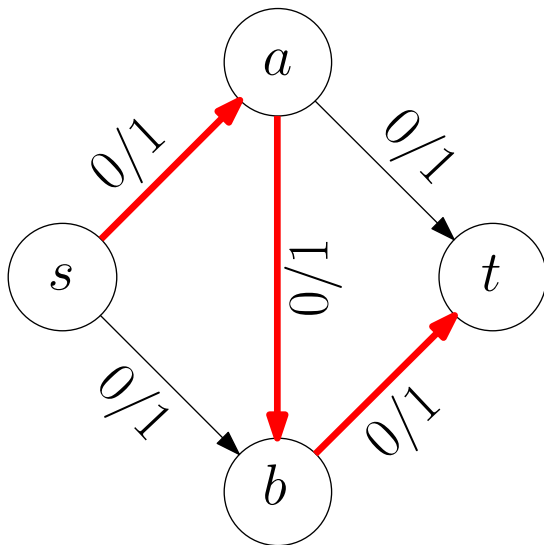
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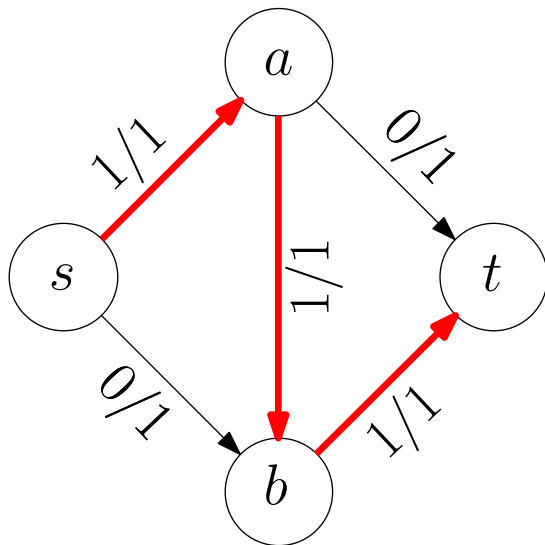
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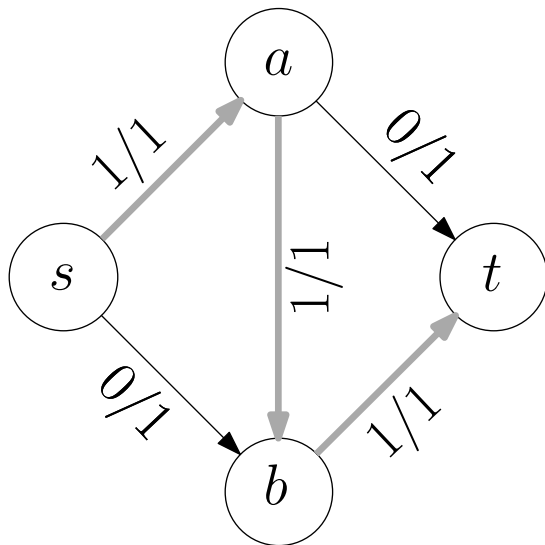
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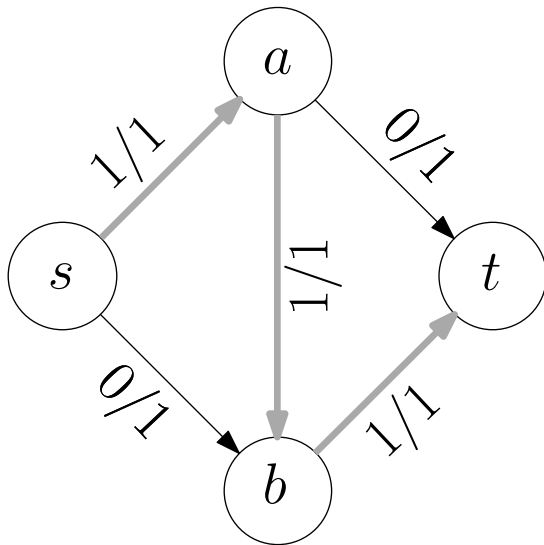
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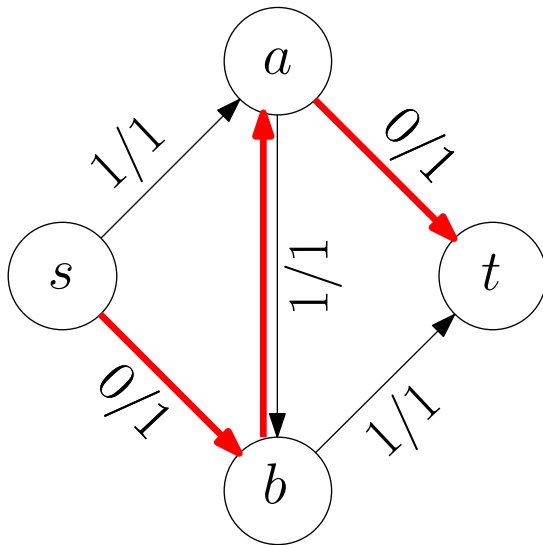
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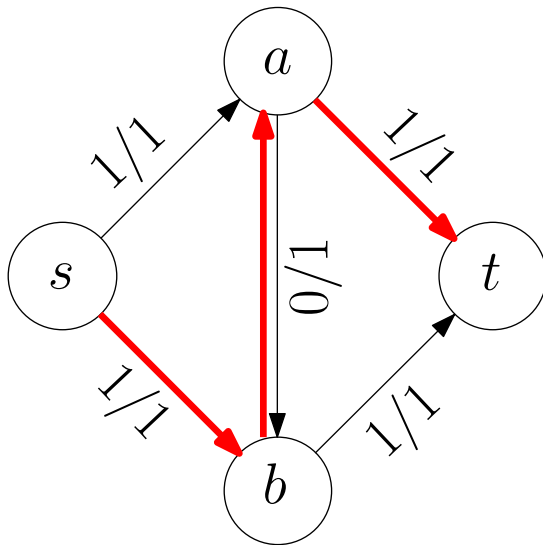
Fix the Issue: Allowing “Undo” Flow Sent



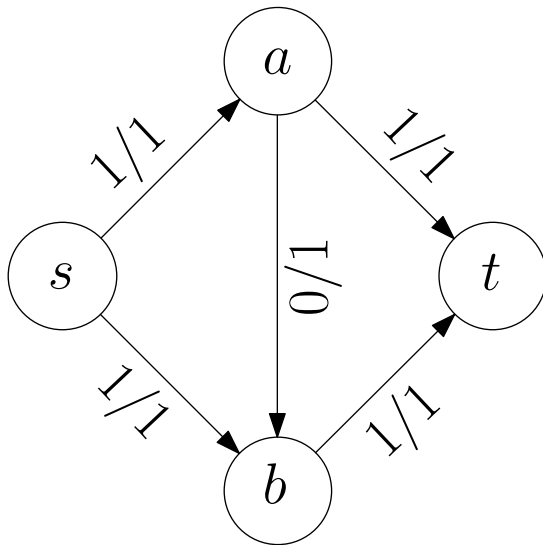
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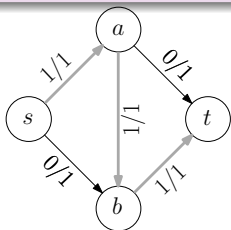
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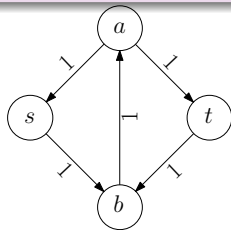
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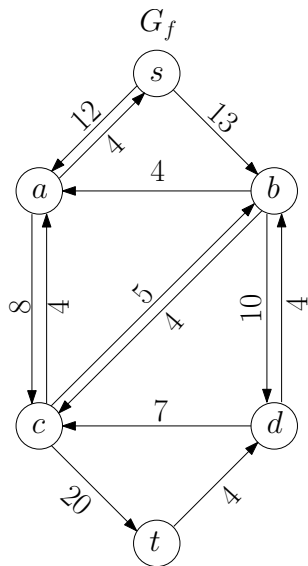
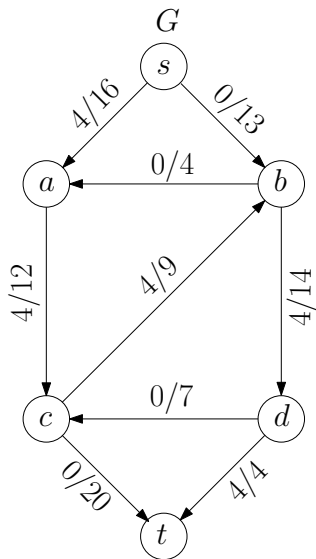


Original graph G and f



Residual Graph G_f

Residual Graph: One More Example



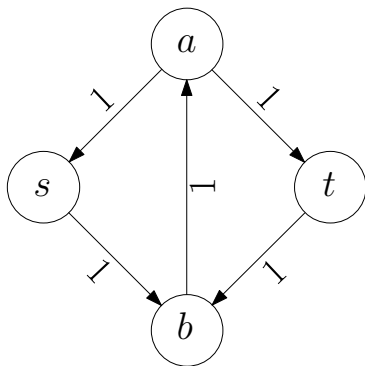
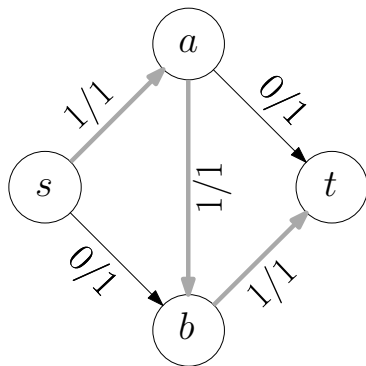
Augmenting Path

Augmenting the flow along a path P from s to t in G_f

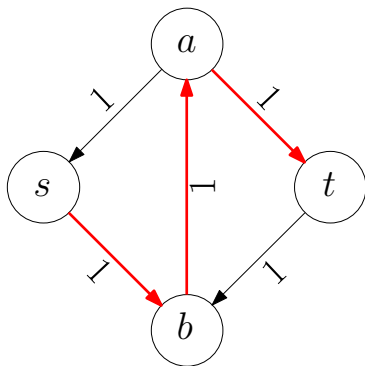
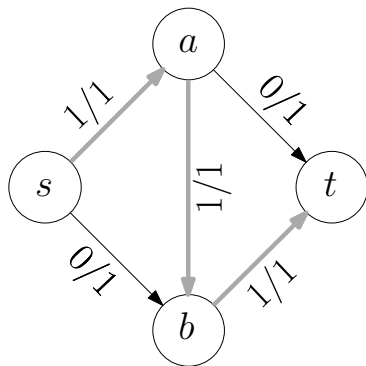
Augment(P)

```
1:  $b \leftarrow \min_{e \in P} c_f(e)$ 
2: for every  $(u, v) \in P$  do
3:   if  $(u, v)$  is a forward edge then
4:      $f(u, v) \leftarrow f(u, v) + b$ 
5:   else  $\triangleright (u, v)$  is a backward edge
6:      $f(v, u) \leftarrow f(v, u) - b$ 
7: return  $f$ 
```

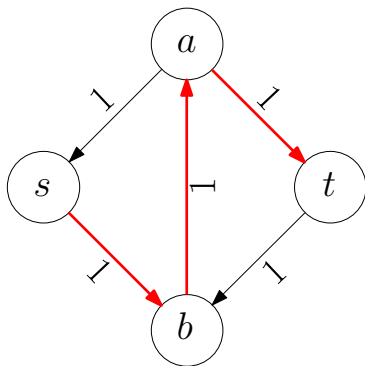
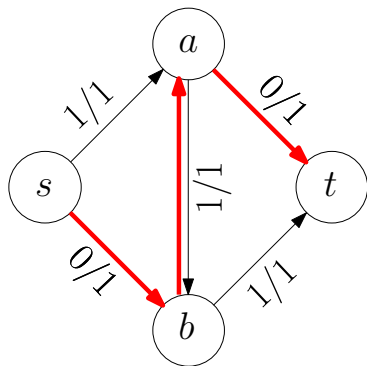
Example for Augmenting Along a Path



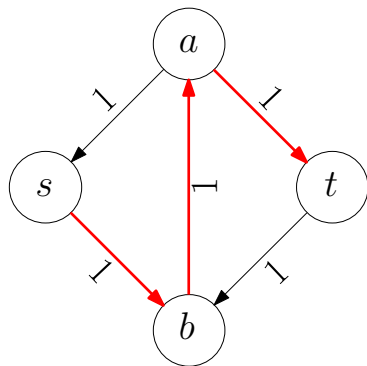
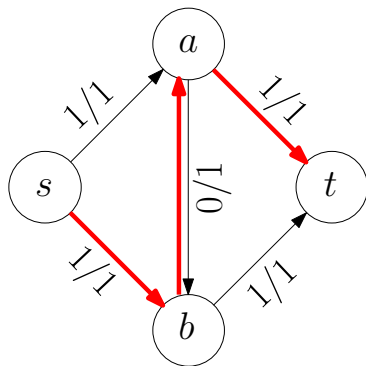
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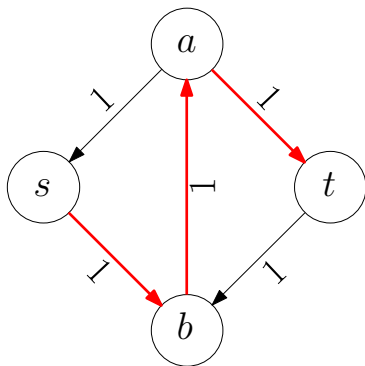
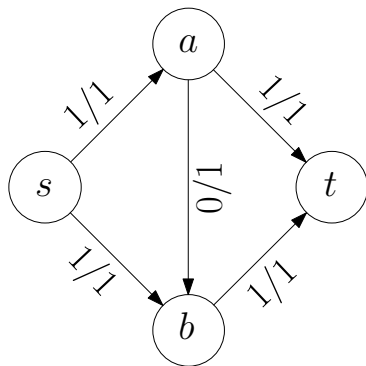
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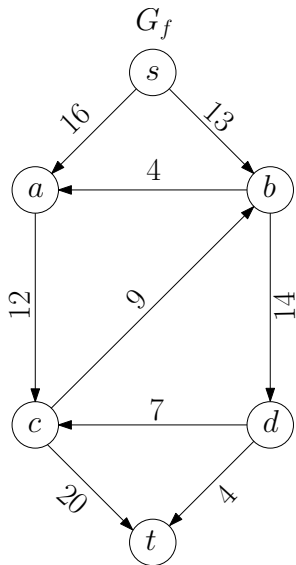
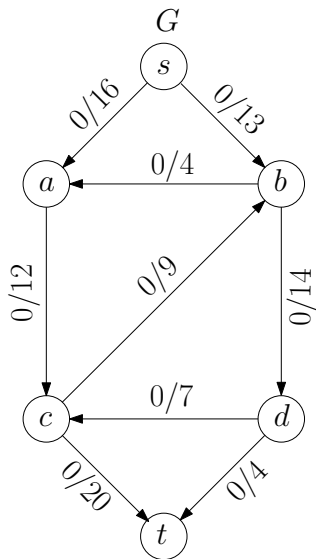


Ford-Fulkerson's Method

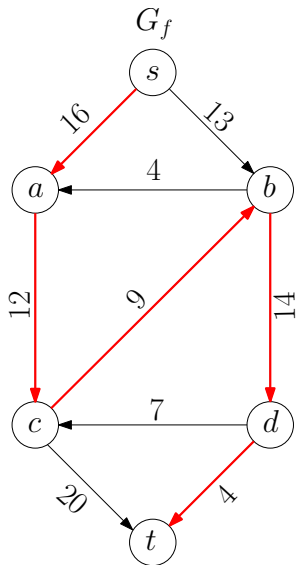
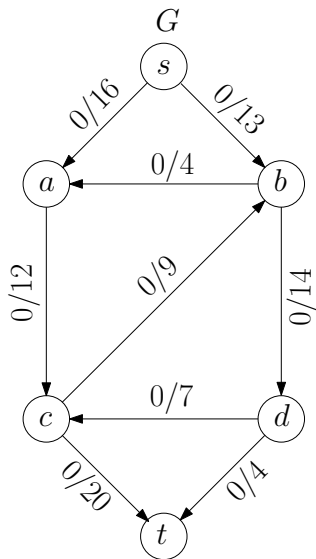
Ford-Fulkerson(G, s, t, c)

- 1: let $f(e) \leftarrow 0$ for every e in G
- 2: **while** there is a path from s to t in G_f **do**
- 3: let P be **any** simple path from s to t in G_f
- 4: $f \leftarrow \text{augment}(f, P)$
- 5: **return** f

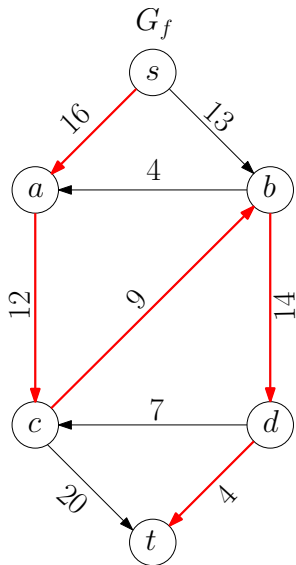
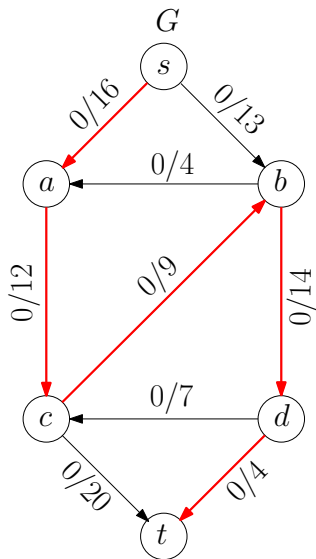
Ford-Fulkerson: Example



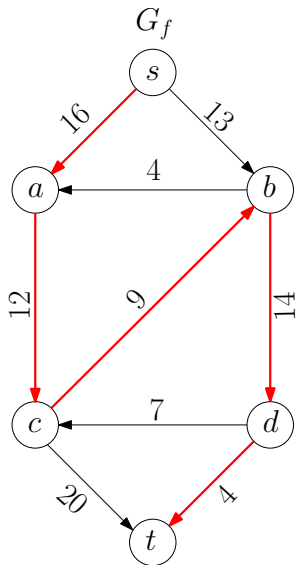
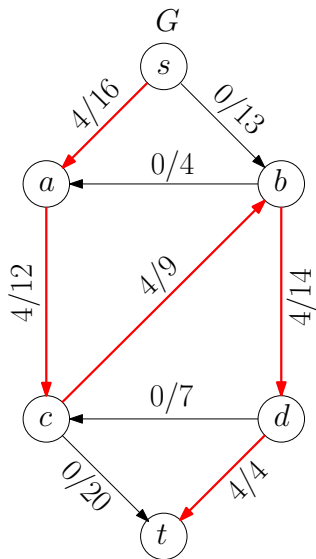
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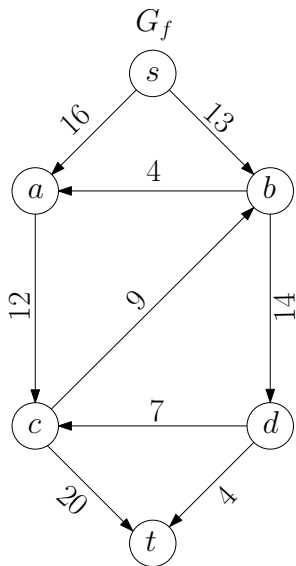
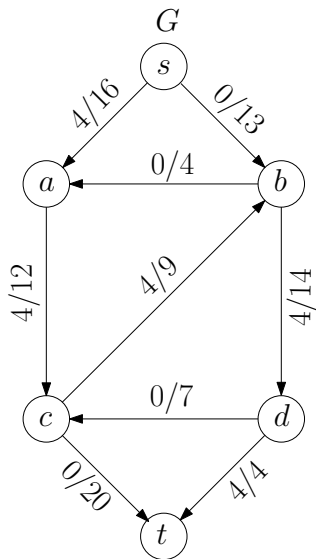
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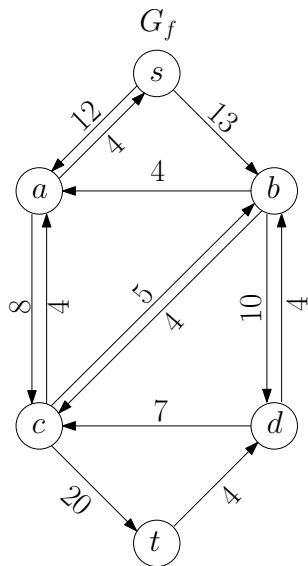
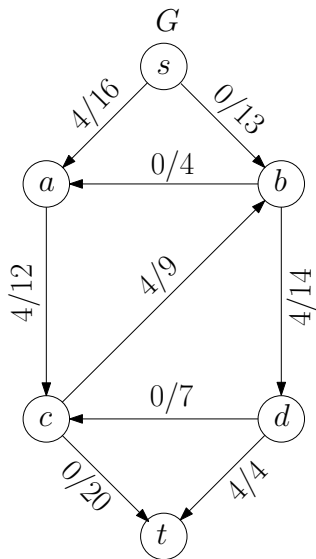
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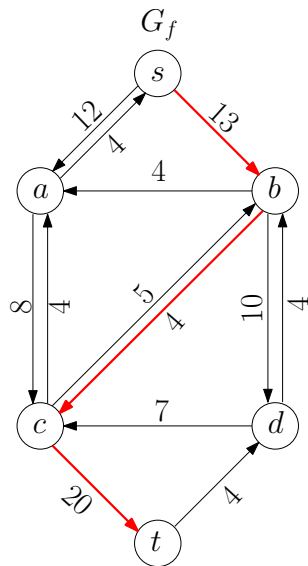
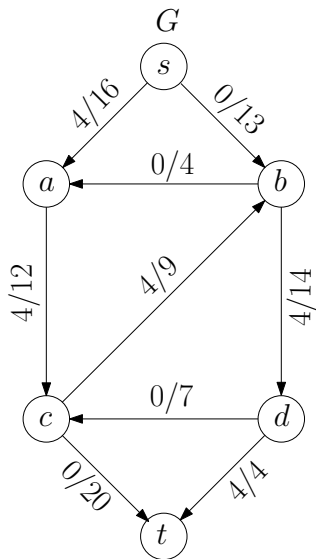
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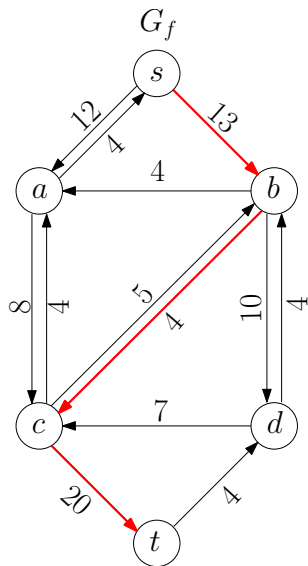
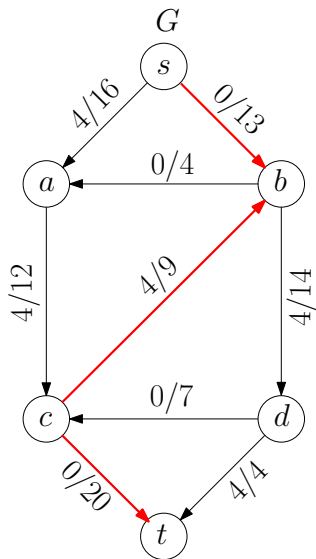
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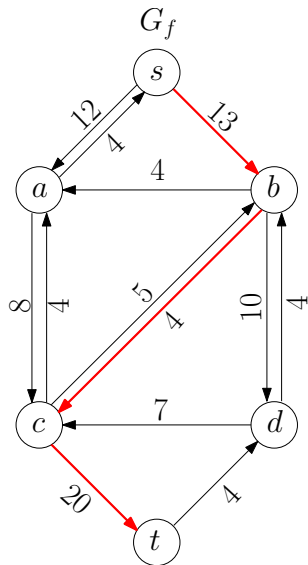
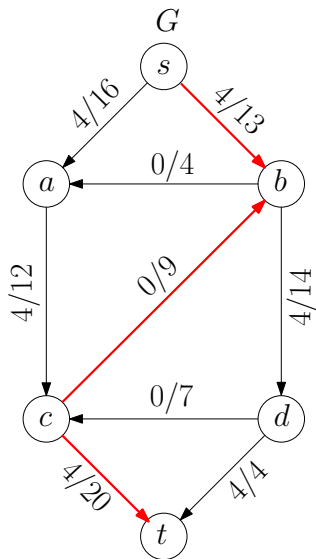
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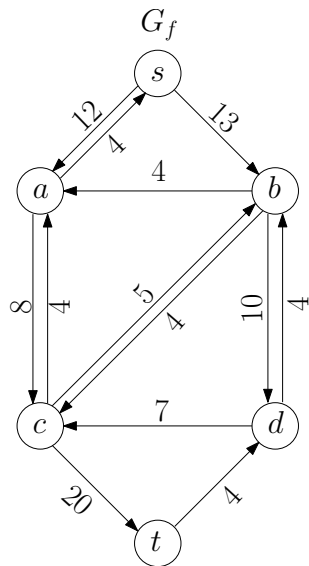
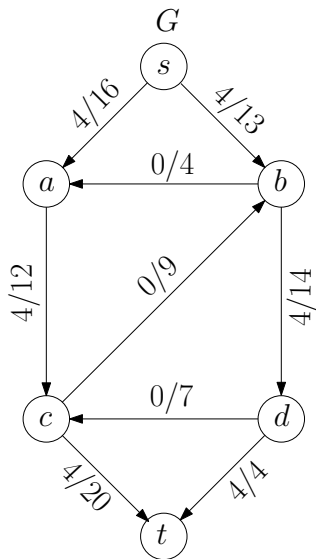
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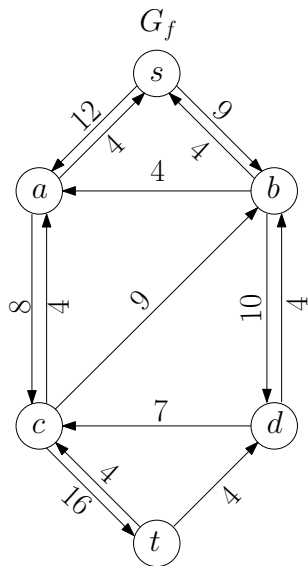
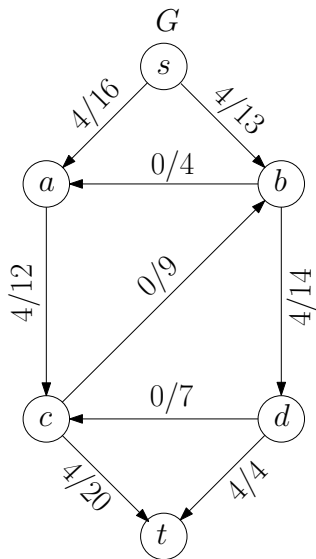
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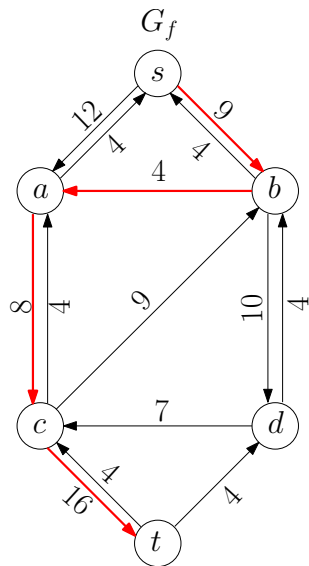
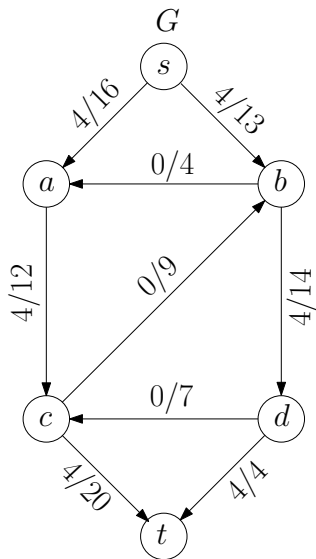
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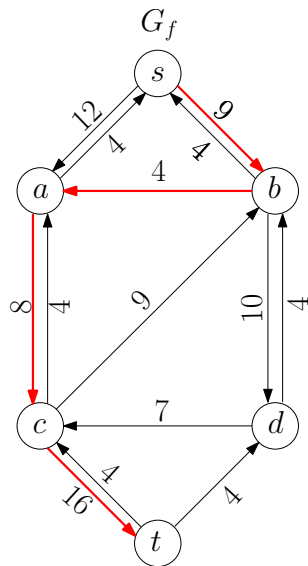
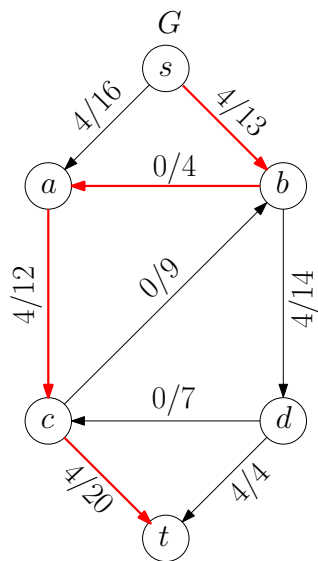
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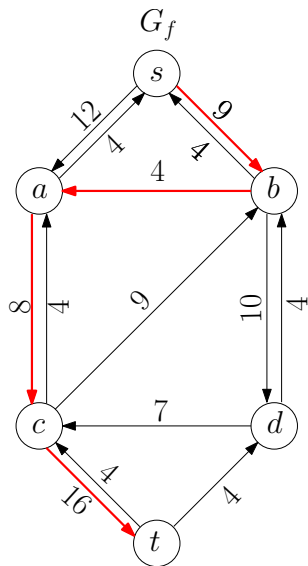
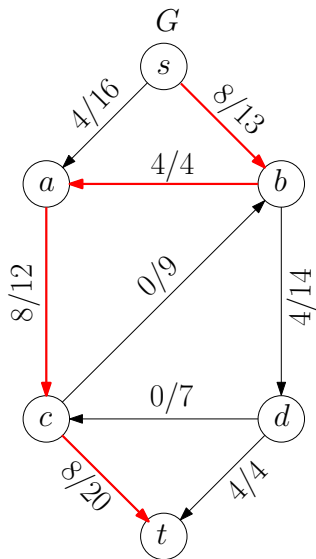
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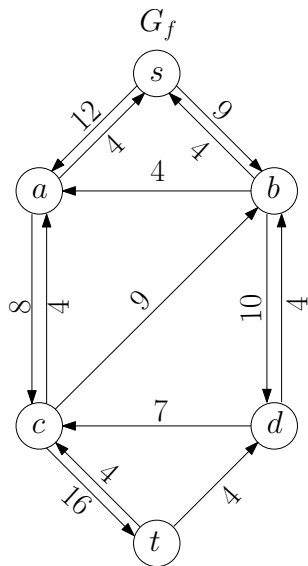
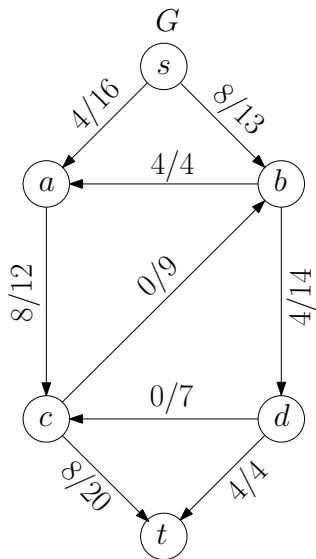
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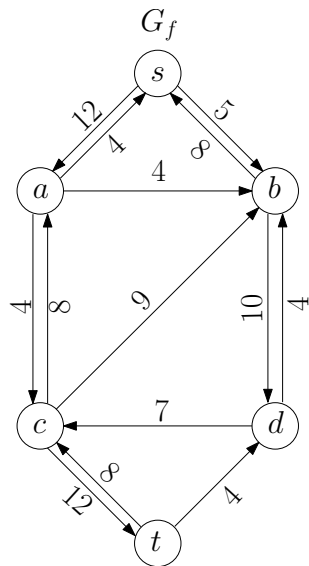
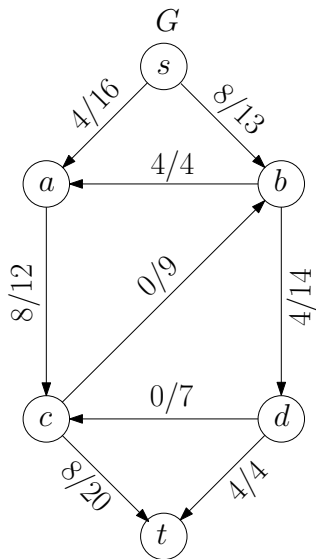
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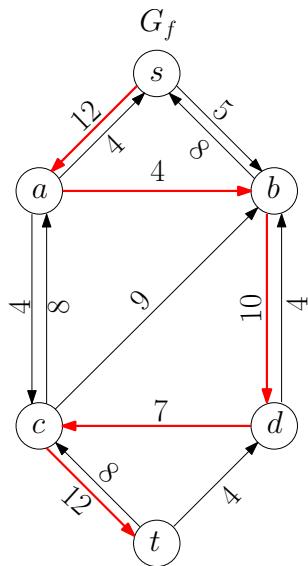
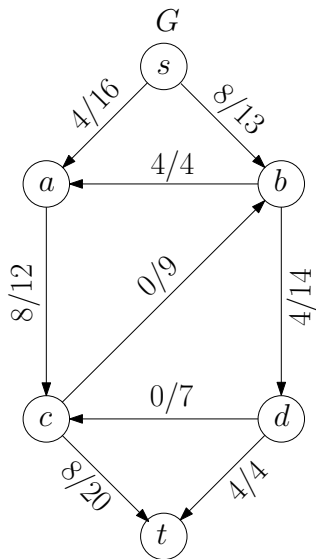
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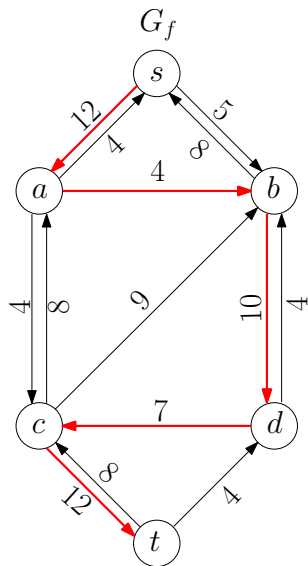
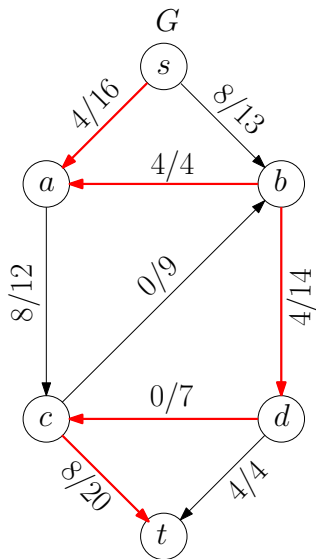
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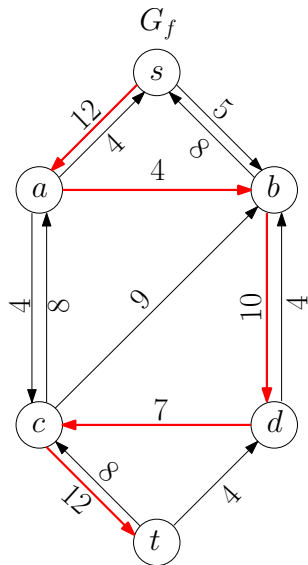
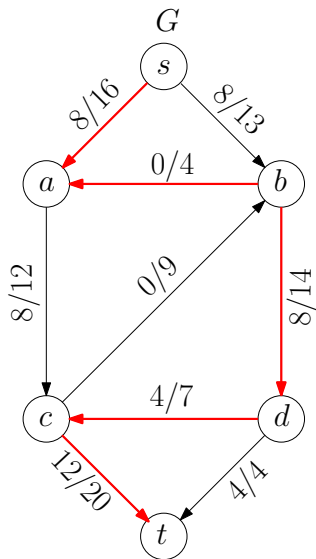
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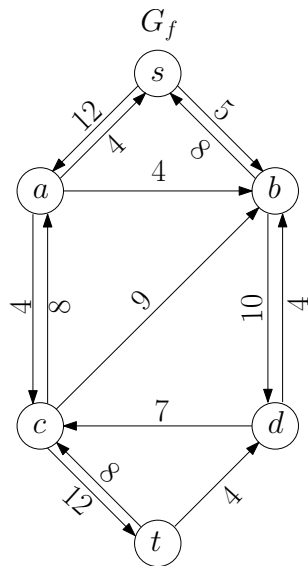
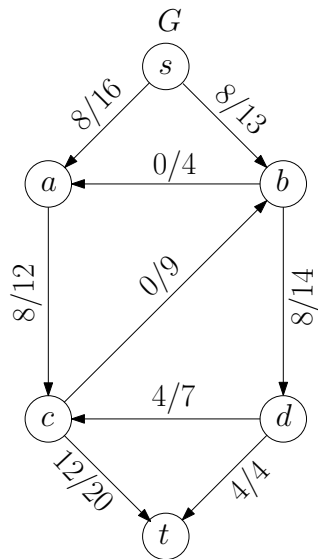
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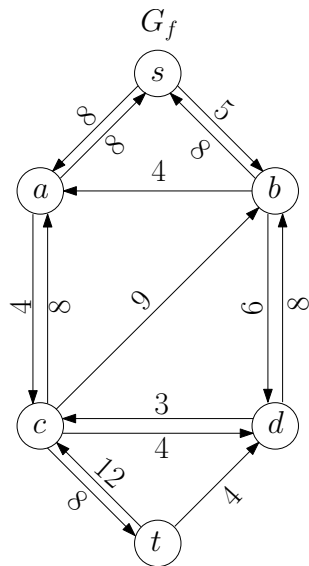
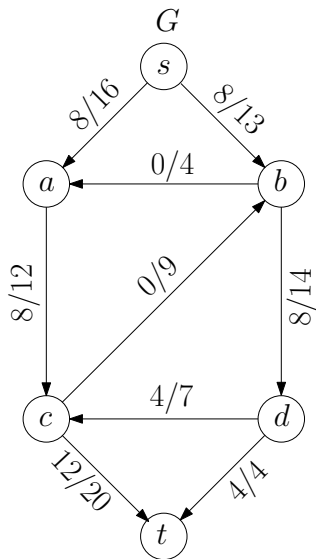
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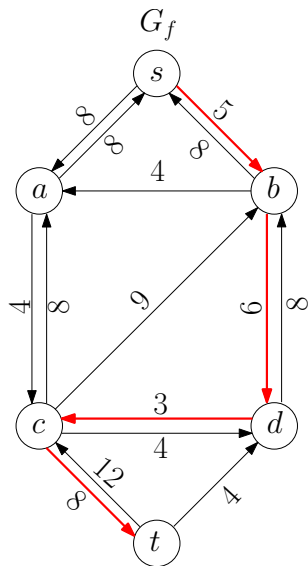
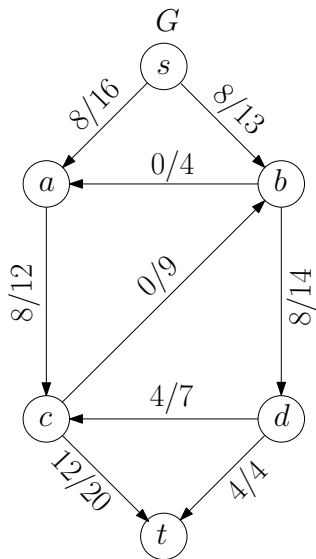
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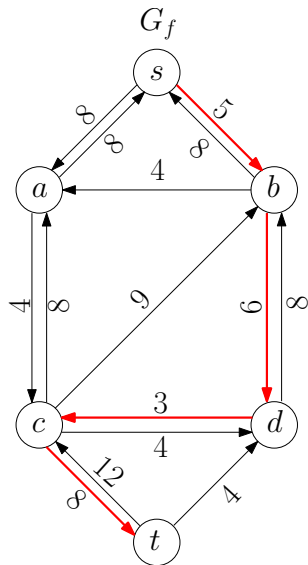
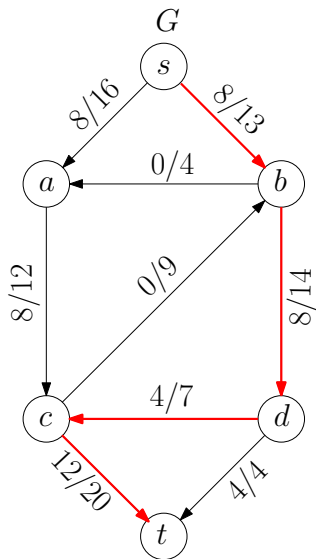
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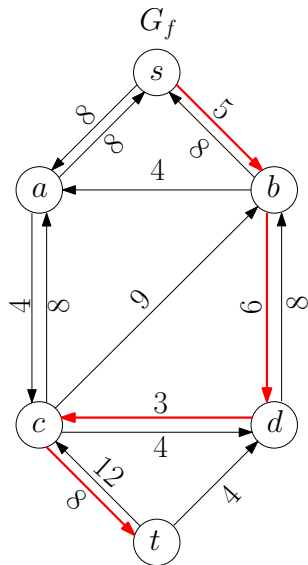
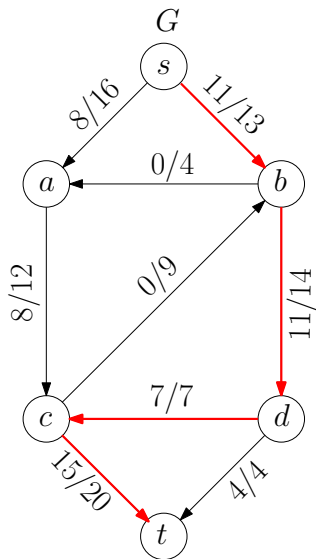
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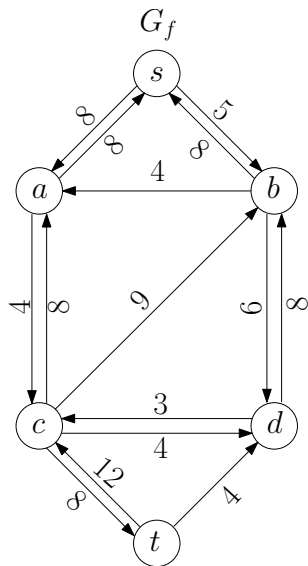
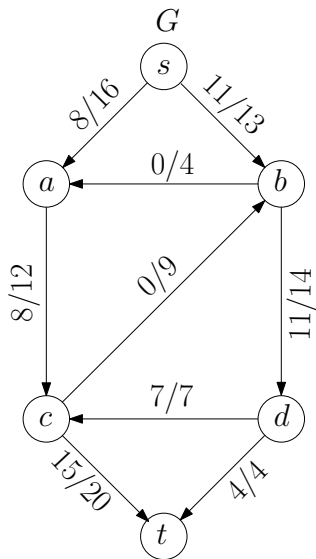
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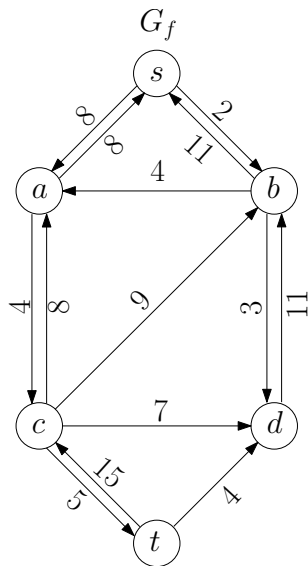
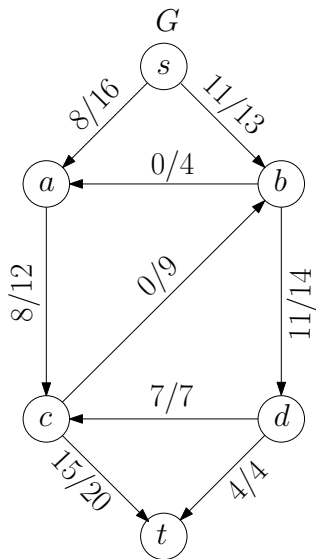
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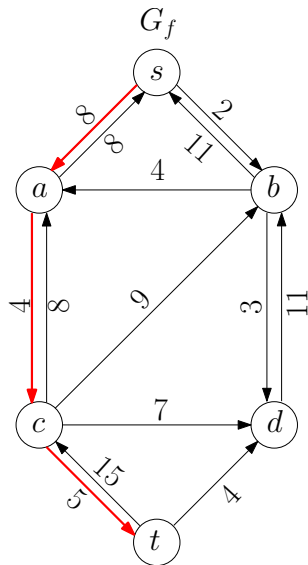
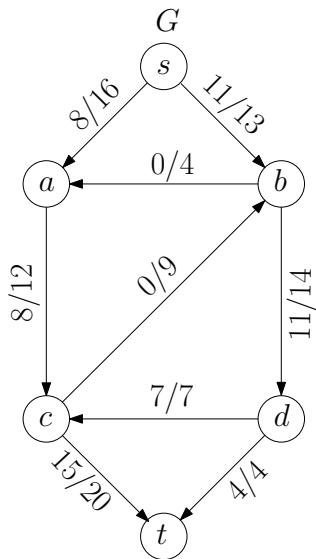
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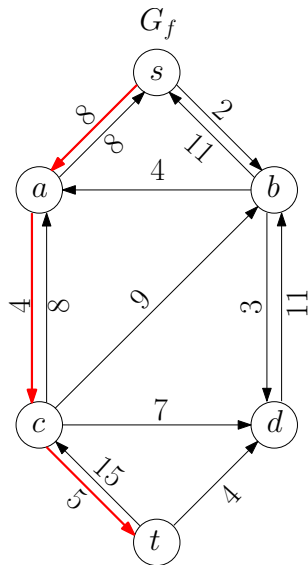
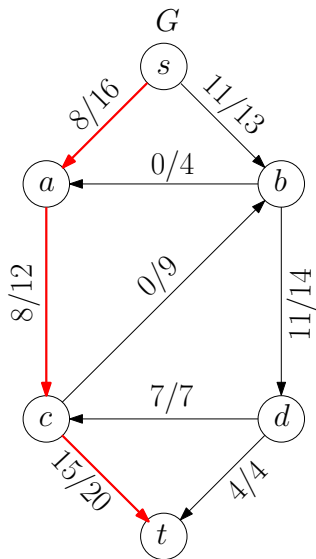
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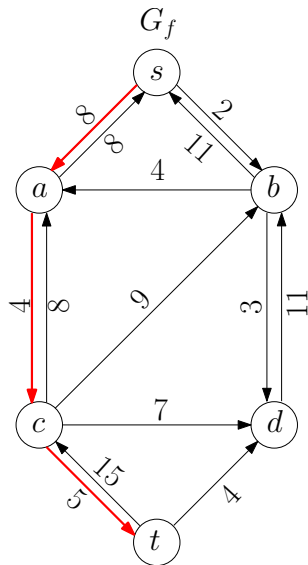
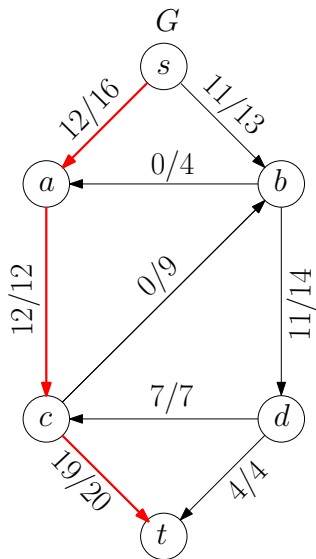
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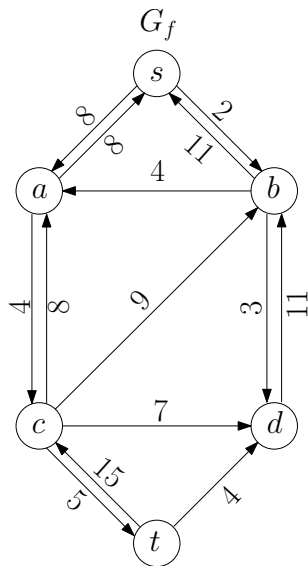
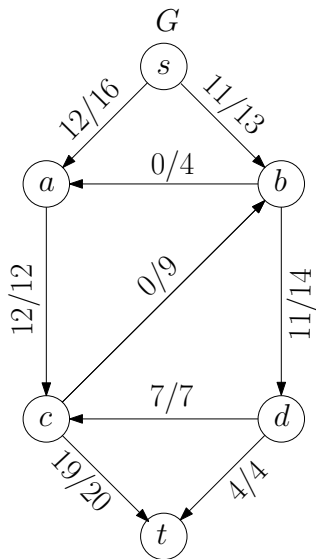
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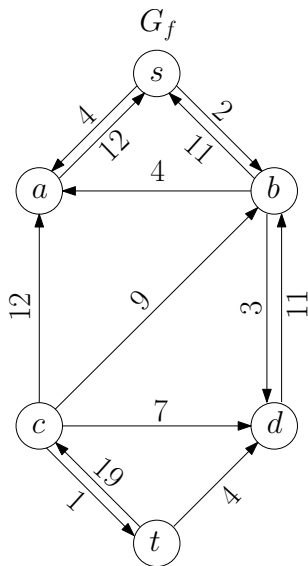
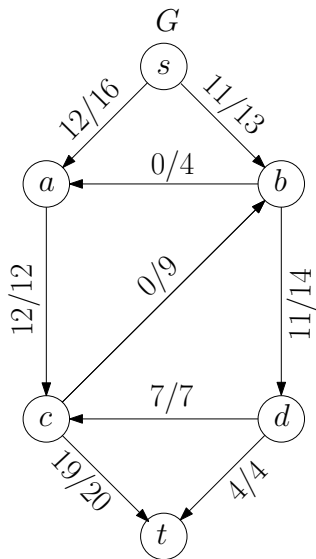
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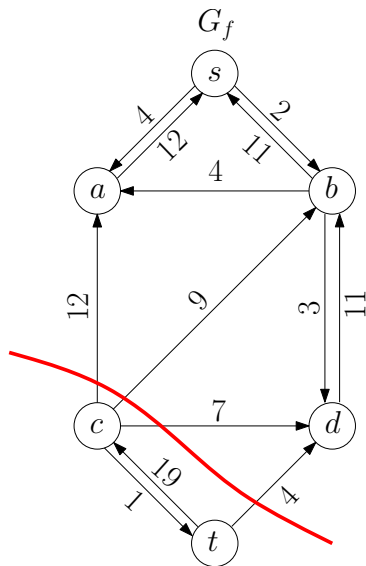
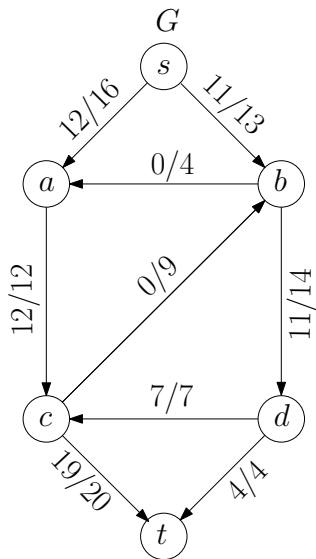
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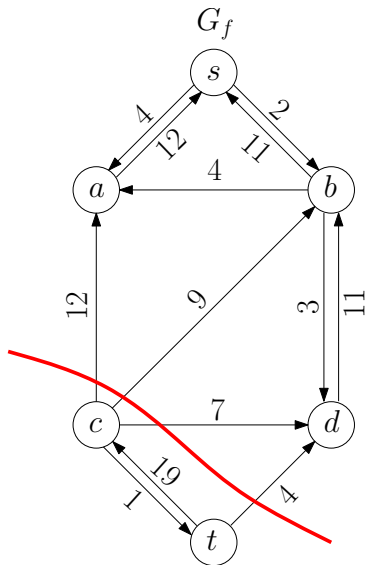
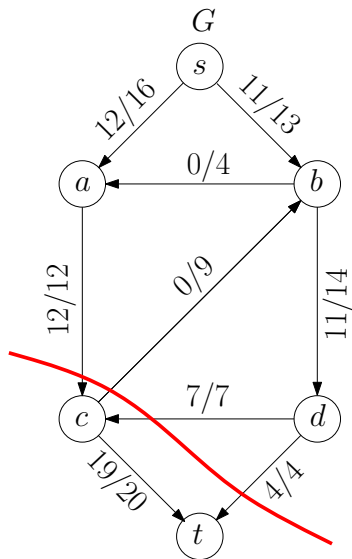
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Outline

- 1 Network Flow
- 2 Ford-Fulkerson Method
- 3 Correctness of Ford-Fulkerson's Method and Maximum Flow Minimum Cut Theorem
- 4 Running Time of Ford-Fulkerson-Type Algorithm
 - Shortest Augmenting Path Algorithm
 - Capacity-Scaling Algorithm
- 5 Bipartite Matching Problem
- 6 s - t Edge-Disjoint Paths Problem
- 7 More Applications

Correctness of Ford-Fulkerson's Method

- ① The procedure $\text{augment}(f, P)$ maintains the two conditions:
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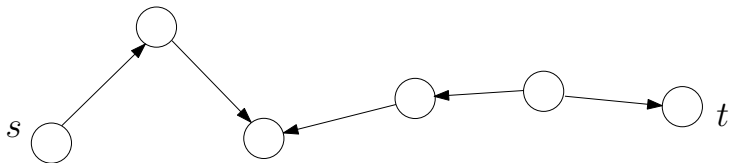
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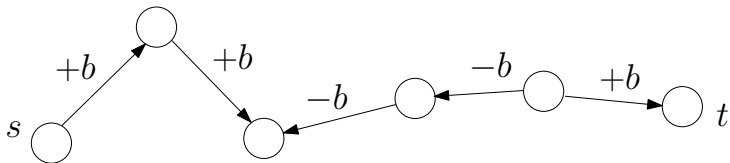
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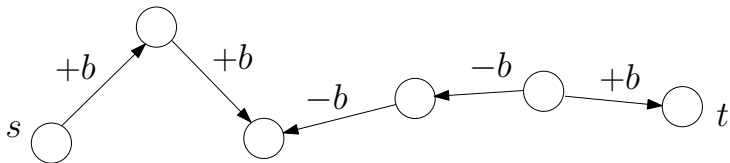
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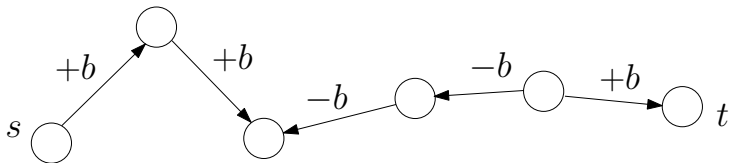
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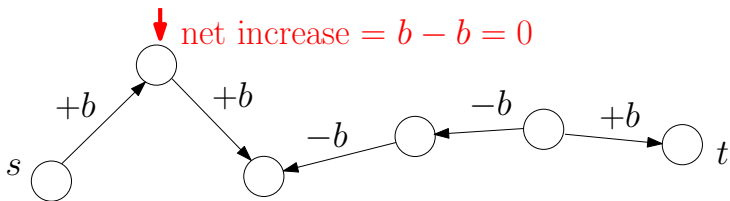
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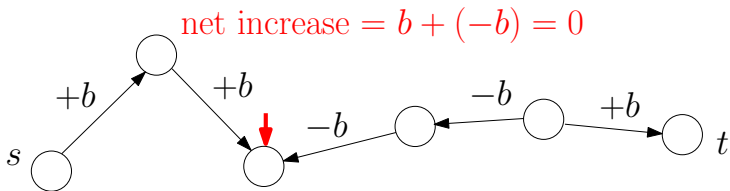
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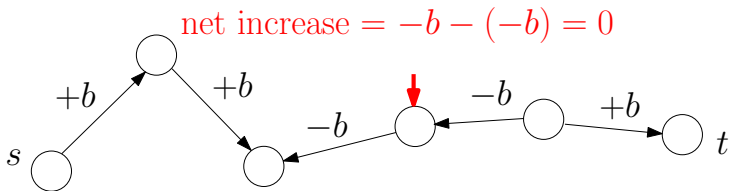
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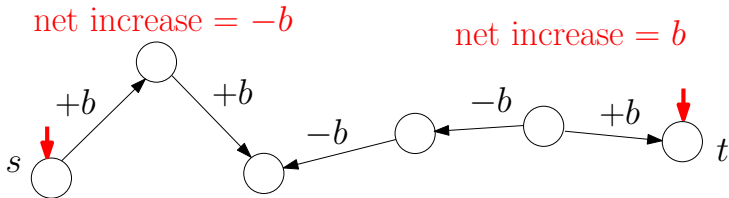
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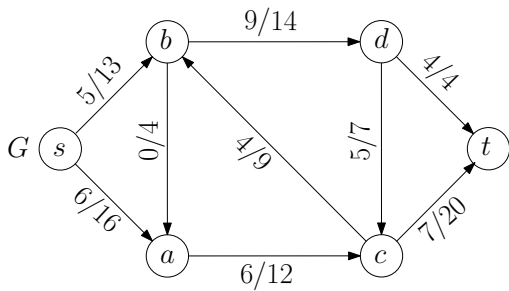
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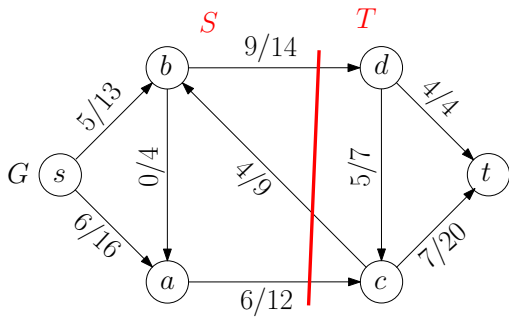
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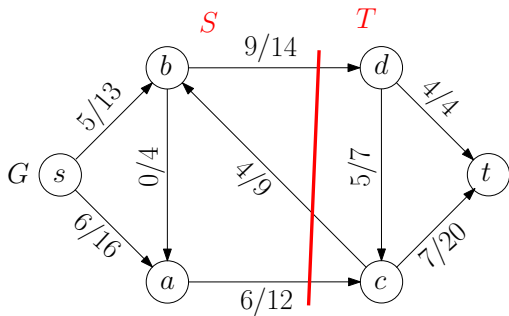
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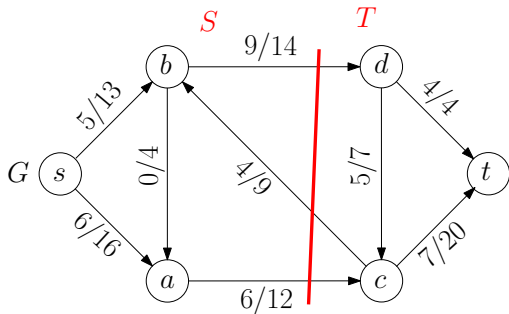






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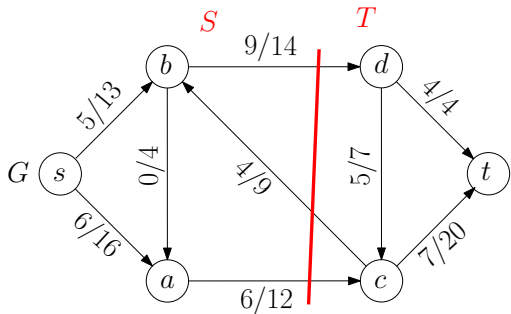
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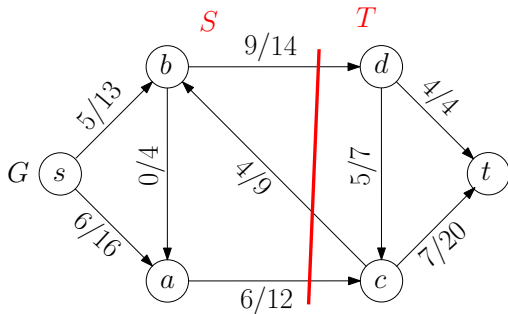


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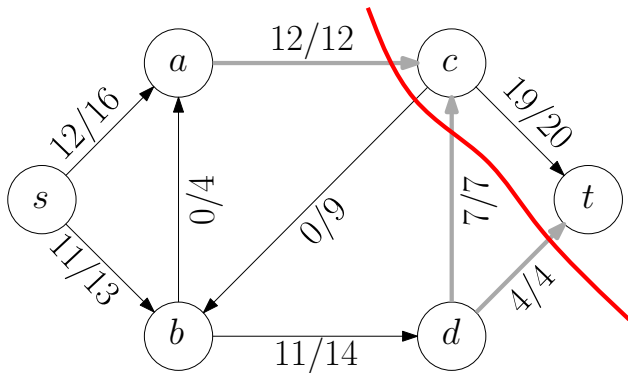
Corollary and Main Lemma implies

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$$\begin{aligned} \text{val}(f) = f(S, T) &= \sum_{e=(u,v) \in E, u \in S, v \in T} f(e) - \sum_{e=(u,v) \in E, u \in T, v \in S} f(e) = \\ &= \sum_{e=(u,v) \in E, u \in S, v \in T} c_e = c(S, T). \end{aligned}$$



Correctness of Ford-Fulkerson's Method

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- for every $e \in E$: $0 \leq f(e) \leq c_e$ (capacity conditions)
 - for every $v \in V \setminus \{s, t\}$:

$$\sum_{e \in \delta_{\text{in}}(v)} f(e) = \sum_{e \in \delta_{\text{out}}(v)} f(e). \quad (\text{conservation conditions})$$

- ② When Ford-Fulkerson's Method terminates, $\text{val}(f)$ is maximized
- ③ Ford-Fulkerson's Method will terminate

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However, the algorithm may not terminate if **some capacities are irrational numbers.** (“Pathological cases”)

Lemma Ford-Fulkerson's Method will terminate if all capacities are integers.

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- Integers can be replaced by rational numbers.

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Running time of the Generic Ford-Fulkerson's Algorithm

Ford-Fulkerson(G, s, t, c)

- 1: let $f(e) \leftarrow 0$ for every e in G
- 2: **while** there is a path from s to t in G_f **do**
- 3: let P be **any** simple path from s to t in G_f
- 4: $f \leftarrow \text{augment}(f, P)$
- 5: **return** f

- $O(m)$ -time for Steps 3 and 4 in each iteration
- Total time = $O(m) \times$ number of iterations

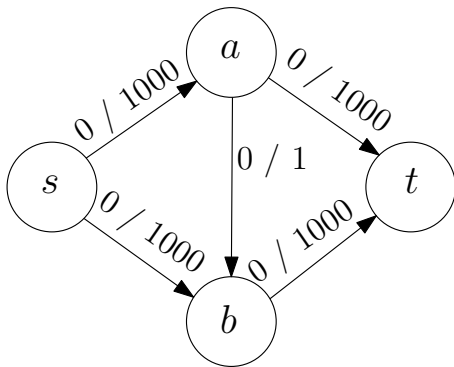
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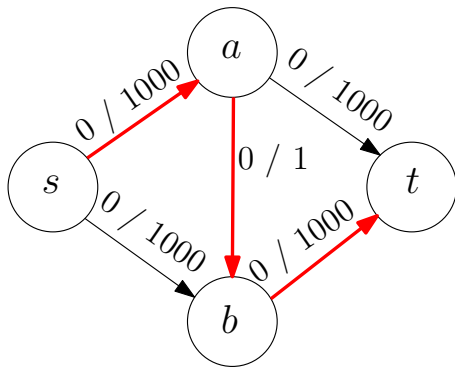
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- $O(m)$ -time for Steps 3 and 4 in each iteration
- Total time = $O(m) \times$ number of iterations
- Assume all capacities are integers, then algorithm may run up to $\text{val}(f^*)$ iterations, where f^* is the optimum flow
- Total time = $O(m \cdot \text{val}(f^*))$
- Running time is “Pseudo-polynomial”

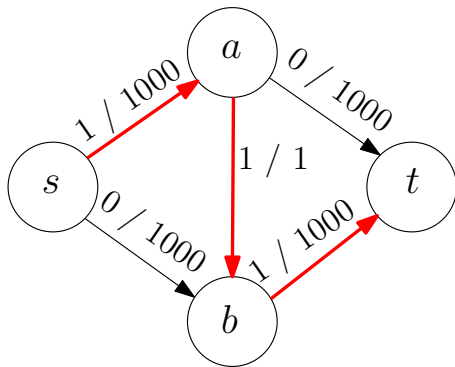
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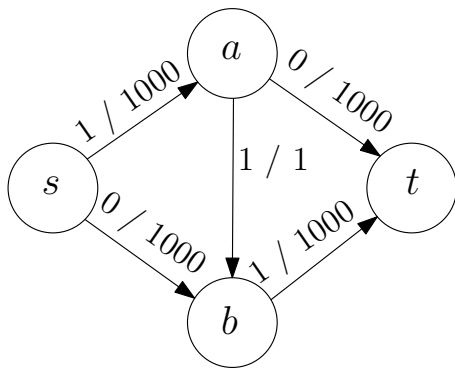
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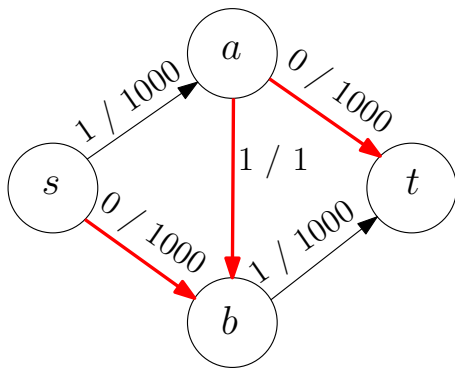
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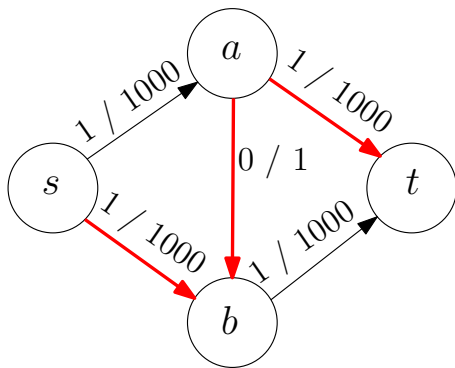
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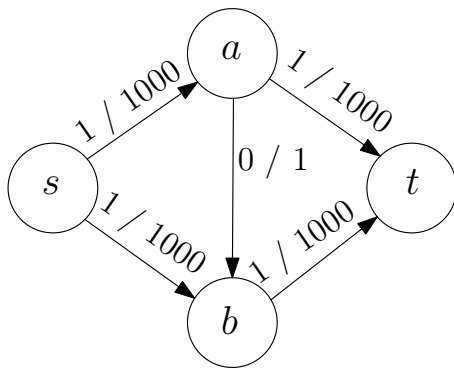
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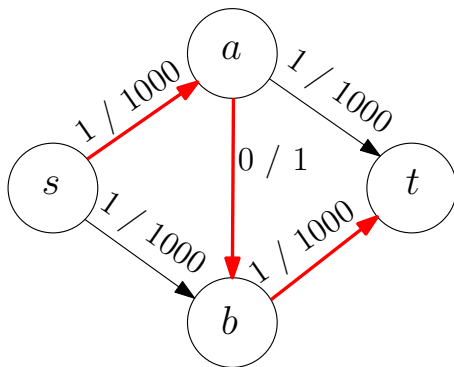
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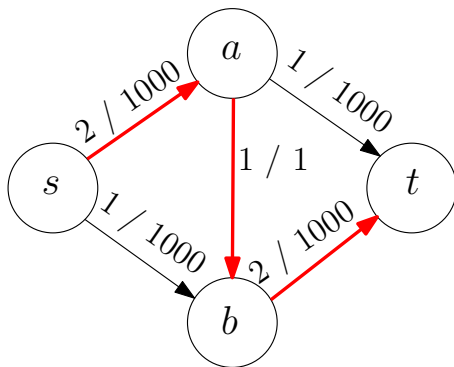
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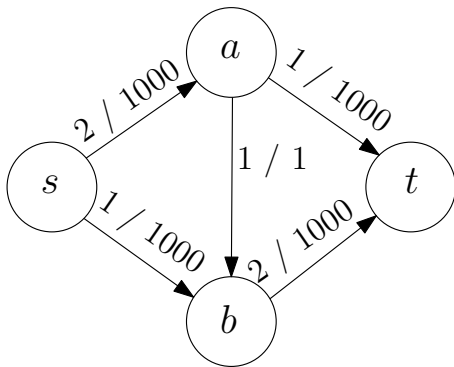
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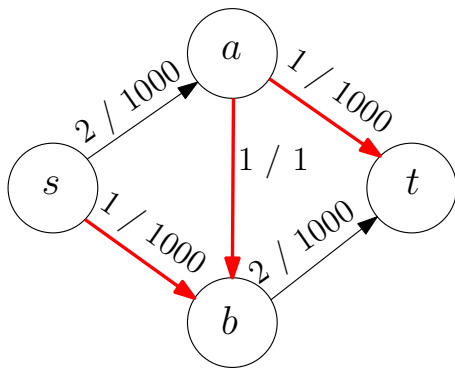
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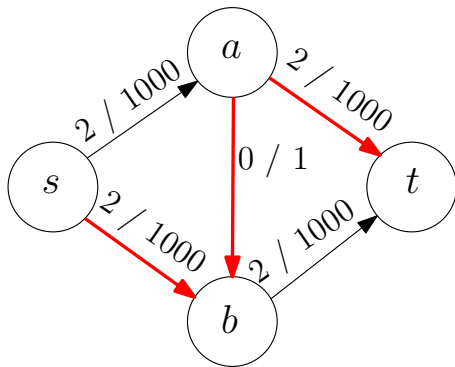
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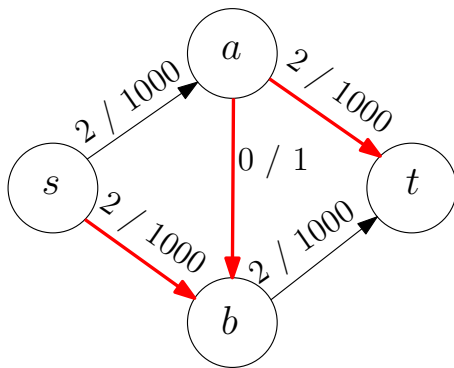
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Better choices for choosing augmentation paths:

- Choose the shortest augmentation path
- Choose the augmentation path with the largest bottleneck capacity

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Shortest Augmenting Path

shortest-augmenting-path(G, s, t, c)

- 1: let $f(e) \leftarrow 0$ for every e in G
- 2: **while** there is a path from s to t in G_f **do**
- 3: $P \leftarrow \text{breadth-first-search}(G_f, s, t)$
- 4: $f \leftarrow \text{augment}(f, P)$
- 5: **return** f

Due to [Dinitz 1970] and [Edmonds-Karp, 1970]

Running Time of Shortest Augmenting Path Algorithm

- Lemma**
1. Throughout the algorithm, length of shortest path from s to t in G_f never decreases.
 2. After at most m shortest path augmentations, the length of the shortest path from s to t in G_f strictly increases.

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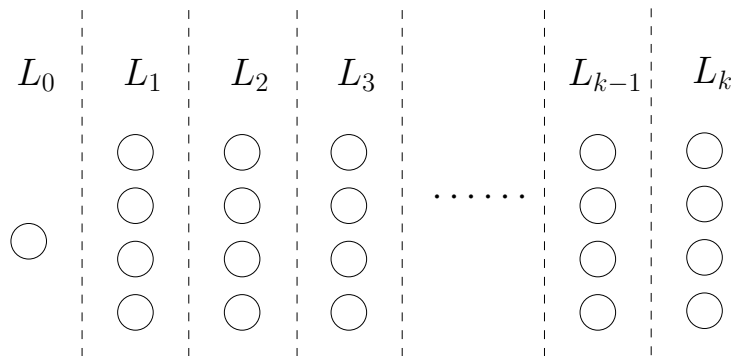
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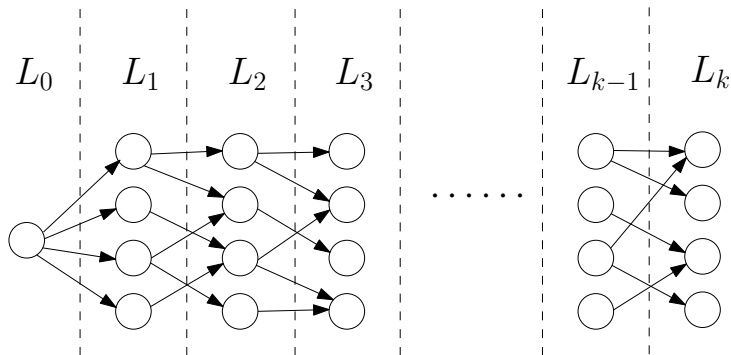
Theorem The shortest-augmenting-path algorithm runs in time $O(m^2n)$.

Proof of Lemma: Focus on G_f



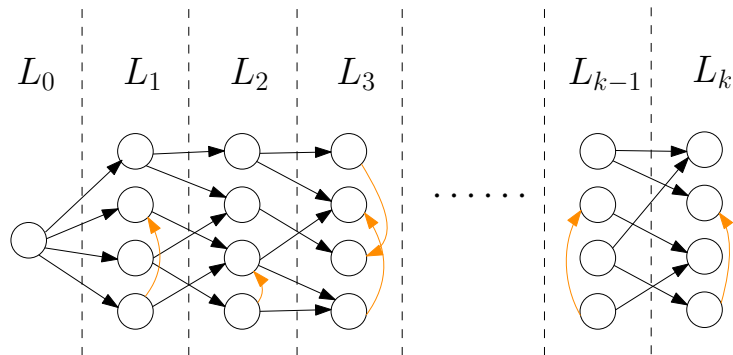
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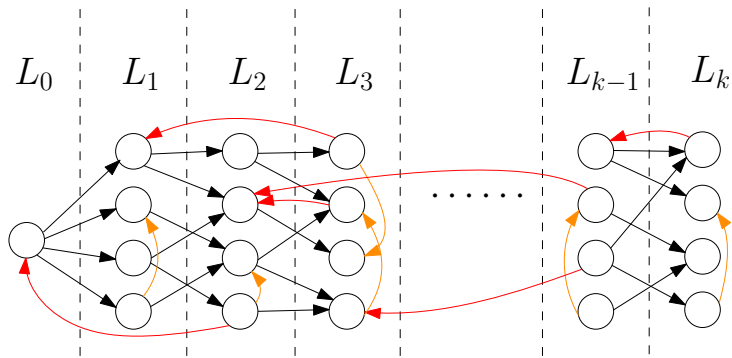
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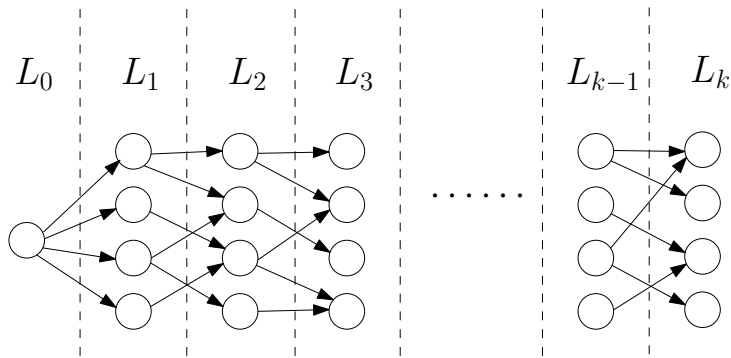
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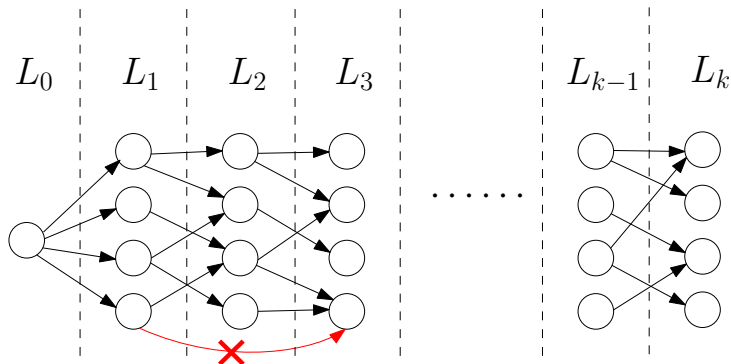
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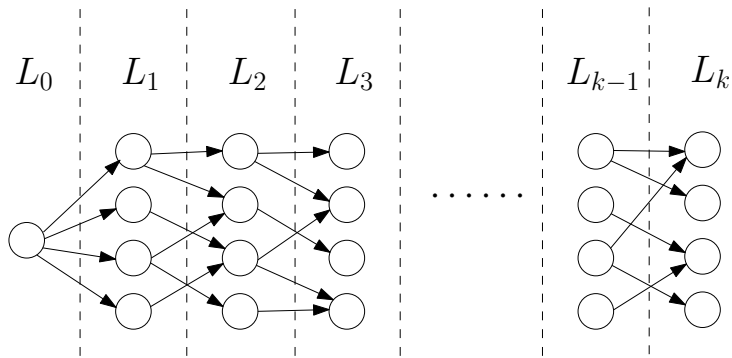
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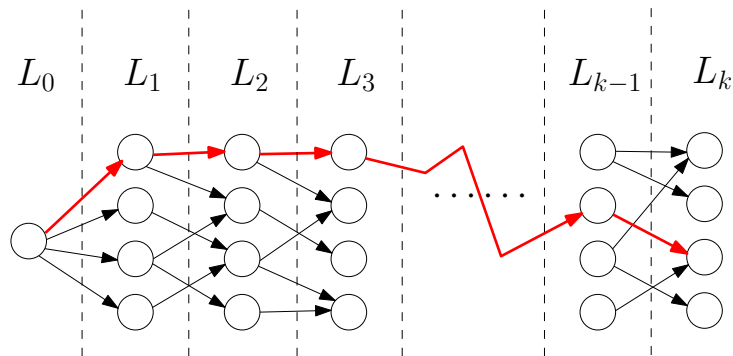
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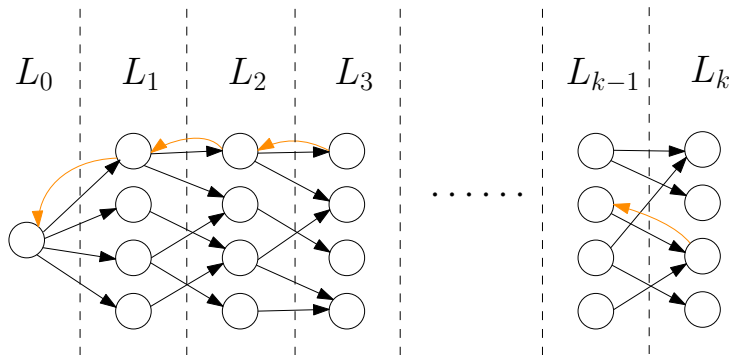
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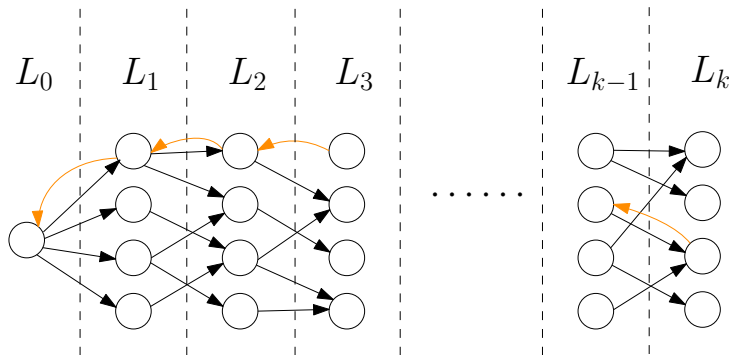
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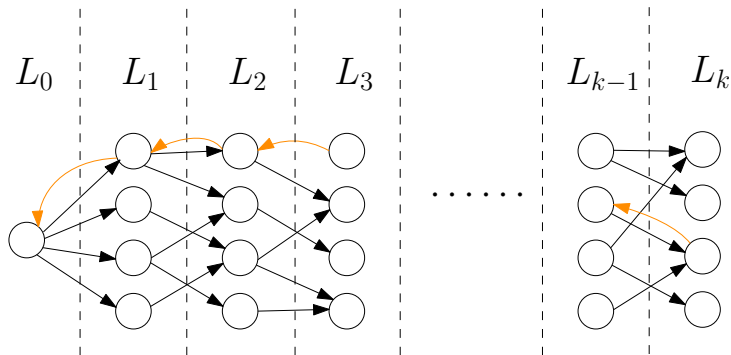
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Proof of Lemma: Focus on G_f



- Assuming $t \in L_k$, shortest $s \rightarrow t$ path uses k forth edges
- After augmenting along the path, back edges will be added to G_f
- One forth edge will be removed from G_f
- In $O(m)$ iterations, there will be no paths from s to t of length k in G_f .

Improving the $O(m^2n)$ Running Time for Shortest Path Augmentation Algorithm

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- Dynamic Trees $\Rightarrow O(mn \log n)$ [Sleator-Tarjan 1983]

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capacity-scaling(G, s, t, c)

- 1: let $f(e) \leftarrow 0$ for every e in G
- 2: $\Delta \leftarrow$ largest power of 2 which is at most C
- 3: **while** $\Delta \geq 1$ **do** **do**
- 4: **while** there exists an augmenting path P with bottleneck capacity at least Δ **do**
- 5: $f \leftarrow \text{augment}(f, P)$
- 6: $\Delta \leftarrow \Delta/2$
- 7: **return** f

Obs. The outer while loop repeats $1 + \lfloor \log_2 C \rfloor$ times.

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Lemma At the beginning of Δ -scale phase, the value of the max-flow is at most $\text{val}(f) + 2m\Delta$.

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- Thus, there are at most $2m$ augmentations for Δ -scale phase.

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- Each augmentation increases the flow value by at least Δ
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Theorem The number of augmentations in the scaling max-flow algorithm is at most $O(m \log C)$. The running time of the algorithm is $O(m^2 \log C)$.

Polynomial Time

Assume all capacities are integers between 1 and C .

Ford-Fulkerson	$O(m^2C)$	pseudo-polynomial
Capacity-scaling:	$O(m^2 \log C)$	weakly-polynomial
Shortest-Path-Augmenting:	$O(m^2n)$	strongly-polynomial

- Polynomial : weakly-polynomial and strongly-polynomial

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Brief History

Algorithm	Year	Time	Description
Ford-Fulkerson	1956	$O(mf)$	Ford-Fulkerson Method.
Edmonds-Karp	1972	$O(nm^2)$	Shortest Augmenting Paths
Dinic	1970	$O(n^2m)$	SAP with blocking Flows
Goldberg-Tarjan	1988	$O(n^3)$	Generic Push-Relabel
Goldberg-Tarjan	1988	$O(n^2\sqrt{m})$	PR using highest-label nodes
Chen et al.	2022	$O(m^{1+o(1)})$	LP-solver, dynamic algorithms

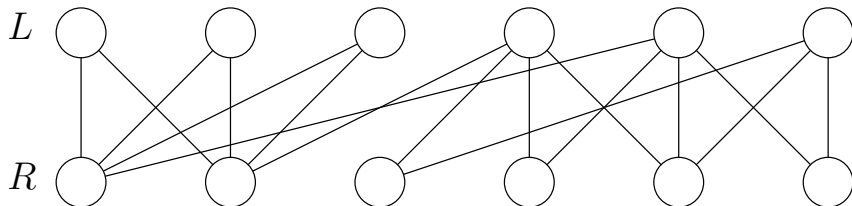
- Chen et al. [[Chen-Kyng-Liu-Peng-Gutenberg-Sachdeva, 2022](#)].

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Bipartite Graphs

Def. A graph $G = (V, E)$ is **bipartite** if the vertices V can be partitioned into two subsets L and R such that every edge in E is between a vertex in L and a vertex in R .



Def. Given a bipartite graph $G = (L \cup R, E)$, a **matching** in G is a set $M \subseteq E$ of edges such that every vertex in V is an endpoint of at most one edge in M .

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Maximum Bipartite Matching Problem

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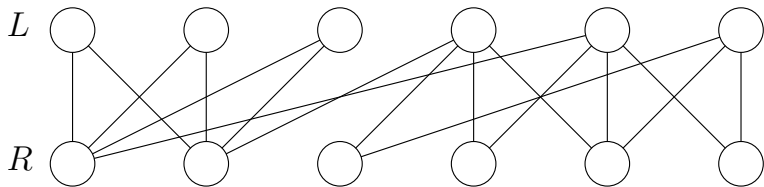
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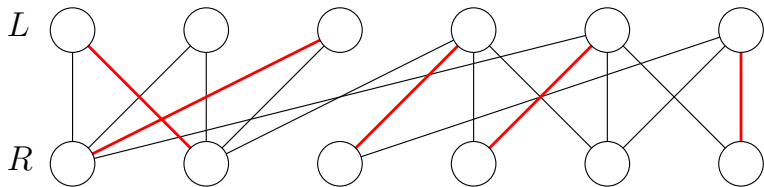


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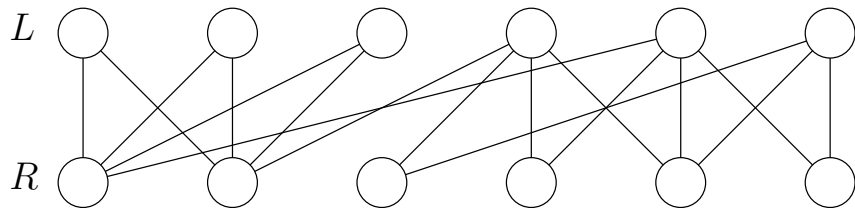
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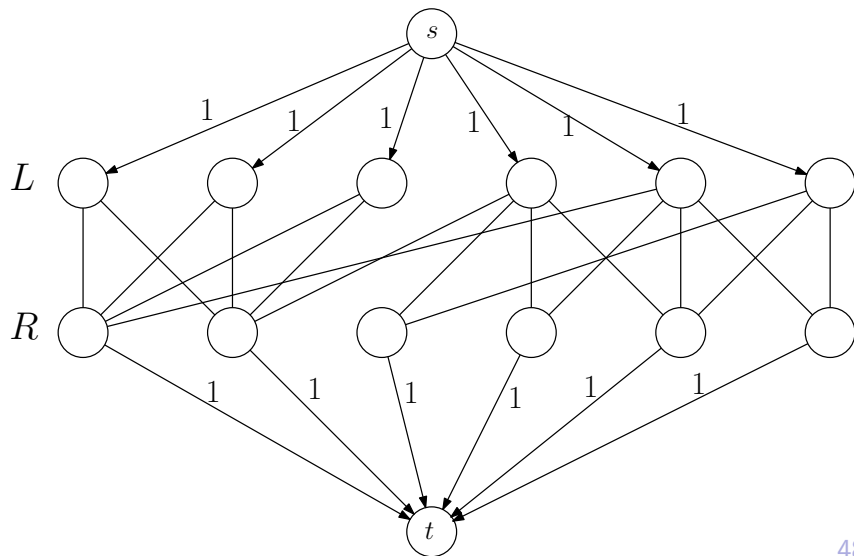
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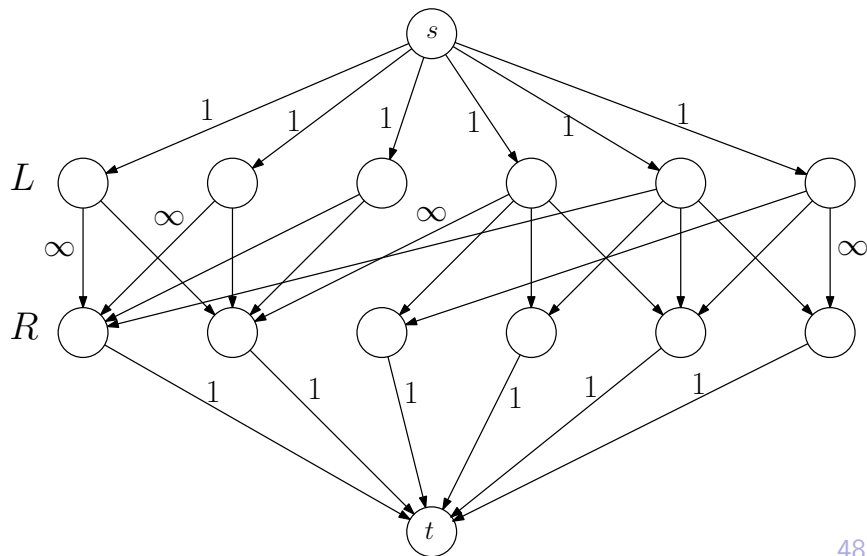
Reduce Maximum Bipartite Matching to Maximum Flow Problem



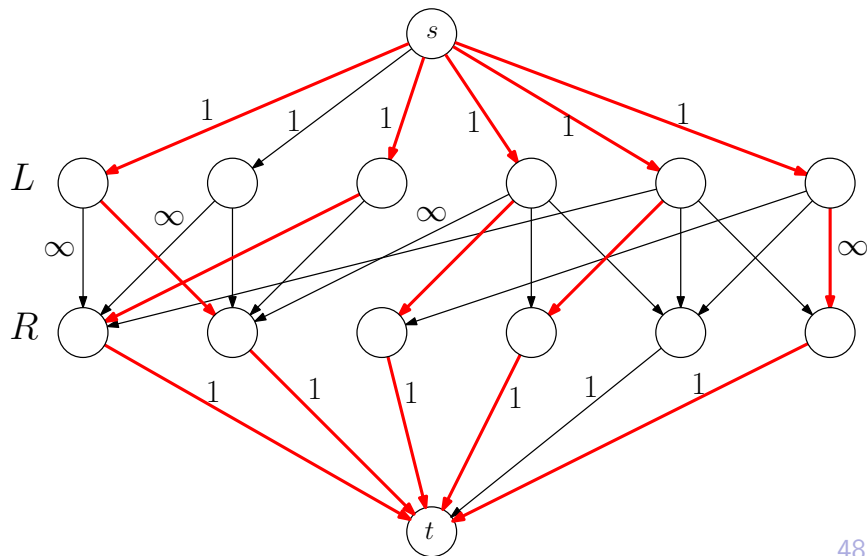
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Reduce Maximum Bipartite Matching to Maximum Flow Problem

- Create a digraph $G' = (L \cup R \cup \{s, t\}, E')$ with capacity $c : E' \rightarrow \mathbb{R}_{\geq 0}$:
 - Add a source s and a sink t
 - Add an edge from s to each vertex $u \in L$ of capacity 1
 - Add an edge from each vertex $v \in R$ to t of capacity 1
 - Direct all edges in E from L to R , and assign ∞ capacity (or capacity 1) to them

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- The maximum flow gives a matching

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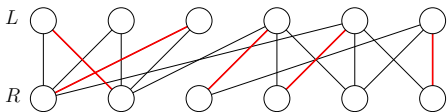
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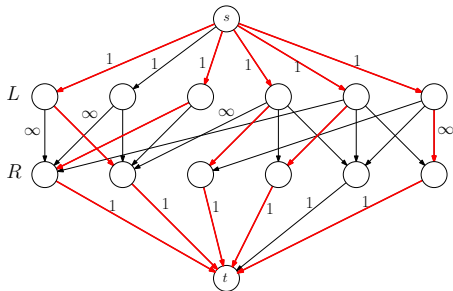
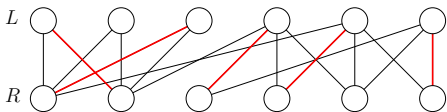
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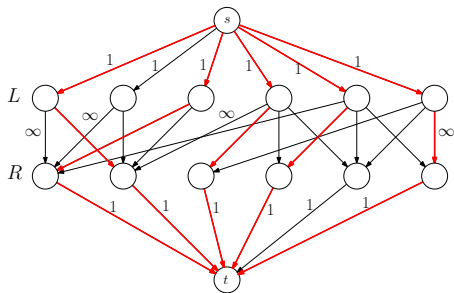
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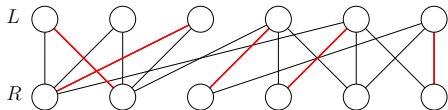
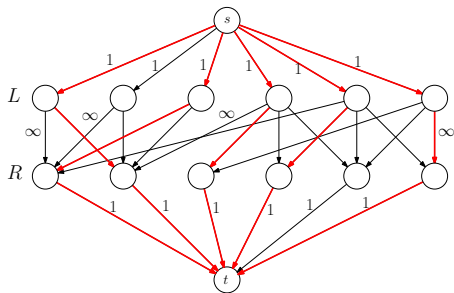
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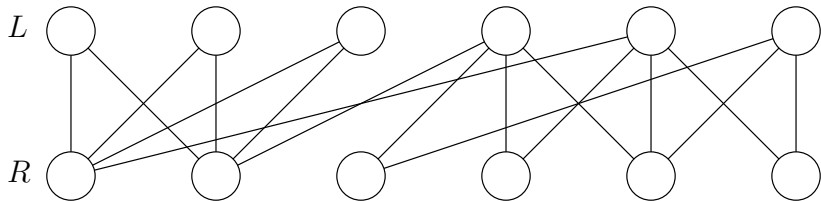
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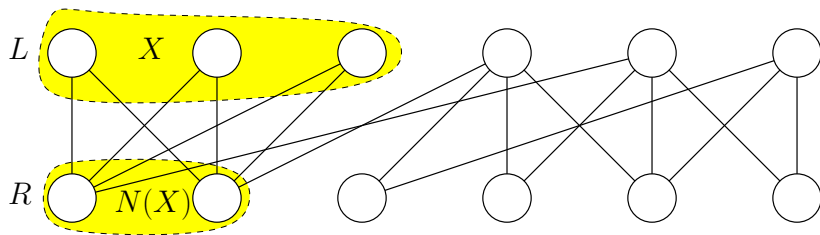
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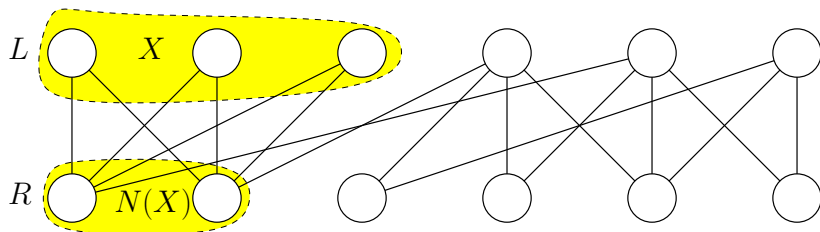


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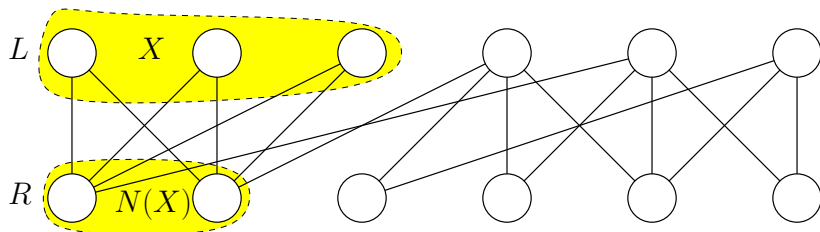


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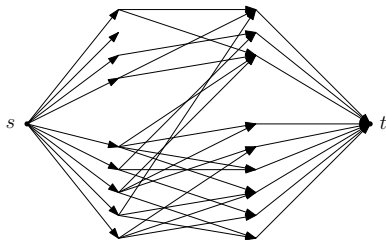
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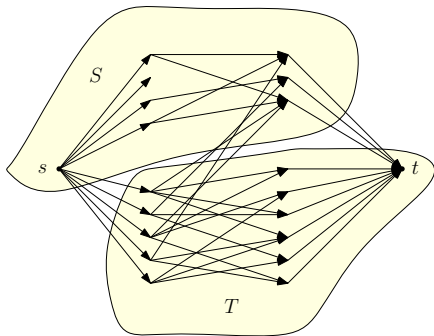
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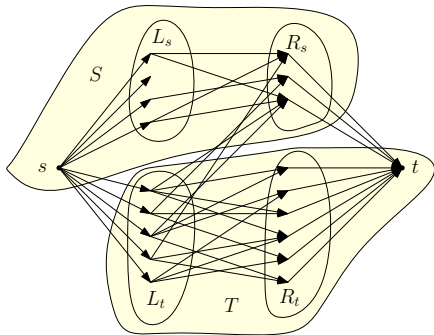
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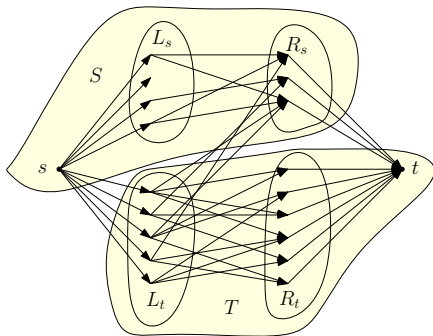
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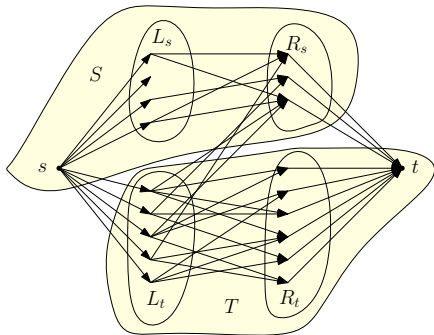
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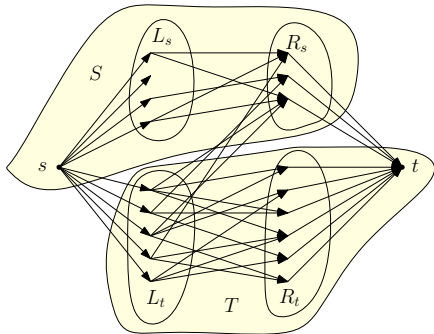
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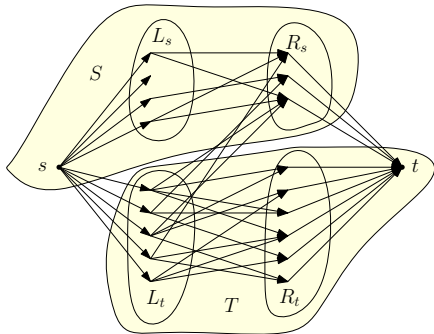
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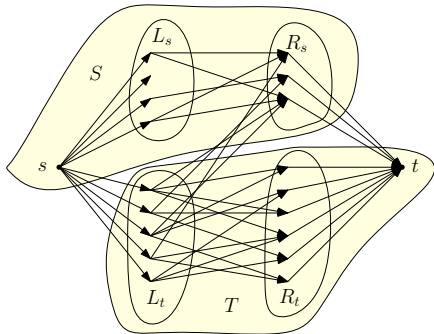
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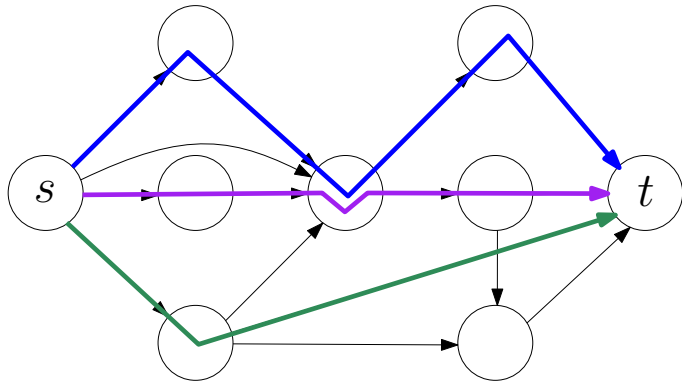
Outline

- 1 Network Flow
- 2 Ford-Fulkerson Method
- 3 Correctness of Ford-Fulkerson's Method and Maximum Flow Minimum Cut Theorem
- 4 Running Time of Ford-Fulkerson-Type Algorithm
 - Shortest Augmenting Path Algorithm
 - Capacity-Scaling Algorithm
- 5 Bipartite Matching Problem
- 6 s - t Edge-Disjoint Paths Problem
- 7 More Applications

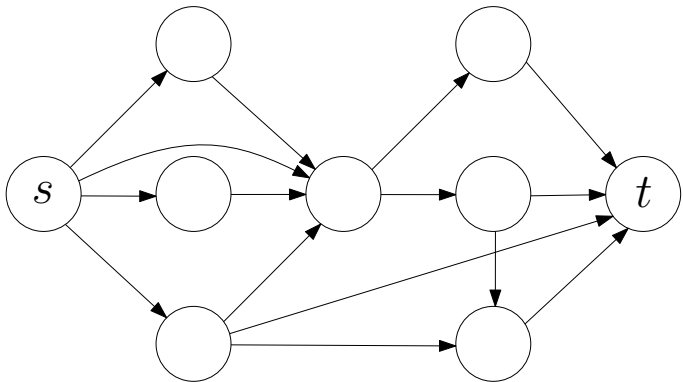
s - t Edge Disjoint Paths

Input: a directed (or undirected) graph $G = (V, E)$ and $s, t \in V$

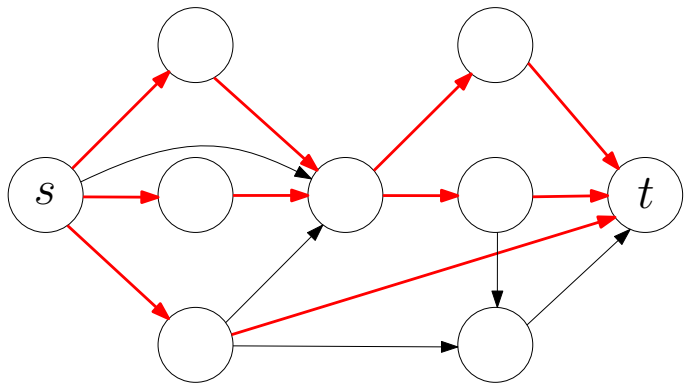
Output: the maximum number of **edge-disjoint** paths from s to t in G



- Solving the maximum flow problem, where all capacities are 1
- All flow values are integral (i.e, either 0 or 1)

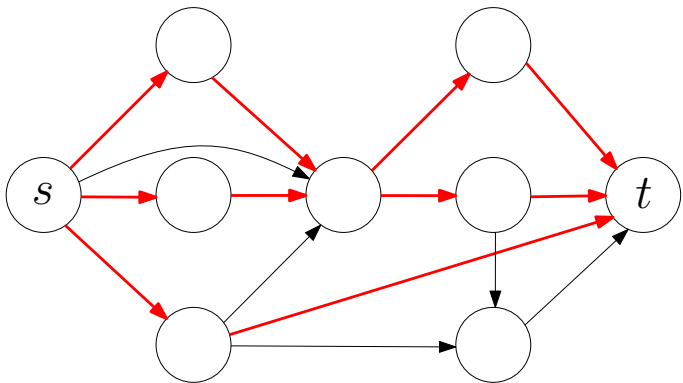


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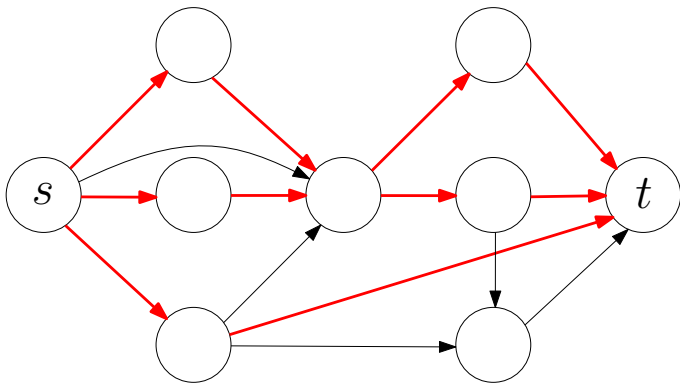
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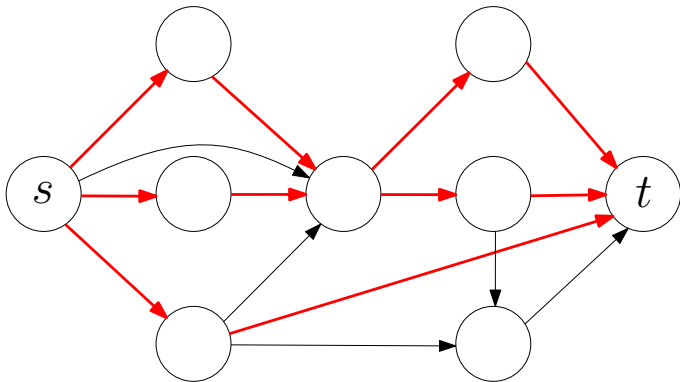
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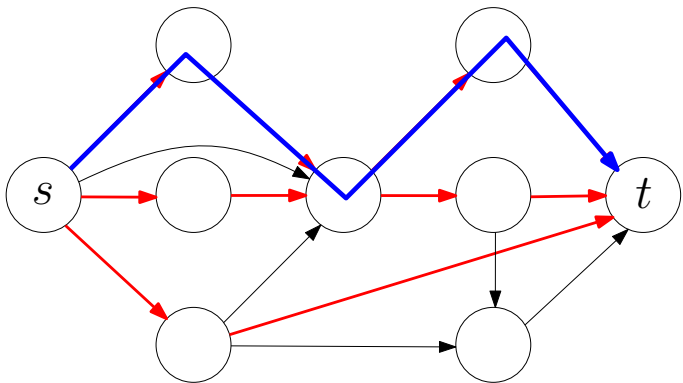
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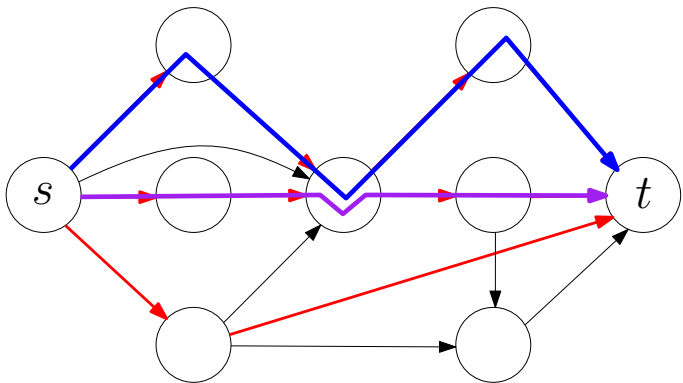
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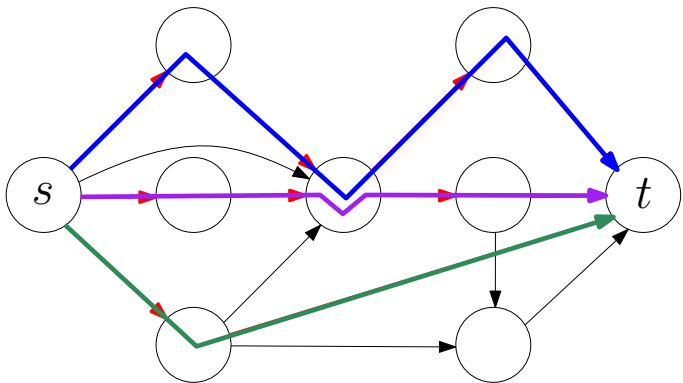
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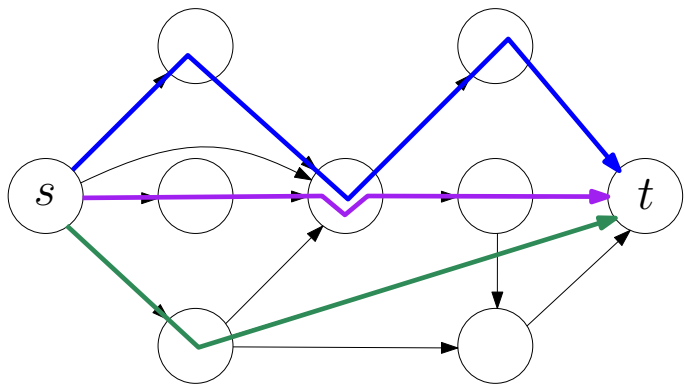
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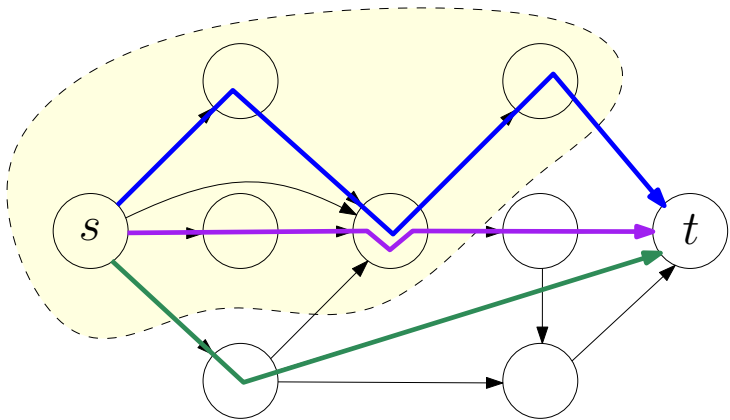
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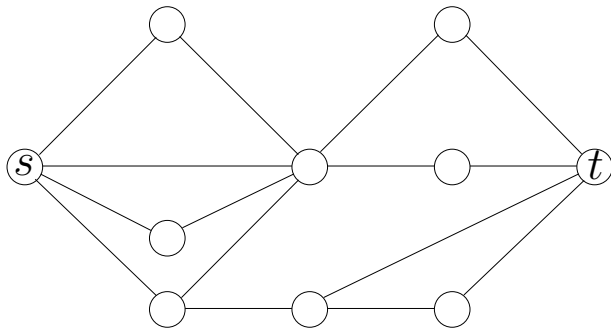
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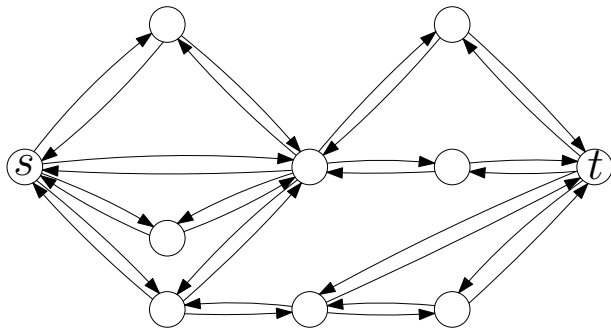
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s - t Edge Disjoint Paths in Undirected Graphs

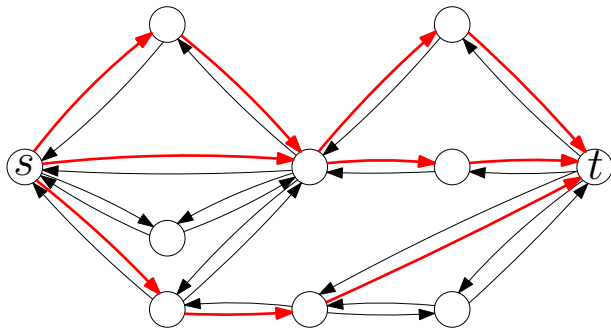


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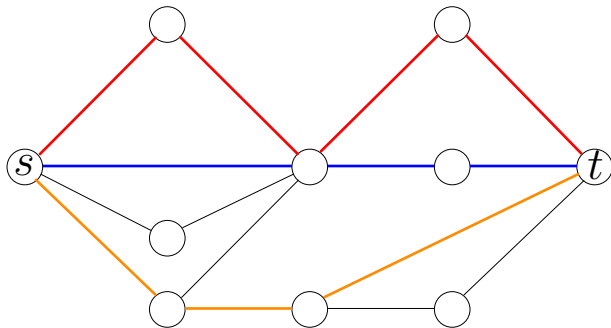
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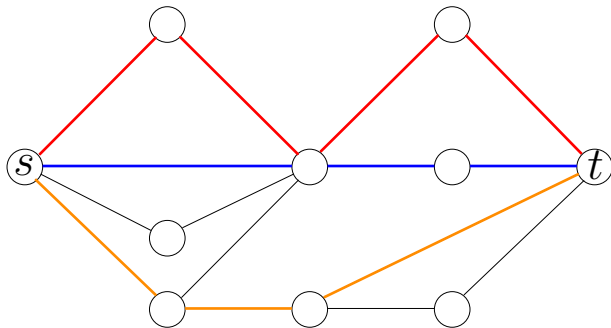
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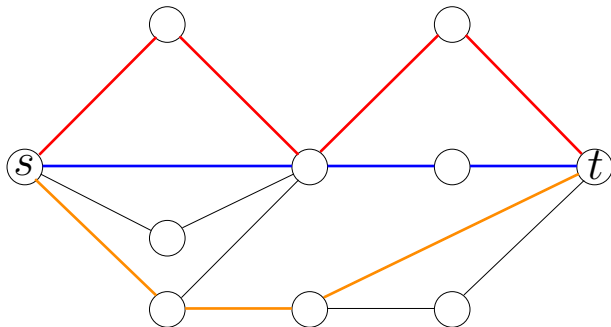
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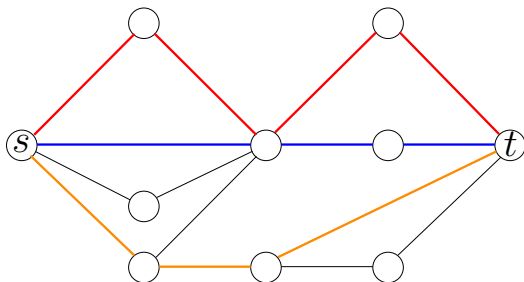
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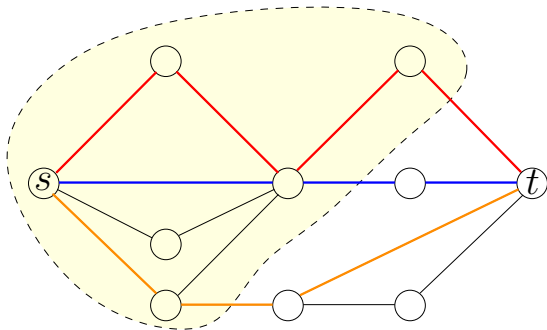
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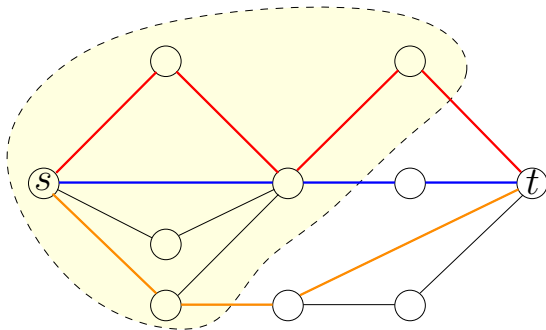
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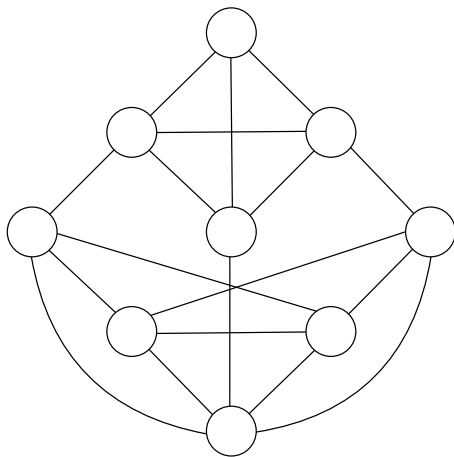


s - t connectivity measures how well s and t are connected.

Global Min-Cut Problem

Input: a connected graph $G = (V, E)$

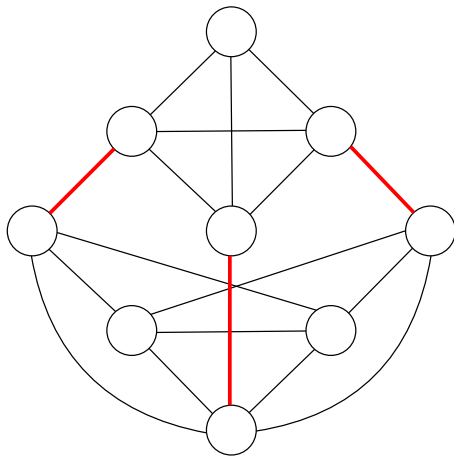
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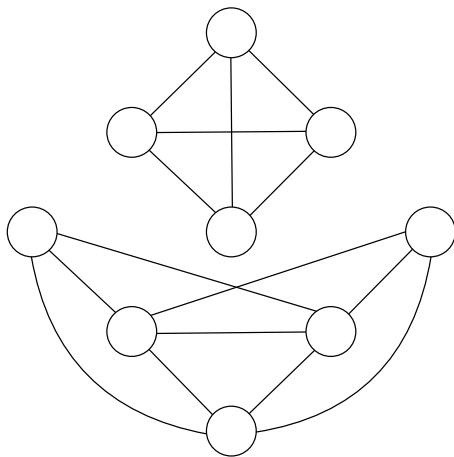
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Solving Global Min-Cut Using Maximum Flow

- 1: let G' be the directed graph obtained from G by replacing every edge with two anti-parallel edges
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- Need to solve $\Theta(n^2)$ maximum flow instances
- Can we do better?
- Yes. We can fix s . We only need to enumerate t

Outline

- 1 Network Flow
- 2 Ford-Fulkerson Method
- 3 Correctness of Ford-Fulkerson's Method and Maximum Flow Minimum Cut Theorem
- 4 Running Time of Ford-Fulkerson-Type Algorithm
 - Shortest Augmenting Path Algorithm
 - Capacity-Scaling Algorithm
- 5 Bipartite Matching Problem
- 6 s - t Edge-Disjoint Paths Problem
- 7 More Applications

Extension of Network Flow: Circulation Problem

Input: A digraph $G = (V, E)$

capacities $c \in \mathbb{Z}_{\geq 0}^E$

supply vector $d \in \mathbb{Z}^V$ with $\sum_{v \in V} d_v = 0$

Output: whether there exists $f : E \rightarrow \mathbb{Z}_{\geq 0}$ s.t.

$$\sum_{e \in \delta^{\text{out}}(v)} f(e) - \sum_{e \in \delta^{\text{in}}(v)} f(e) = d_v \quad \forall v \in V$$

$$0 \leq f(e) \leq c_e \quad \forall e \in E$$

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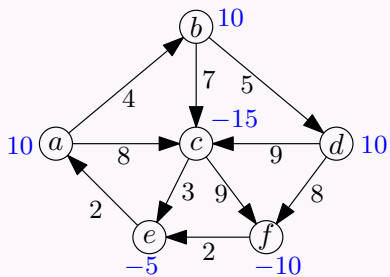
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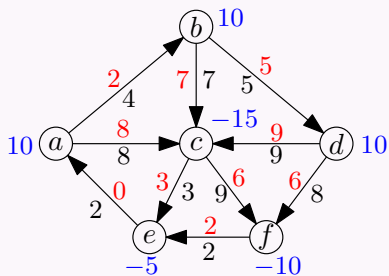
$$\begin{aligned} \sum_{e \in \delta^{\text{out}}(v)} f(e) - \sum_{e \in \delta^{\text{in}}(v)} f(e) &= d_v & \forall v \in V \\ 0 \leq f(e) &\leq c_e & \forall e \in E \end{aligned}$$

- d_v denotes the net supply of a good
- $d_v > 0$: there is a **supply** of d_v at v
- $d_v < 0$: there is a **demand** of $-d_v$ at v
- problem: whether we can match the supplies and demands without violating capacity constraints

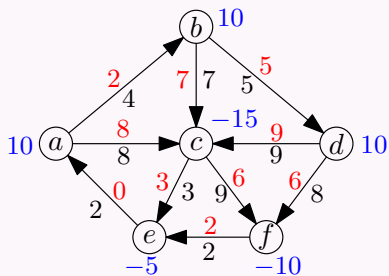
Example



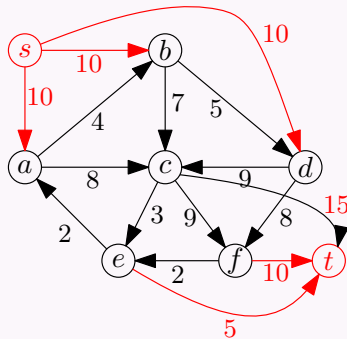
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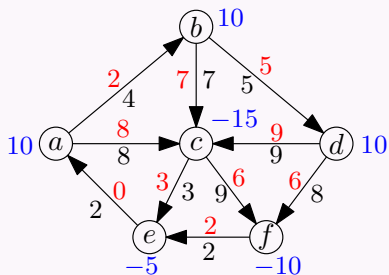
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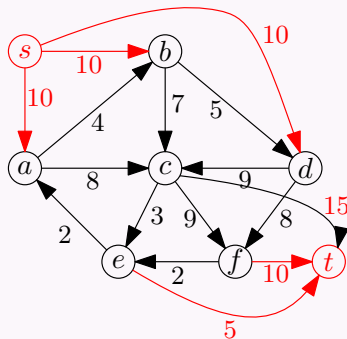
Reduction



Example



Reduction



Reduction to maximum flow

- add a super-source s and a super-sink t to network
- for every $v \in V$ with $d_v > 0$: add edge (s, v) of capacity d_v
- for every $v \in V$ with $d_v < 0$: add edge (v, t) of capacity $-d_v$
- check if maximum flow has value $\sum_{v:d_v>0} d_v$

- $d(S) := \sum_{v \in S} d_v, \forall S \subseteq V.$
- $c(S, V \setminus S) := \sum_{(u,v) \in E: u \in S, v \notin S} c_{(u,v)}.$

Lemma The instance is feasible if and only if for every $S \subseteq V$, $d(S) \leq c(S, V \setminus S).$

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Proof of “only if” direction.

- if for some $S \subseteq V$, $c(S, V \setminus S) < d(S)$, then the demand in S can not be sent out of S . □

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- It remains to consider the “if” direction

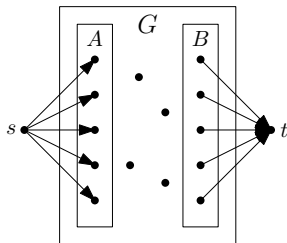
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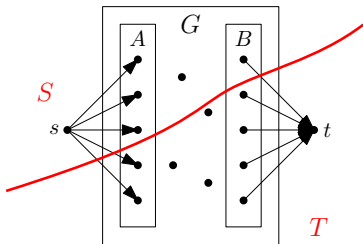
- assume instance is infeasible:
max-flow $< d(A)$
- $A := \{v \in V : d_v > 0\}$
- $B := \{v \in V : d_v < 0\}$



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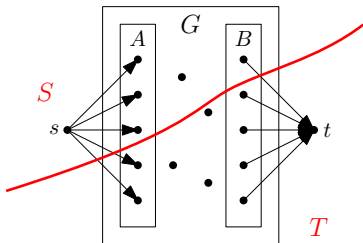
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$$d(T \cap A) + |d(S \cap B)| + c(S \setminus \{s\}, T \setminus \{t\}) < d(A)$$

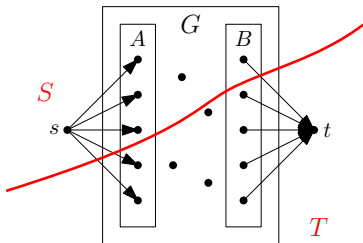
$$d(T \cap A) - d(S \cap B) + c(S \setminus \{s\}, T \setminus \{t\}) < d(A)$$

$$c(S \setminus \{s\}, T \setminus \{t\}) < d(S \cap A) + d(S \cap B) = d(S \setminus \{s\})$$

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- Define $S' = S \setminus \{s\}$: $d(S') > c(S', V \setminus S')$.

Circulation Problem with Capacity Lower Bounds

Input: A digraph $G = (V, E)$

capacities $c \in \mathbb{Z}_{\geq 0}^E$

capacity lower bounds $l \in \mathbb{Z}_{\geq 0}^E$, $0 \leq l_e \leq c_e$

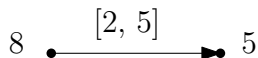
supply vector $d \in \mathbb{Z}^V$ with $\sum_{v \in V} d_v = 0$

Output: whether there exists $f : E \rightarrow \mathbb{Z}_{\geq 0}$ s.t.

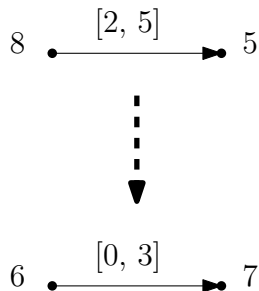
$$\sum_{e \in \delta^{\text{out}}(v)} f(e) - \sum_{e \in \delta^{\text{in}}(v)} f(e) = d_v \quad \forall v \in V$$

$$l_e \leq f(e) \leq c_e \quad \forall e \in E$$

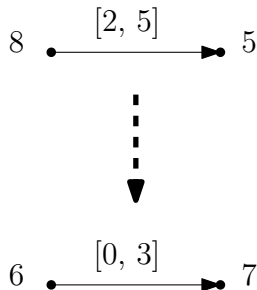
Removing Capacity Lower Bounds



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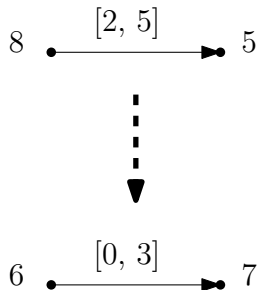
Removing Capacity Lower Bounds



handling $e = (u, v)$ with $l_e > 0$

- $d'_u \leftarrow d_u - l_e$
- $d'_v \leftarrow d_v + l_e$
- $c'_e \leftarrow c_e - l_e$
- $l'_e \leftarrow 0$

Removing Capacity Lower Bounds



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- $d'_u \leftarrow d_u - l_e$
- $d'_v \leftarrow d_v + l_e$
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- in old instance: flow is $f(e) \in [l_e, c_e] \implies f(e) - l_e \in [0, c_e - l_e]$
- in new instance: flow is $f(e) - l_e \in [0, c'_e]$

Survey Design

Input: integers $n, k \geq 1$ and $E \subseteq [n] \times [k]$

integers $0 \leq c_i \leq c'_i, \forall i \in [n]$

integers $0 \leq p_j \leq p'_j, \forall j \in [k]$

Survey Design

Input: integers $n, k \geq 1$ and $E \subseteq [n] \times [k]$

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Output: $E' \subseteq E$ s.t.

$$c_i \leq |\{j \in [k] : (i, j) \in E'\}| \leq c'_i, \quad \forall i \in [n]$$

$$p_j \leq |\{i \in [m] : (i, j) \in E'\}| \leq p'_j, \quad \forall j \in [k]$$

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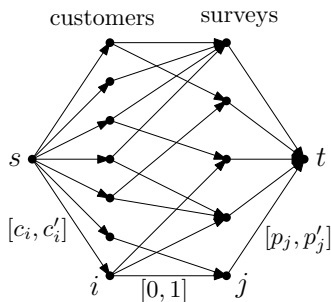
$$p_j \leq |\{i \in [m] : (i, j) \in E'\}| \leq p'_j, \quad \forall j \in [k]$$

Background

- $[n]$: customers, $[k]$: products
- $ij \in E$: customer i purchased product j and can do a survey
- every customer i needs to do between c_i and c'_i surveys
- every product j needs to collect between p_j and p'_j surveys

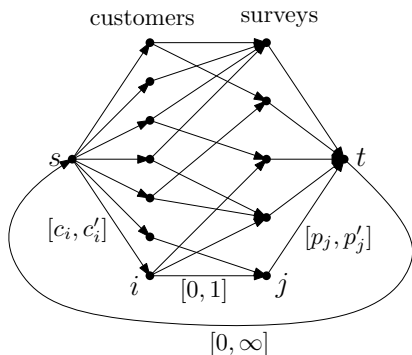
Reduction to Circulation

- vertices $\{s, t\} \uplus [n] \uplus [k]$,
- $(i, j) \in E$: (i, j) with bounds $[0, 1]$
- $\forall i$: (s, i) with bounds $[c_i, c'_i]$
- $\forall j$: (j, t) with bounds $[p_j, p'_j]$



Reduction to Circulation

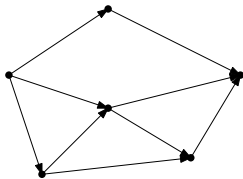
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- $\forall j$: (j, t) with bounds $[p_j, p'_j]$
- (t, s) with bounds $[0, \infty]$



Airline Scheduling

Input: a DAG $G = (V, E)$

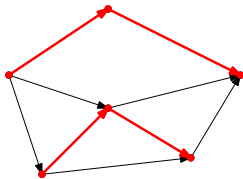
Output: the minimum number of disjoint paths in G to cover all vertices



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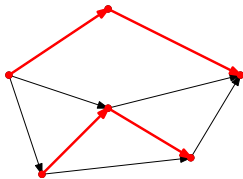
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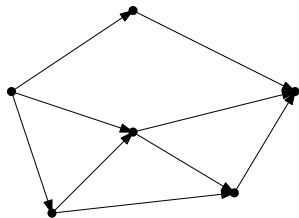
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Background

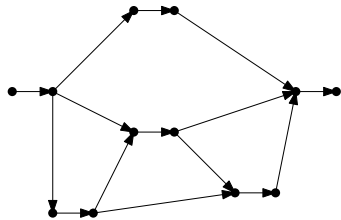
- vertex : a flight
- edge (u, v) : an aircraft that serves u can serve v immediately
- goal: minimize the number of aircrafts

Reduction to the Circulation Problem



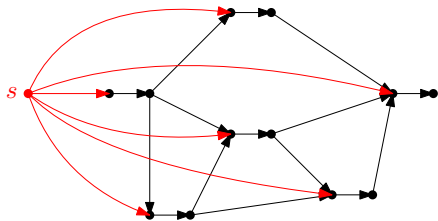
Reduction to the Circulation Problem

- split v into $(v_{\text{in}}, v_{\text{out}})$



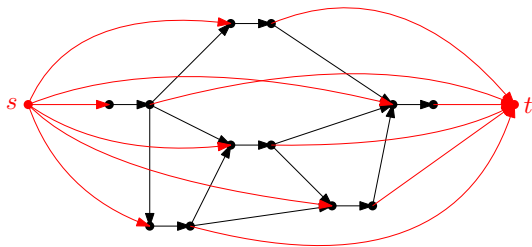
Reduction to the Circulation Problem

- split v into $(v_{\text{in}}, v_{\text{out}})$
- add s , and $(s, v_{\text{in}}), \forall v$



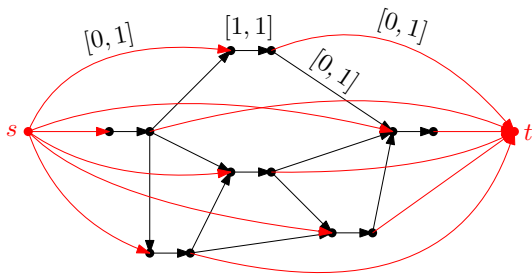
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- split v into $(v_{\text{in}}, v_{\text{out}})$
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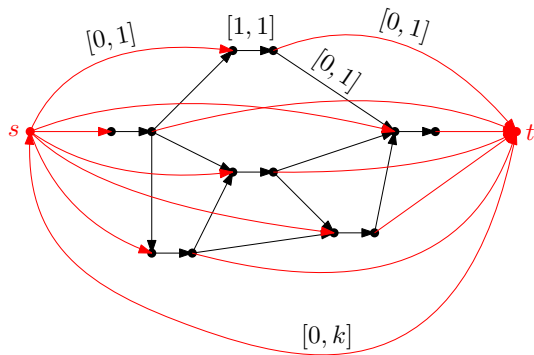
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- split v into $(v_{\text{in}}, v_{\text{out}})$
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- set lower and upper bounds



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- add $t \rightarrow s$ of capacity k



Reduction to the Circulation Problem

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- set lower and upper bounds
- add $t \rightarrow s$ of capacity k
- find minimum k s.t. instance is feasible

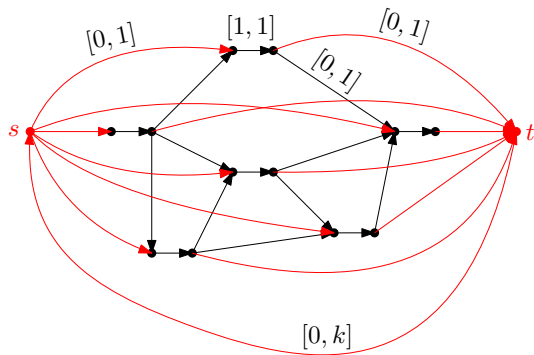


Image Segmentation

Input: A graph $G = (V, E)$, with edge costs $c \in \mathbb{Z}_{\geq 0}^E$
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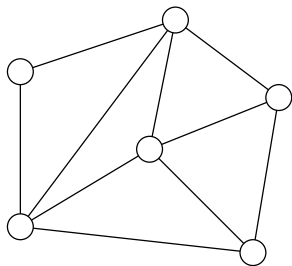
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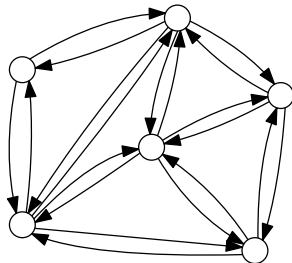
- a_v : the likelihood of v being a foreground pixel
- b_v : the likelihood of v being a background pixel
- $c_{(u,v)}$: the penalty for separating u and v
- need to maximize total reward - total penalty

Reduction to Network Flow



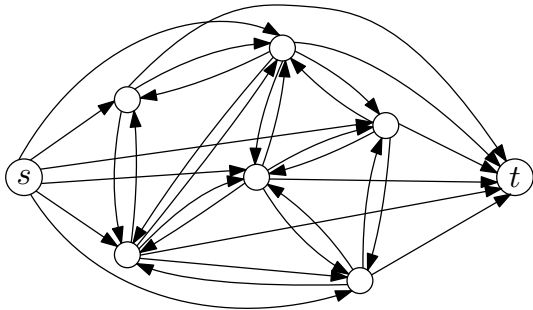
Reduction to Network Flow

- replace (u, v) with two anti-parallel arcs



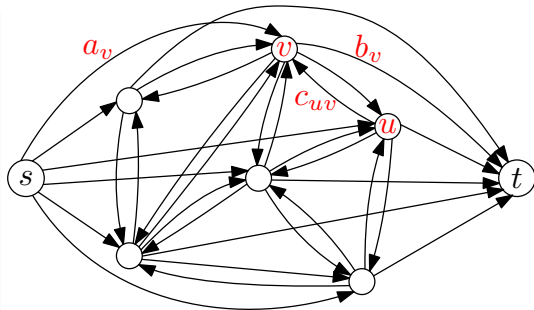
Reduction to Network Flow

- replace (u, v) with two anti-parallel arcs
- add source s and arcs $(s, v), \forall v$
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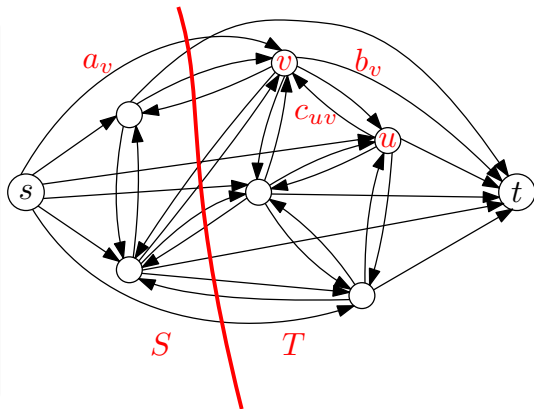
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- set capacities



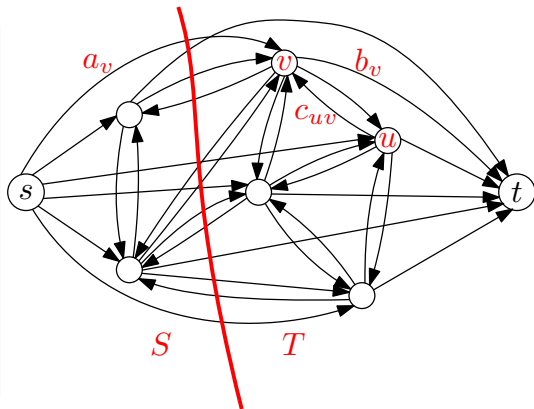
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- The cut value of $(S = \{s\} \cup A, \{t\} \cup B)$ is

$$\sum_{v \in B} a_v + \sum_{v \in A} b_v + \sum_{(u,v) \in E: |\{u,v\} \cap A| = 1} c_{(u,v)}$$

$$= \sum_{v \in V} (a_v + b_v) - \left(\sum_{v \in A} a_v + \sum_{v \in B} b_v - \sum_{(u,v) \in E: |\{u,v\} \cap A| = 1} c_{(u,v)} \right)$$

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- So, maximizing the objective of (A, B) is equivalent to minimizing the cut value.

Project Selection

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revenue on vertices: $p \in \mathbb{Z}^V$; p_v 's could be negative.

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- Goal: maximize the revenue subject to the precedence constraint.

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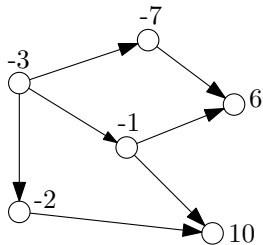
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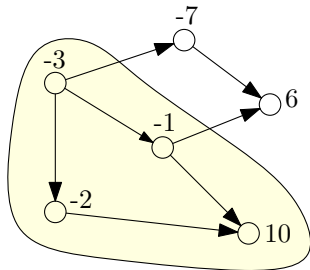
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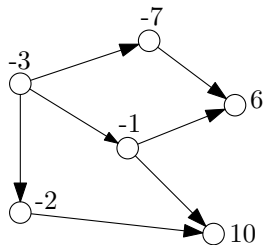
$$v \in B \implies u \in B, \quad \forall (u, v) \in E$$

Motivation

- Motivation: $(u, v) \in E$: u is a prerequisite of v , to select v , we must select u
- Goal: maximize the revenue subject to the precedence constraint.

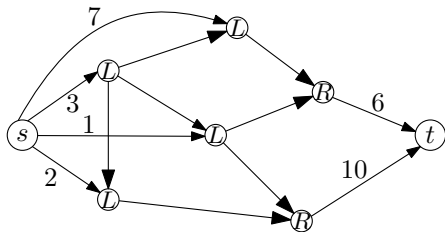


Reduction



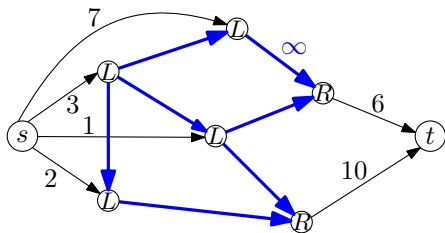
Reduction

- add source s and sink t
- $p_v < 0$: (s, v) of capacity $-p_v$
- $p_v > 0$: (v, t) of capacity p_v
- $L = \{v : p_v < 0\}$
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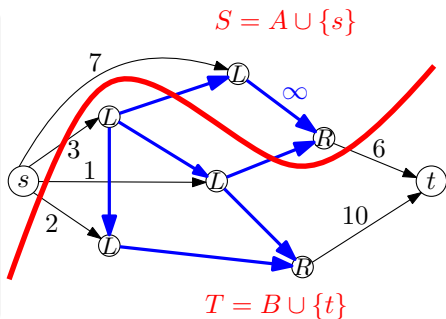
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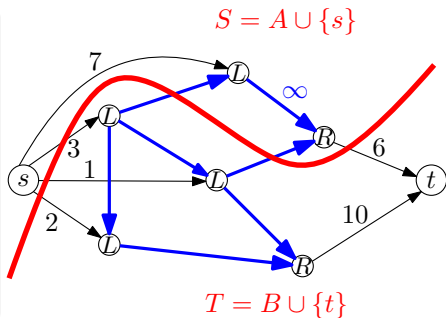
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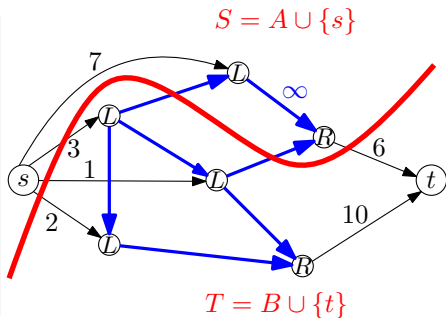
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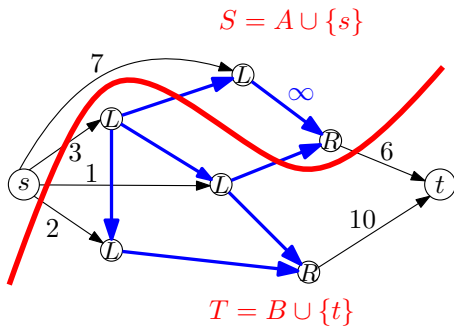
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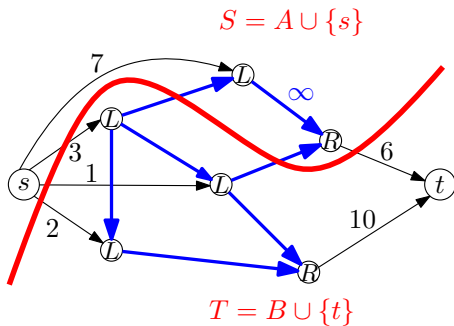


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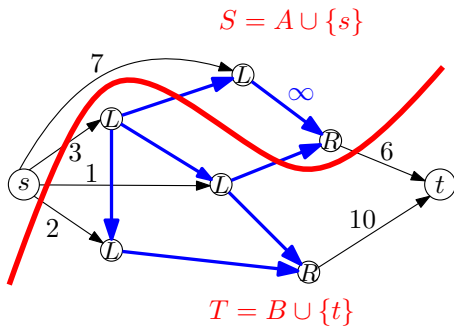
$$\begin{aligned} \sum_{v \in B \cap L} (-p_v) + \sum_{v \in A \cap R} p_v &= - \sum_{v \in B \cap L} p_v - \sum_{v \in B \cap R} p_v + \sum_{v \in R} p_v \\ &= \sum_{v \in R} p_v - \sum_{v \in B} p_v \end{aligned}$$



- B is a valid solution $\iff c(S, T) \neq \infty$



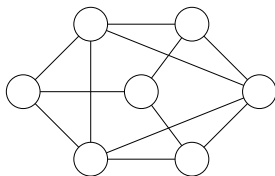
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- so, to maximize $\sum_{v \in B} p_v$, we need to minimize $c(S, T)$.

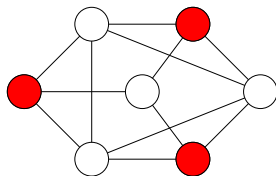
Maximum Independent Set Problem

Def. An **independent set** of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in I are adjacent in G .



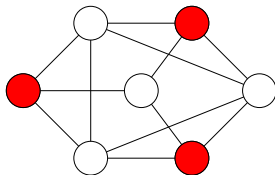
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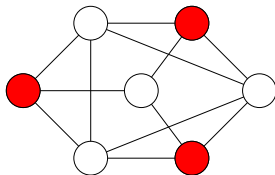
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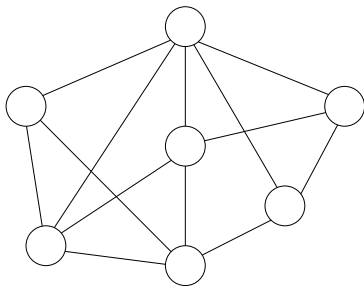
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- Maximum Independent Set is NP-hard

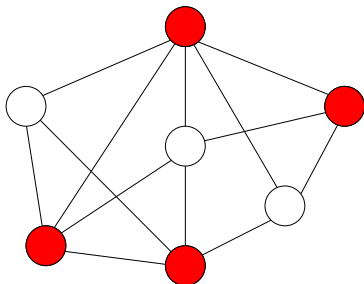
Vertex-Cover

Def. Given a graph $G = (V, E)$, a **vertex cover** of G is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$.



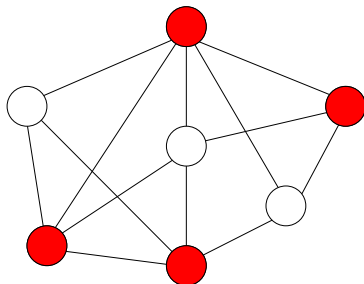
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Vertex-Cover Problem

Input: $G = (V, E)$ and integer k

Output: whether there is a vertex cover of G of size at most k

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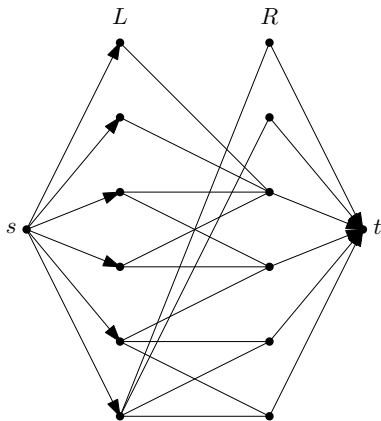
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Lemma In a **bipartite** graph $G = (L \cup R, E)$, we have

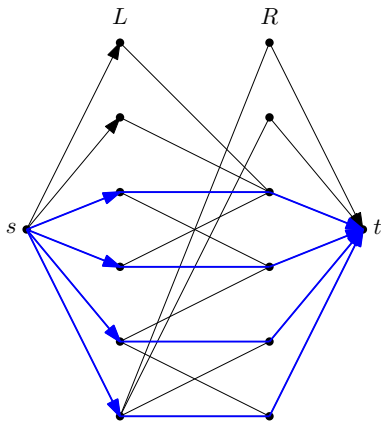
- **MaxM = MinVC** = $n - \text{MaxIS}$
- MaxM: size of maximum matching

- First, $\text{MaxM} \leq \text{MinVC}$, for any graph.

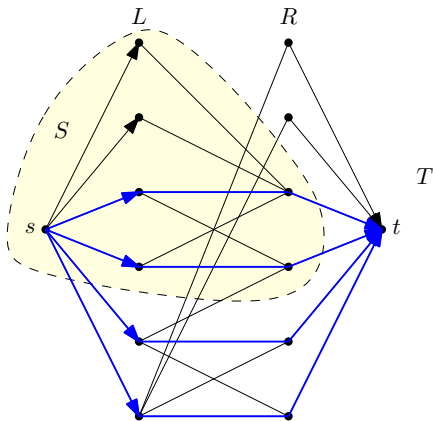
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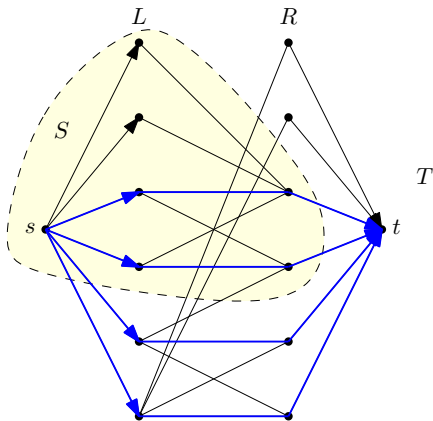


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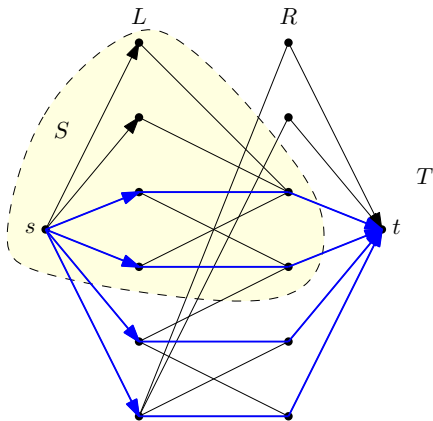
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- $(L \cap T) \cup (R \cap S)$ is a VC, whose size is $|M|$

