A 1.488 Approximation Algorithm for the Uncapacitated Facility Location Problem

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Abstract

We present a 1.488-approximation algorithm for the metric uncapacitated facility location (UFL) problem. Previously, the best algorithm was due to Byrka (2007). Byrka proposed an algorithm parametrized by γ and used it with $\gamma \approx 1.6774$. By either running his algorithm or the algorithm proposed by Jain, Mahdian and Saberi (STOC '02), Byrka obtained an algorithm that gives expected approximation ratio 1.5. We show that if γ is randomly selected, the approximation ratio can be improved to 1.488. Our algorithm cuts the gap with the 1.463 approximability lower bound by almost 1/3.

Keywords: Approximation, Facility Location Problem, Theory

1. Introduction

In this paper, we present an improved approximation algorithm for the (metric) uncapacitated facility location (UFL) problem. In the UFL problem, we are given a set of potential facility locations $\mathcal{F}$, each $i \in \mathcal{F}$ with a facility cost f_i, a set of clients $\mathcal{C}$ and a metric d over $\mathcal{F} \cup \mathcal{C}$. The goal is to find a subset $\mathcal{F}' \subseteq \mathcal{F}$ of locations to open facilities, to minimize the sum of the total facility cost and the connection cost. The total facility cost is $\sum_{i \in \mathcal{F}'} f_i$, and the connection cost is $\sum_{j \in \mathcal{C}} d(j, i_j)$, where i_j is the closest facility in $\mathcal{F}'$ to j.

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The UFL problem is NP-hard and has received a lot of attention. In 1982, Hochbaum [5] presented a greedy algorithm with $O(\log n)$-approximation guarantee. Shmoys, Tardos and Aardal [12] used the filtering technique of Lin and Vitter [10] to give a 3.16-approximation algorithm, which is the first constant factor approximation. After that, a large number of constant factor approximation algorithms were proposed [4, 9, 3, 7, 8, 11]. The current best known approximation ratio for the UFL problem is 1.50, given by Byrka [1].

On the negative side, Guha and Kuller [6] showed that there is no ρ-approximation for the UFL problem if $\rho < 1.463$, unless $\mathbf{NP} \subseteq \mathbf{DTIME}\left(n^{O(\log\log n)}\right)$. Later, Sviridenko [13] strengthened the result by changing the condition to "unless $\mathbf{NP} = \mathbf{P}$". Jain et al. [8] generalized the result to show that no (γ_f, γ_c)-bifactor approximation exists for $\gamma_c < 1 + 2e^{-\gamma_f}$ unless $\mathbf{NP} \subseteq \mathbf{DTIME}\left(n^{O(\log\log n)}\right)$. An algorithm is a (γ_f, γ_c)-approximation algorithm if the solution given by the algorithm has expected total cost at most $\gamma_f F^* + \gamma_c C^*$, where F^* and C^* are the facility and the connection cost of an optimal solution for the linear programming relaxation of the UFL problem described later, respectively.

Building on the work of Byrka [1], we give a 1.488-approximation algorithm for the UFL problem. Byrka presented an algorithm $A_1(\gamma)$ which gives the optimal bifactor approximation $(\gamma, 1+2e^{-\gamma})$ for $\gamma \geq \gamma_0 \approx 1.6774$. By either running $A_1(\gamma_0)$ or the $(1.11, 1.78)$-approximation algorithm A_2 proposed by Jain, Mahdian and Saberi [8], Byrka was able to give a 1.5-approximation algorithm. We show that the approximation ratio can be improved to 1.488 if γ is randomly selected. To be more specific, we show

Theorem 1. *There is a distribution over $[1, \infty) \cup \{\perp\}$ such that the following random algorithm for the UFL problem gives a solution whose expected cost is at most 1.488 times the cost of the optimal solution : we randomly choose a γ from the distribution; if $\gamma = \perp$, return the solution given by A_2; otherwise, return the solution given by $A_1(\gamma)$.*

Due to the $(\gamma, 1+2e^{-\gamma})$-hardness result given by [8], there is a hard instance for the algorithm $A_1(\gamma)$ for every γ. Roughly speaking, we show that a fixed instance can not be hard for two different γ's. Guided by this fact, we first give a bifactor approximation ratio for $A_1(\gamma)$ that depends on the input instance and then introduce a 0-sum game that characterizes the approximation ratio of our algorithm. The game is between an algorithm designer and an adversary. The algorithm designer plays either $A_1(\gamma)$ for some $\gamma \geq 1$ or A_2, while the adversary plays an input instance for the UFL problem. By giving an explicit (mixed) strategy for the algorithm designer, we show that the value of the game is at most 1.488.

In Section 2, we review the algorithm $A_1(\gamma)$, and then give our improvement in Section 3.

2. Review of the Algorithm $A_1(\gamma)$ in [1]

In this section, we review the $(\gamma, 1+2e^{-\gamma})$-bifactor approximation algorithm $A_1(\gamma)$ for $\gamma \geq \gamma_0 \approx 1.67736$ in [1].

In $A_1(\gamma)$ we first solve the following natural linear programming relaxation for the UFL problem.

$$\min \quad \sum_{i \in \mathcal{F}, j \in \mathcal{C}} d(i,j) x_{i,j} + \sum_{i \in \mathcal{F}} f_i y_i \quad \text{s.t.}$$

$$\sum_{i \in \mathcal{F}} x_{i,j} = 1 \quad \forall j \in \mathcal{C} \tag{1}$$

$$x_{i,j} - y_i \leq 0 \quad \forall i \in \mathcal{F}, j \in \mathcal{C} \tag{2}$$

$$x_{i,j}, y_i \geq 0 \quad \forall i \in \mathcal{F}, j \in \mathcal{C} \tag{3}$$

In the integer programming correspondent to the above LP relaxation, $x_{i,j}, y_i \in \{0,1\}$ for every $i \in \mathcal{F}$ and $j \in \mathcal{C}$. y_i indicates whether the facility i is open and $x_{i,j}$ indicates whether the client j is connected to the facility i. Equation (1) says that the client j must be connected to some facility and Inequality (2) says that a client j can be connected to a facility i only if i is open.

If the y-variables are fixed, x-variables can be assigned greedily in the following way. Initially, $x_{i,j} = 0$. For each client $j \in \mathcal{C}$, run the following steps.

Sort facilities by their distances to j; then for each facility i in the order, assign $x_{ij} = y_i$ if $\sum_{i' \in \mathcal{F}} x_{i',j} + y_i \le 1$ and $x_{i,j} = 1 - \sum_{i'} x_{i',j}$ otherwise.

After obtaining a solution (x, y), we modify it by scaling the y-variables up by a constant factor $\gamma \ge 1$. Let $\overline{y}$ be the scaled vector of y-variables. We reassign x-variables using the above greedy process to obtain a new solution $(\overline{x}, \overline{y})$.

Without loss of generality, we can assume that the following conditions hold for every $i \in \mathcal{C}$ and $j \in \mathcal{F}$:

1. $x_{i,j} \in \{0, y_i\}$;
2. $\overline{x}_{i,j} \in \{0, \overline{y}_i\}$;
3. $\overline{y}_i \le 1$.

Indeed, the above conditions can be guaranteed by splitting facilities. To guarantee the first condition, we split i into 2 co-located facilities i' and i'' and let $x_{i',j} = y_{i'} = x_{i,j}, y_{i''} = y_i - x_{i,j}$ and $x_{i'',j} = 0$, if we find some facility $i \in \mathcal{F}$ and client $j \in \mathcal{C}$ with $0 < x_{i,j} < y_i$. The other x variables associated with i' and i'' can be assigned naturally. We update $\overline{x}$ and $\overline{y}$ variables accordingly. Similarly, we can guarantee the second condition. To guarantee the third condition, we can split a facility i into 2 co-located facilities i' and i'' with $\overline{y}_{i'} = 1$ and $\overline{y}_{i''} = \overline{y}_i - 1$, if we find some facility $i \in \mathcal{F}$ with $\overline{y}_i > 1$.

Definition 2 (volume). For some subset $\mathcal{F}' \subseteq \mathcal{F}$ of facilities, define the *volume* of $\mathcal{F}'$, denoted by $\mathsf{vol}(\mathcal{F}')$, to be the sum of y_i over all facilities $i \in \mathcal{F}'$. i.e, $\mathsf{vol}(\mathcal{F}') = \sum_{i \in \mathcal{F}'} \overline{y}_i$.

Definition 3 (close and distant facilities). For a client $j \in \mathcal{C}$, we say a facility i is one of its *close facilities* if $\overline{x}_{i,j} > 0$. If $\overline{x}_{i,j} = 0$, but $x_{i,j} > 0$, then we say i is a *distant facility* of client j. Let $\mathcal{F}_j^C$ and $\mathcal{F}_j^D$ be the set of close and distant facilities of j, respectively. Let $\mathcal{F}_j = \mathcal{F}_j^C \cup \mathcal{F}_j^D$.

Definition 4. For a client $j \in \mathcal{C}$ and a subset $\mathcal{F}' \subseteq \mathcal{F}$ of facilities, define $d(j, \mathcal{F}')$ to be the average distance of j to facilities in $\mathcal{F}'$, with respect to the weights $\overline{y}$. Recalling that $\overline{y}$ is a scaled vector of y, the average distances are also with respect to the weights y. i.e,

$$d(j, \mathcal{F}') = \frac{\sum_{i \in \mathcal{F}'} \overline{y}_i d(j,i)}{\sum_{i \in \mathcal{F}'} \overline{y}_i} = \frac{\sum_{i \in \mathcal{F}'} y_i d(j,i)}{\sum_{i \in \mathcal{F}'} y_i}.$$

Definition 5 **($d^C_{\mathsf{ave}}(j), d^D_{\mathsf{ave}}(j), d_{\mathsf{ave}}(j)$ and $d^C_{\max}(j)$).** For a client $j \in \mathcal{C}$, define $d^C_{\mathsf{ave}}(j), d^D_{\mathsf{ave}}(j)$ and $d_{\mathsf{ave}}(j)$ to be the average distance from j to $\mathcal{F}^C_j, \mathcal{F}^D_j$ and $\mathcal{F}_j$ respectively. i.e, $d^C_{\mathsf{ave}}(j) = d(j, \mathcal{F}^C_j), d^D_{\mathsf{ave}}(j) = d(j, \mathcal{F}^D_j)$ and $d_{\mathsf{ave}}(j) = d(j, \mathcal{F}_j)$. Define $d^C_{\max}(j)$ to be maximum distance from j to a facility in $\mathcal{F}^C_j$.

By definition, $d_{\mathsf{ave}}(j)$ is the connection cost of j in the optimal fractional solution. See Figure 1 for illustration of the above 3 definitions. The following claims hold by the definitions of the corresponding quantities, and will be used repeatedly later :

Claim 6. *$d^C_{\mathsf{ave}}(j) \le d^C_{\max}(j) \le d^D_{\mathsf{ave}}(j)$ and $d^C_{\mathsf{ave}}(j) \le d_{\mathsf{ave}}(j) \le d^D_{\mathsf{ave}}(j)$, for all clients $j \in \mathcal{C}$.*

Claim 7. *$d_{\mathsf{ave}}(j) = \frac{1}{\gamma} d^C_{\mathsf{ave}}(j) + \frac{\gamma - 1}{\gamma} d^D_{\mathsf{ave}}(j)$, for all clients $j \in \mathcal{C}$.*

Claim 8. *$\mathsf{vol}(\mathcal{F}^C_j) = 1, \mathsf{vol}(\mathcal{F}^D_j) = \gamma - 1$ and $\mathsf{vol}(\mathcal{F}_j) = \gamma$, for all clients $j \in \mathcal{C}$.*

Recall that y_i indicates that whether the facility i is open. If we are aiming at a (γ, γ)-bifactor approximation, we can open i with probability $\gamma y_i = \overline{y}_i$. Then, the expected opening cost is exactly γ times that of the optimal fractional solution. If by luck the sets $\mathcal{F}^C_j, j \in \mathcal{C}$ are disjoint, the following simple algorithm gives a $(\gamma, 1)$-bifactor approximation. Open exactly 1 facility in $\mathcal{F}^C_j$, with $\overline{y}_i$ being the probability of opening i. (Recall that $\sum_{i \in \mathcal{F}^C_j} \overline{y}_i = \mathsf{vol}(\mathcal{F}^C_j) = 1$.) Connect each client j to its closest open facility. This is indeed a $(\gamma, 1)$-bifactor approximation since the expected connection cost of j given by the algorithm is $d^C_{\mathsf{ave}}(j) \le d_{\mathsf{ave}}(j)$.

In general, the sets $\mathcal{F}^C_j, j \in \mathcal{C}$ may not be disjoint. In this case, we can not randomly select 1 open facility from every $\mathcal{F}^C_j$, since a facility i belonging to two different $\mathcal{F}^C_j$'s would be open with probability more than $\overline{y}_i$. To overcome this problem, we shall select a subset $\mathcal{C}' \subseteq \mathcal{C}$ of clients such that the sets $\mathcal{F}^C_j, j \in \mathcal{C}'$ are disjoint. We randomly open a facility in $\mathcal{F}^C_j$ only for facilities $j \in \mathcal{C}'$. To

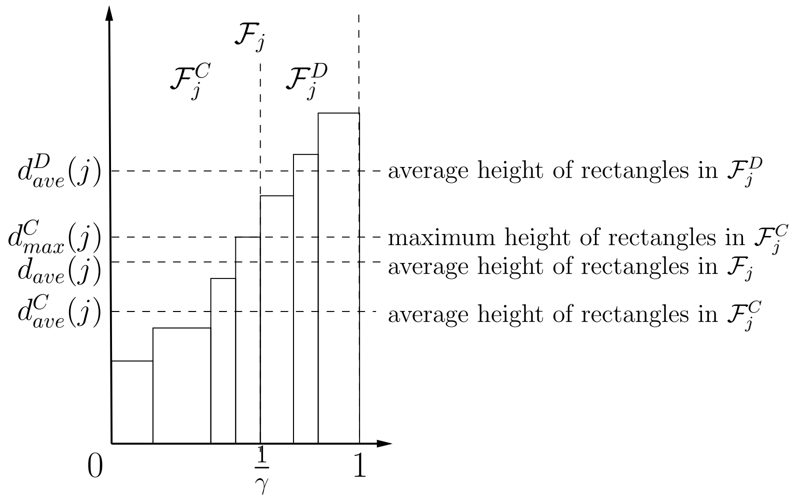

Figure 1: Illustration for $d^C_{\mathsf{ave}}(j), d^D_{\mathsf{ave}}(j), d_{\mathsf{ave}}(j), d^C_{\max}(j), \mathcal{F}_j, \mathcal{F}^C_j$ and $\mathcal{F}^D_j$. Each rectangle is a facility : its width corresponds to y value and its height corresponds to its distance to j. The heights of the rectangles are non-decreasing from the left to the right. The average heights shown in the figure are weighted with respect to the widths of rectangles.

allow us to bound the expected connection cost of clients not in $\mathcal{C}'$, $\mathcal{C}'$ also has the property that every client $j \in \mathcal{C}\backslash\mathcal{C}'$ is close to some client in $\mathcal{C}'$. The exact definition of being close is given later in Claim 10.

The subset of clients $\mathcal{C}'$ is selected greedily in the following way. Initially let $\mathcal{C}'' = \mathcal{C}$ and $\mathcal{C}' = \emptyset$. While $\mathcal{C}''$ is not empty, select the client j in $\mathcal{C}''$ with the minimum $d^C_{\mathsf{ave}}(j) + d^C_{\max}(j)$, add j to $\mathcal{C}'$ and remove j and all clients j' satisfying $\mathcal{F}^C_j \cap \mathcal{F}^C_{j'} \neq \emptyset$ from $\mathcal{C}''$. The following two claims hold for $\mathcal{C}'$.

Claim 9. *$\mathcal{F}^C_j \cap \mathcal{F}^C_{j'} = \emptyset$, for two distinct clients $j, j' \in \mathcal{C}'$.*

As already mentioned, this claim allows us to randomly open 1 facility from each set $\mathcal{F}^C_j, j \in \mathcal{C}'$.

Claim 10. *For every client $j \notin \mathcal{C}'$, there exists a client $j' \in \mathcal{C}'$ such that $\mathcal{F}^C_j \cap \mathcal{F}^C_{j'} \neq \emptyset$ and $d^C_{\mathsf{ave}}(j') + d^C_{\max}(j') \le d^C_{\mathsf{ave}}(j) + d^C_{\max}(j)$.*

Notice that $\mathcal{F}^C_j \cap \mathcal{F}^C_{j'} \neq \emptyset$ implies $d(j,j') \le d^C_{\max}(j) + d^C_{\max}(j')$. This property and that $d^C_{\mathsf{ave}}(j') + d^C_{\max}(j') \le d^C_{\mathsf{ave}}(j) + d^C_{\max}(j)$ will be used to bound the expected connection cost of j.

Definition 11 (cluster center). For a client $j \notin \mathcal{C}'$, the client $j' \in \mathcal{C}$ that makes Claim 10 hold is called the *cluster center* of j.

We now randomly round the fractional solution $(\overline{x}, \overline{y})$. As we already mentioned, for each $j \in \mathcal{C}'$, we open exactly one of its close facilities randomly with probabilities $\overline{y}_i$. For each facility i that is not a close facility of any client in $\mathcal{C}'$, we open it independently with probability $\overline{y}_i$. Connect each client j to its closest open facility and let C_j be its connection cost.

Since i is open with probability exactly $\overline{y}_i$ for every facility $i \in \mathcal{F}$, the expected opening cost of the solution given by the above algorithm is exactly γ times that of the optimal fractional solution. If $j \in \mathcal{C}'$ then $\mathbb{E}[C_j] = d^C_{\mathsf{ave}}(j) \leq d_{\mathsf{ave}}(j)$.

Byrka in [1] showed that for every client $j \notin \mathcal{C}'$,

1. The probability that some facility in $\mathcal{F}^C_j$ (resp., $\mathcal{F}^D_j$) is open is at least $1 - e^{-\mathsf{vol}(\mathcal{F}^C_j)} = 1 - e^{-1}$(resp., $1 - e^{-\mathsf{vol}(\mathcal{F}^D_j)} = 1 - e^{-(\gamma-1)}$), and under the condition that the event happens, the expected distance between j and the closest open facility in $\mathcal{F}^C_j$ (resp., $\mathcal{F}^D_j$) is at most $d^C_{\mathsf{ave}}(j)$ (resp., $d^D_{\mathsf{ave}}(j)$);
2. $d(j, \mathcal{F}^C_{j'} \backslash \mathcal{F}_j) \leq d^D_{\mathsf{ave}}(j) + d^C_{\max}(j) + d^C_{\mathsf{ave}}(j)$(recall that $d(j, \mathcal{F}^C_{j'} \backslash \mathcal{F}_j)$ is the weighted average distance from j to facilities in $\mathcal{F}^C_{j'} \backslash \mathcal{F}_j$), where j' is the cluster center of j; or equivalently, under the condition that there is no open facility in $\mathcal{F}_j$, the expected distance between j and the unique open facility in $\mathcal{F}^C_{j'}$ is at most $d^D_{\mathsf{ave}}(j) + d^C_{\max}(j) + d^C_{\mathsf{ave}}(j)$.

Since $d^C_{\mathsf{ave}}(j) \leq d^D_{\mathsf{ave}}(j) \leq d^D_{\mathsf{ave}}(j) + d^C_{\max}(j) + d^C_{\mathsf{ave}}(j)$, we have

$$\begin{aligned}
\mathbb{E}[C_j] &\leq (1 - e^{-1}) d^C_{\mathsf{ave}}(j) + e^{-1}(1 - e^{-(\gamma-1)}) d^D_{\mathsf{ave}}(j) \\
&\qquad + e^{-1} e^{-(\gamma-1)} (d^D_{\mathsf{ave}}(j) + d^C_{\max}(j) + d^C_{\mathsf{ave}}(j)) \\
&= (1 - e^{-1} + e^{-\gamma}) d^C_{\mathsf{ave}}(j) + e^{-1} d^D_{\mathsf{ave}}(j) + e^{-\gamma} d^C_{\max}(j) \\
&\leq (1 - e^{-1} + e^{-\gamma}) d^C_{\mathsf{ave}}(j) + (e^{-1} + e^{-\gamma}) d^D_{\mathsf{ave}}(j). \qquad (4)
\end{aligned}$$

Notice that the connection cost of j in the optimal fractional solution is $d_{\mathsf{ave}}(j) = \dfrac{1}{\gamma} d^C_{\mathsf{ave}}(j) + \dfrac{\gamma - 1}{\gamma} d^D_{\mathsf{ave}}(j)$. We compute the maximum ratio between $(1 - e^{-1} +$

$e^{-\gamma})d^C_{\mathsf{ave}}(j) + (e^{-1}+e^{-\gamma})d^D_{\mathsf{ave}}(j)$ and $\frac{1}{\gamma}d^C_{\mathsf{ave}}(j) + \frac{\gamma-1}{\gamma}d^D_{\mathsf{ave}}(j)$. Since $d^C_{\mathsf{ave}}(j) \le d^D_{\mathsf{ave}}(j)$, the ratio is maximized when $d^C_{\mathsf{ave}}(j) = d^D_{\mathsf{ave}}(j) > 0$ or $d^D_{\mathsf{ave}}(j) > d^C_{\mathsf{ave}}(j) = 0$. For $\gamma \ge \gamma_0$, the maximum ratio is achieved when $d^C_{\mathsf{ave}}(j) = d^D_{\mathsf{ave}}(j) > 0$, in which case the maximum is $1 + 2e^{-\gamma}$. Thus, the algorithm $A_1(\gamma_0)$ gives a $(\gamma_0 \approx 1.67736, 1 + 2e^{-\gamma_0} \approx 1.37374)$-bifactor approximation.[2]

3. A 1.488 Approximation Algorithm for the UFL Problem

In this section, we give our approximation algorithm for the UFL problem. Our algorithm is also based on the combination of $A_1(\gamma)$ and A_2. However, instead of running $A_1(\gamma)$ for a fixed γ, we randomly select γ from some distribution described later.

To understand why this approach can reduce the approximation ratio, we list some necessary conditions that the upper bound in (4) is tight.

1. The facilities in $\mathcal{F}_j$ have tiny weights. In other words, $\max_{i \in \mathcal{F}_j} \overline{y}_i$ tends to 0. Moreover, all these facilities were independently sampled in the algorithm. These conditions are necessary to tighten the $1 - e^{-1}$ (resp., $1 - e^{-(\gamma-1)}$) upper bound for the probability that at least 1 facility in $\mathcal{F}^C_j$ (resp., $\mathcal{F}^D_j$) is open.
2. The distances from j to all the facilities in $\mathcal{F}^C_j$ (resp., $\mathcal{F}^D_j$) are the same. Otherwise, the expected distance from j to the closest open facility in $\mathcal{F}^C_j$ (resp., $\mathcal{F}^D_j$), under the condition that it exists, is strictly smaller than $d^C_{\mathsf{ave}}(j)$ (resp., $d^D_{\mathsf{ave}}(j)$).
3. $d^C_{\max}(j) = d^D_{\mathsf{ave}}(j)$. This is also required since we used $d^D_{\mathsf{ave}}(j)$ as an upper bound of $d^C_{\max}(j)$ in (4).

To satisfy all the above conditions, the distances from j to $\mathcal{F}_j$ must be distributed as follows. $1/(\gamma + \epsilon)$ fraction of facilities in $\mathcal{F}_j$ (the fraction is with

[2]Byrka's analysis in [1] was slightly different; it used some variables from the dual LP. Later, Byrka et al. [2] gave an analysis without using the dual LP, which is the one we explain in our paper.

respect to the weights y_i.) have distances a to j, and the other $1 - 1/(\gamma + \epsilon)$ fraction have distances $b \geq a$ to j. For ϵ tending to 0, $d^C_{\mathsf{ave}}(j) = a$ and $d^C_{\max}(j) = d^D_{\mathsf{ave}}(j) = b$.

As discussed earlier, if $a = b$, then $\mathbb{E}[C_j]/d_{\mathsf{ave}}(j) \leq 1 + 2e^{-\gamma}$. Intuitively, the bad cases should have $a \ll b$. However, if we replace γ with $\gamma + 1.01\epsilon$ (say), then $d^C_{\max}(j)$ will equal $d^C_{\mathsf{ave}}(j) = a$, instead of $d^D_{\mathsf{ave}}(j) = b$. Thus, we can greatly reduce the approximation ratio if the distributions for all j's are of the above form.

Hence, using only two different γ's, we are already able to make an improvement. To give a better analysis, we first give in Section 3.1 an upper bound on $\mathbb{E}[C_j]$, in terms of the distribution of distances from j to $\mathcal{F}_j$, not just $d^C_{\mathsf{ave}}(j)$ and $d^D_{\mathsf{ave}}(j)$ and then give in Section 3.2 an explicit distribution for γ by introducing a 0-sum game.

3.1. Upper-Bounding the Expected Connection Cost of a Client

We bound $\mathbb{E}[C_j]$ in this subsection. It suffices to assume $j \notin \mathcal{C}'$, since we can think of a client $j \in \mathcal{C}'$ as a client $j \notin \mathcal{C}'$ which has a co-located client $j' \in \mathcal{C}'$. Similar to [1], we first give an upper bound on $d(j, \mathcal{F}^C_{j'} \backslash \mathcal{F}_j)$ in Lemma 12. The bound and the proof are the same as the counterparts in [1], except that we made a slight improvement. The improvement is not essential to the final approximation ratio; however, it will simplify the analytical proof in Section 3.2.

Lemma 12. *For some client $j \notin \mathcal{C}'$, let j' be the cluster center of j. So $j' \in \mathcal{C}', \mathcal{F}^C_j \cap \mathcal{F}^C_{j'} \neq \emptyset$ and $d^C_{\mathsf{ave}}(j') + d^C_{\max}(j') \leq d^C_{\mathsf{ave}}(j) + d^C_{\max}(j)$. We have,*

$$d(j, \mathcal{F}^C_{j'} \backslash \mathcal{F}_j) \leq (2 - \gamma) d^C_{\max}(j) + (\gamma - 1) d^D_{\mathsf{ave}}(j) + d^C_{\max}(j') + d^C_{\mathsf{ave}}(j'). \quad (5)$$

Proof. Figure 2 illustrates the sets of facilities we are going to use. Figure 3 shows the dependence of equations we shall prove and can be viewed as the outline of the proof.

If $d(j, j') \leq (2 - \gamma) d^C_{\max}(j) + (\gamma - 1) d^D_{\mathsf{ave}}(j) + d^C_{\mathsf{ave}}(j')$, the remaining $d^C_{\max}(j')$ is enough for the distance between j' and any facility in $\mathcal{F}^C_{j'}$. So, we will assume

$$d(j, j') \geq (2 - \gamma) d^C_{\max}(j) + (\gamma - 1) d^D_{\mathsf{ave}}(j) + d^C_{\mathsf{ave}}(j'). \quad (6)$$

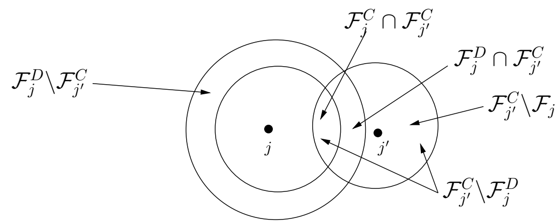

Figure 2: Sets of facilities used in the proof of lemma 12.

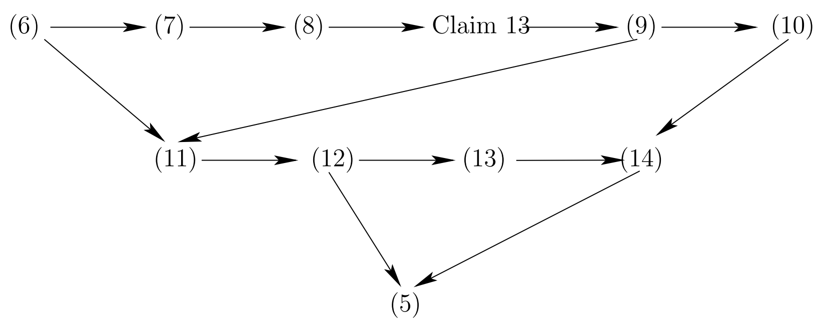

Figure 3: The dependence graph for the equations in the proof of lemma 12. An equation is implied by its predecessor-equations.

Since $d^D_{\mathsf{ave}}(j) \geq d^C_{\max}(j)$ and $\gamma - 1 \geq 0$, (6) implies

$$d(j, j') \geq d^C_{\max}(j) + d^C_{\mathsf{ave}}(j'). \tag{7}$$

By triangle inequality,

$$d(j', \mathcal{F}_j^C \cap \mathcal{F}_{j'}^C) \geq d(j, j') - d(j, \mathcal{F}_j^C \cap \mathcal{F}_{j'}^C),$$

and by (7) and $d(j, \mathcal{F}_j^C \cap \mathcal{F}_{j'}^C) \leq d^C_{\max}(j)$,

$$d(j', \mathcal{F}_j^C \cap \mathcal{F}_{j'}^C) \geq d^C_{\max}(j) + d^C_{\mathsf{ave}}(j') - d^C_{\max}(j) = d^C_{\mathsf{ave}}(j'). \tag{8}$$

Claim 13. *If $d(j', \mathcal{F}_j^D \cap \mathcal{F}_{j'}^C) \geq d^C_{\mathsf{ave}}(j')$, then (5) holds.*

Proof. Notice that $d^C_{\mathsf{ave}}(j') = d(j', \mathcal{F}_{j'}^C)$ and $\mathcal{F}_{j'}^C$ is the union of the following 3 disjoint set : $\mathcal{F}_j^C \cap \mathcal{F}_{j'}^C, \mathcal{F}_j^D \cap \mathcal{F}_{j'}^C, \mathcal{F}_{j'}^C \backslash \mathcal{F}_j$. If $d(j', \mathcal{F}_j^D \cap \mathcal{F}_{j'}^C) \geq d^C_{\mathsf{ave}}(j')$, then by (8), we have $d(j', \mathcal{F}_{j'}^C \backslash \mathcal{F}_j) \leq d^C_{\mathsf{ave}}(j')$. Then, by triangle inequality,

$$d(j, \mathcal{F}_{j'}^C \backslash \mathcal{F}_j) \leq d(j, j') + d(j', \mathcal{F}_{j'}^C \backslash \mathcal{F}_j) \leq d(j, j') + d^C_{\mathsf{ave}}(j').$$

Since $\mathcal{F}_j^C \cap \mathcal{F}_{j'}^C \neq \emptyset$,

$$\begin{aligned}
d(j, \mathcal{F}_{j'}^C \backslash \mathcal{F}_j) &\leq d_{\max}^C(j) + d_{\max}^C(j') + d_{\mathsf{ave}}^C(j') \\
&\leq (2-\gamma) d_{\max}^C(j) + (\gamma - 1) d_{\mathsf{ave}}^D(j) + d_{\max}^C(j') + d_{\mathsf{ave}}^C(j').
\end{aligned}$$

This proves the claim. $\square$

So, we can also assume

$$d(j', \mathcal{F}_j^D \cap \mathcal{F}_{j'}^C) = d_{\mathsf{ave}}^C(j') - z \tag{9}$$

for some positive z. Let $\hat{y} = \mathsf{vol}(\mathcal{F}_j^D \cap \mathcal{F}_{j'}^C)$. Notice that $\hat{y} \leq \max\{\gamma - 1, 1\}$. Since $d_{\mathsf{ave}}^C(j') = \hat{y} d(j', \mathcal{F}_j^D \cap \mathcal{F}_{j'}^C) + (1 - \hat{y}) d(j', \mathcal{F}_{j'}^C \backslash \mathcal{F}_j^D)$, we have

$$d(j', \mathcal{F}_{j'}^C \backslash \mathcal{F}_j^D) = d_{\mathsf{ave}}^C(j') + \frac{\hat{y}}{1 - \hat{y}} z. \tag{10}$$

By (6), (9) and triangle inequality, we have

$$\begin{aligned}
d(j, \mathcal{F}_j^D \cap \mathcal{F}_{j'}^C) &\geq d(j, j') - d(j', \mathcal{F}_{j'}^C \cap \mathcal{F}_j^D) \\
&\geq (2-\gamma) d_{\max}^C(j) + (\gamma - 1) d_{\mathsf{ave}}^D(j) + d_{\mathsf{ave}}^C(j') - (d_{\mathsf{ave}}^C(j') - z) \\
&= d_{\mathsf{ave}}^D(j) - (2-\gamma)(d_{\mathsf{ave}}^D(j) - d_{\max}^C(j)) + z. \qquad (11)
\end{aligned}$$

Noticing that $d_{\mathsf{ave}}^D(j) = \dfrac{\hat{y}}{\gamma - 1} d(j, \mathcal{F}_j^D \cap \mathcal{F}_{j'}^C) + \dfrac{\gamma - 1 - \hat{y}}{\gamma - 1} d(j, \mathcal{F}_j^D \backslash \mathcal{F}_{j'}^C)$, we have

$$-\frac{\hat{y}}{\gamma - 1} \left(d(j, \mathcal{F}_j^D \cap \mathcal{F}_{j'}^C) - d_{\mathsf{ave}}^D(j) \right) = \frac{\gamma - 1 - \hat{y}}{\gamma - 1} \left(d(j, \mathcal{F}_j^D \backslash \mathcal{F}_{j'}^C) - d_{\mathsf{ave}}^D(j) \right).$$

Thus,

$$d(j, \mathcal{F}_j^D \backslash \mathcal{F}_{j'}^C) = d_{\mathsf{ave}}^D(j) - \frac{\hat{y}}{\gamma - 1 - \hat{y}} \left(d(j, \mathcal{F}_j^D \cap \mathcal{F}_{j'}^C) - d_{\mathsf{ave}}^D(j) \right).$$

Then, by $d_{\max}^C(j) \leq d(j, \mathcal{F}_j^D \backslash \mathcal{F}_{j'}^C)$ and (11),

$$d_{\max}^C(j) \leq d(j, \mathcal{F}_j^D \backslash \mathcal{F}_{j'}^C) \leq d_{\mathsf{ave}}^D(j) - \frac{\hat{y}}{\gamma - 1 - \hat{y}} \left(z - (2-\gamma)(d_{\mathsf{ave}}^D(j) - d_{\max}^C(j)) \right).$$

So,

$$d_{\mathsf{ave}}^D(j) - d_{\max}^C(j) \geq \frac{\hat{y}}{\gamma - 1 - \hat{y}} \left(z - (2-\gamma)(d_{\mathsf{ave}}^D(j) - d_{\max}^C(j)) \right),$$

and since $1+\dfrac{(2-\gamma)\hat{y}}{\gamma-1-\hat{y}}=\dfrac{(\gamma-1)(1-\hat{y})}{\gamma-1-\hat{y}}\geq 0$,

$$d^D_{\mathsf{ave}}(j)-d^C_{\max}(j)\geq \frac{\hat{y}}{\gamma-1-\hat{y}}z\Big/\left(1+\frac{(2-\gamma)\hat{y}}{\gamma-1-\hat{y}}\right)=\frac{\hat{y}z}{(\gamma-1)(1-\hat{y})}. \tag{12}$$

By triangle inequality,

$$d(j',\mathcal{F}^C_j\cap\mathcal{F}^C_{j'})\geq d(j,j')-d(j,\mathcal{F}^C_j\cap\mathcal{F}^C_{j'}),$$

and by (6) and $d(j,\mathcal{F}^C_j\cap\mathcal{F}^C_{j'})\leq d^C_{\max}(j)$,

$$\begin{aligned}d(j',\mathcal{F}^C_j\cap\mathcal{F}^C_{j'})&\geq(2-\gamma)d^C_{\max}(j)+(\gamma-1)d^D_{\mathsf{ave}}(j)+d^C_{\mathsf{ave}}(j')-d^C_{\max}(j)\\&=(\gamma-1)(d^D_{\mathsf{ave}}(j)-d^C_{\max}(j))+d^C_{\mathsf{ave}}(j').\end{aligned}$$

Then by (12), we have

$$d(j',\mathcal{F}^C_j\cap\mathcal{F}^C_{j'})\geq\frac{\hat{y}}{1-\hat{y}}z+d^C_{\mathsf{ave}}(j'). \tag{13}$$

Notice that $\mathcal{F}^C_{j'}\backslash\mathcal{F}^D_j$ is the union of the following two sets : $\mathcal{F}^C_j\cap\mathcal{F}^C_{j'}$ and $\mathcal{F}^C_{j'}\backslash\mathcal{F}_j$. Combining (10) and (13), we have

$$d(j',\mathcal{F}^C_{j'}\backslash\mathcal{F}_j)\leq d^C_{\mathsf{ave}}(j')+\frac{\hat{y}}{1-\hat{y}}z. \tag{14}$$

So,

$$\begin{aligned}d(j,\mathcal{F}^C_{j'}\backslash\mathcal{F}_j)&\leq d^C_{\max}(j)+d^C_{\max}(j')+d(j',\mathcal{F}^C_{j'}\backslash\mathcal{F}_j)\\&=(2-\gamma)d^C_{\max}(j)+(\gamma-1)d^C_{\max}(j)+d^C_{\max}(j')+d(j',\mathcal{F}^C_{j'}\backslash\mathcal{F}_j)\end{aligned}$$

by (12) and (14),

$$\begin{aligned}&\leq(2-\gamma)d^C_{\max}(j)+(\gamma-1)\left(d^D_{\mathsf{ave}}(j)-\frac{\hat{y}z}{(\gamma-1)(1-\hat{y})}\right)\\&\quad+d^C_{\max}(j')+d^C_{\mathsf{ave}}(j')+\frac{\hat{y}z}{1-\hat{y}}\\&=(2-\gamma)d^C_{\max}(j)+(\gamma-1)d^D_{\mathsf{ave}}(j)+d^C_{\max}(j')+d^C_{\mathsf{ave}}(j').\end{aligned}$$

$\square$

Lemma 14.

$$d(j, \mathcal{F}_{j'}^C \backslash \mathcal{F}_j) \le \gamma d_{\mathsf{ave}}(j) + (3-\gamma) d_{\max}^C(j). \tag{15}$$

Proof. Noticing that $d_{\max}^C(j') + d_{\mathsf{ave}}^C(j') \le d_{\max}^C(j) + d_{\mathsf{ave}}^C(j)$, the proof is straightforward.

$$\begin{aligned} d(j, \mathcal{F}_{j'}^C \backslash \mathcal{F}_j) &\le (2-\gamma) d_{\max}^C(j) + (\gamma - 1) d_{\mathsf{ave}}^D(j) + d_{\max}^C(j') + d_{\mathsf{ave}}^C(j') \\ &\le (2-\gamma) d_{\max}^C(j) + (\gamma - 1) d_{\mathsf{ave}}^D(j) + d_{\max}^C(j) + d_{\mathsf{ave}}^C(j) \\ &= \gamma \left(\frac{1}{\gamma} d_{\mathsf{ave}}^C(j) + \frac{\gamma - 1}{\gamma} d_{\mathsf{ave}}^D(j) \right) + (3-\gamma) d_{\max}^C(j) \\ &= \gamma d_{\mathsf{ave}}(j) + (3-\gamma) d_{\max}^C(j). \end{aligned}$$

□

Definition 15 (characteristic function). Given a UFL instance and its optimal fractional solution (x, y), the *characteristic function* $h_j : [0,1] \to \mathbb{R}$ of some client $j \in \mathcal{C}$ is defined as follows. Let $i_1, i_2, \cdots, i_m$ the facilities in $\mathcal{F}_j$, in the non-decreasing order of distances to j. Then $h_j(p) = d(i_t, j)$, where t is the minimum number such that $\displaystyle\sum_{s=1}^{t} y_{i_s} \ge p$. The *characteristic function* of the instance is defined as $h = \sum_{j \in \mathcal{C}} h_j$.

Notice that $h_j, j \in \mathcal{C}$ and h are defined using y vector, not $\overline{y}$ vector, and is thus independent of γ.

Claim 16. *h is a non-decreasing piece-wise constant function. That is, there exists points $0 = p_0 < p_1 < p_2 < \cdots < p_m = 1$ and values $0 = c_0 \le c_1 < c_2 < \cdots < c_m$ such that $h(p) = c_t$ where t is the minimum integer such that $p \le p_t$.*

The above claim will be used in the proof of Lemma 23. (Although the monotonicity of h is enough to prove Lemma 23, it requires us to use measure theory. Using the piece-wise-constant property will simplify the proof.)

Claim 17.

$$d_{\mathsf{ave}}(j) = \int_0^1 h_j(p) \mathbf{d}p, \qquad d_{\max}^C(j) = h_j(1/\gamma).$$

This claim together with Lemma 14 implies

$$d(j, \mathcal{F}_{j'}^C \backslash \mathcal{F}_j) \le \gamma \int_0^1 h_j(p)\mathbf{d}p + (3-\gamma) h_j\left(\frac{1}{\gamma}\right). \tag{16}$$

Lemma 18. *For any client j,*

$$\mathbb{E}[C_j] \le \int_0^1 h_j(p) e^{-\gamma p}\gamma \mathbf{d}p + e^{-\gamma}\left(\gamma\int_0^1 h_j(p)\mathbf{d}p + (3-\gamma)h_j\left(\frac{1}{\gamma}\right)\right). \tag{17}$$

Proof. Let $j' \in \mathcal{C}'$ be the cluster center of j. We connect j to the closest open facility in $\mathcal{F}_j \cup \mathcal{F}_{j'}^C$. The proof is outlined as follows: we first prove the lemma for a special case; then we show that any general case can be converted to the special case by a sequence of operations which can only increase $\mathbb{E}[C_j]$.

We first consider the special case where all the facilities in $\mathcal{F}_j$ have infinitely small $\overline{y}$ values (say, $\overline{y}_i = \epsilon$ and ϵ tends to 0) and they were independently sampled in the algorithm. Let $i_1, i_2, \cdots, i_m$ be the facilities in $\mathcal{F}_j$, in the order of increasing distances to j. Notice that $m\epsilon = \gamma$. Then, the probability we connect j to i_t is $(1-\epsilon)^{t-1}\epsilon$. Under the condition that no facilities in $\mathcal{F}_j$ is open, the expected connection cost of j is at most $D = \gamma\int_0^1 h_j(p)\mathbf{d}p + (3-\gamma)h_j\left(\frac{1}{\gamma}\right)$ by (16).

$$\begin{aligned}
\mathbb{E}[C_j] &\le \sum_{t=1}^m \epsilon(1-\epsilon)^{t-1} d(j, i_t) + (1-\epsilon)^m D \\
&= \sum_{t=1}^m \frac{\gamma}{m}\left(1-\frac{\gamma}{m}\right)^{t-1} h_j(t/m) + \left(1-\frac{\gamma}{m}\right)^m D.
\end{aligned}$$

Let m tend to ∞, we have

$$\begin{aligned}
\mathbb{E}[C_j] &\le \int_0^1 \gamma \mathbf{d}p e^{-\gamma p} h(p) + e^{-\gamma} D \\
&= \int_0^1 h_j(p) e^{-\gamma p}\gamma\mathbf{d}p + e^{-\gamma}\left(\gamma\int_0^1 h_j(p)\mathbf{d}p + (3-\gamma)h_j\left(\frac{1}{\gamma}\right)\right).
\end{aligned}$$

Define a partition $\mathcal{P}$ of $\mathcal{F}_j \cup \mathcal{F}_{j'}^C$ as follows. For every $j'' \in \mathcal{C}'$ such that $\left(\mathcal{F}_j \cup \mathcal{F}_{j'}^C\right) \cap \mathcal{F}_{j''}^C \ne \emptyset$, $\left(\mathcal{F}_j \cup \mathcal{F}_{j'}^C\right) \cap \mathcal{F}_{j''}^C \in \mathcal{P}$. For every facility $i \in \mathcal{F}_j \cup \mathcal{F}_{j'}^C$ that is not inside any $\mathcal{F}_{j''}^C, j'' \in \mathcal{C}'$, we have $\{i\} \in \mathcal{P}$.

$\mathcal{P}$ defines the distribution for the opening status of $\mathcal{F}_j \cup \mathcal{F}_{j''}^C$: for every $\mathcal{F}' \in \mathcal{P}$, open 1 facility in $\mathcal{F}'$ with probability $\mathsf{vol}(\mathcal{F}')$ and open no facility with probability $1 - \mathsf{vol}(\mathcal{F}')$; the probability of opening $i \in \mathcal{F}'$ is $\overline{y}_i$; the sets in $\mathcal{P}$ are handled independently.

Claim 19. *If some subset $\mathcal{F}' \subseteq \mathcal{F}_j \backslash \mathcal{F}_{j'}^C, \mathcal{F}' \in \mathcal{P}$ has $|\mathcal{F}'| \geq 2$, removing $\mathcal{F}'$ from $\mathcal{P}$ and add $|\mathcal{F}'|$ singular sets, each containing one facility in mF', to $\mathcal{P}$ can only make $\mathbb{E}[C_j]$ smaller.*

Proof. For the sake of description, we only consider the case when $|\mathcal{F}'| = 2$. The proof can be easily extended to the case where $|\mathcal{F}'| \geq 3$.

Assume $\mathcal{F}' = \{i, i'\}$, where i and i' are two distinct facilities in $\mathcal{F}_j \backslash \mathcal{F}_{j'}^C$ and $d(j, i) \leq d(j, i')$. Focus on the distribution for the distance D between j and its closest open facility in $\{i, i'\}$ ($D = \infty$ if it does not exist) before and after splitting the set $\{i, i'\}$.

Before the splitting operation, the distribution is : with probability $\overline{y}_i$, $D = d(j, i)$, with probability $\overline{y}_{i'}$, $D = d(j, i')$, and with probability $1 - \overline{y}_i - \overline{y}_{i'}$, $D = \infty$. After splitting, the distribution is: with probability $\overline{y}_i$, $D = d(j, i)$, with probability $(1 - \overline{y}_i)\overline{y}_{i'}$, $D = d(j, i')$ and with the probability $(1 - \overline{y}_i)(1 - \overline{y}_{i'})$, $D = \infty$. So, the distribution before splitting strictly dominates the distribution after splitting. Thus, splitting $\{i, i'\}$ can only increase $\mathbb{E}[C_j]$. $\square$

Thus, we can assume $\mathcal{P} = \left\{\mathcal{F}_{j'}^C\right\} \bigcup \left\{\{i\} : i \in \mathcal{F}_j \backslash \mathcal{F}_{j'}^C\right\}$. That is, facilities in $\mathcal{F}_j \backslash \mathcal{F}_{j'}^C$ are independently sampled. Then, we show that the following sequence of operations can only increase $\mathbb{E}[C_j]$.

1. Split the set $\mathcal{F}_{j'}^C \in \mathcal{P}$ into two subsets: $\mathcal{F}_{j'}^C \cap \mathcal{F}_j$, and $\mathcal{F}_{j'}^C \backslash \mathcal{F}_j$;
2. Scale up $\bar{y}$ values in $\mathcal{F}_{j'}^C \backslash \mathcal{F}_j$ so that the volume of $\mathcal{F}_{j'}^C \backslash \mathcal{F}_j$ becomes 1.

Indeed, consider distribution for the distance between j and the closest open facility in $\mathcal{F}_{j'}^C$. The distribution does not change after we performed the operations, since $d_{max}(j, \mathcal{F}_{j'}^C \cap \mathcal{F}_j) \leq d_{min}(j, \mathcal{F}_{j'}^C \backslash \mathcal{F}_j)$, where d_{max} and d_{min} denotes the maximum and the minimum distance from a client to a set of facilities, respectively.

Again, by Claim 19, we can split $\mathcal{F}_{j'}^C \cap \mathcal{F}_j$ into singular sets. By the same argument, splitting a facility $i \in \mathcal{F}_j$ into 2 facilities i' and i'' with $\overline{y}_i = \overline{y}_{i'} + \overline{y}_{i''}$

can only increase $\mathbb{E}[C_j]$. Now, we are in a situation where, facilities in $\mathcal{F}_j$ are independently sampled, each facility $i \in \mathcal{F}_j$ has $\overline{y}_i = \epsilon$ with $\epsilon \to 0$. This is exactly the special case defined in the beginning of the proof. Thus, (17) holds. $\square$

Lemma 20. *The expected connection cost of the integral solution is*

$$\mathbb{E}[C] \leq \int_0^1 h(p)e^{-\gamma p}\gamma \mathbf{d}p + e^{-\gamma}\left(\gamma\int_0^1 h(p)\mathbf{d}p + (3-\gamma)h\left(\frac{1}{\gamma}\right)\right). \tag{18}$$

Proof. Summing up (17) over all clients j will give us the lemma. $\square$

3.2. An Explicit Distribution for γ

In this subsection, we give an explicit distribution for γ by introducing a 0-sum game.

Definition 21. Let $h : [0,1)$ be the characteristic function of some UFL instance and $\gamma \geq 1$. Define

$$\alpha(\gamma, h) = \int_0^1 h(p)e^{-\gamma p}\gamma \mathbf{d}p + e^{-\gamma}\left(\gamma\int_0^1 h(p)\mathbf{d}p + (3-\gamma)h\left(\frac{1}{\gamma}\right)\right). \tag{19}$$

By Lemma 20, $\alpha(\gamma, h)$ is an upper bound for the connection cost of the solution given by $A_1(\gamma)$ when the characteristic function of the input instance is h.

We can scale the distances of input instance so that $\int_0^1 h(p)\mathbf{d}p = 1$. Then,

$$\alpha(\gamma, h) = \int_0^1 h(p)e^{-\gamma p}\gamma \mathbf{d}p + e^{-\gamma}\left(\gamma + (3-\gamma)h\left(\frac{1}{\gamma}\right)\right). \tag{20}$$

We consider a 0-sum game between an algorithm designer A and an adversary B. The strategy of A is a pair (μ, θ), where $0 \leq \theta \leq 1$ and μ is $1-\theta$ times a probability density function for γ. i.e,

$$\theta + \int_1^\infty \mu(\gamma)\mathbf{d}\gamma = 1. \tag{21}$$

The pair (μ, θ) corresponds to running A_2 with probability θ and running $A_1(\gamma)$ with probability $1-\theta$, where γ is randomly selected according to the density function $\mu/(1-\theta)$. The strategy for B is non-decreasing piece-wise constant function $h : [0,1] \to \mathbb{R}^*$ such that $\int_0^1 h(p)\mathbf{d}p = 1$.

Definition 22. The value of the game, when A plays (μ, θ) and B plays h, is defined as

$$\nu(\mu, \theta, h) = \max\left\{\int_1^\infty \gamma\mu(\gamma)\mathbf{d}\gamma + 1.11\theta, \int_1^\infty \alpha(\gamma, h)\mu(\gamma)\mathbf{d}\gamma + 1.78\theta\right\}. \quad (22)$$

Let $h_q : [0, 1] \to \mathbb{R}, 0 \le q < 1$ be a threshold function defined as follows :

$$h_q(p) = \begin{cases} 0 & p \le q \\ \frac{1}{1-q} & p > q \end{cases}. \quad (23)$$

Lemma 23. *For a fixed strategy (θ, μ) for A, there is a best strategy for B that is a threshold function h_q.*

Proof. Let h^* be a best strategy for B. Notice that h^* is a piece-wise constant function. Let $0 = p_0 < p_1 < p_2 < \cdots < p_m, 0 = c_0 \le c_1 < c_2 < \cdots < c_m$ be the values that makes Claim 16 true for h^*. Then,

$$h^* = \sum_{i=0}^{m-1}(c_{i+1} - c_i)(1 - p_i)h_{p_i}, \quad (24)$$

and

$$\int_1^\infty \alpha(\gamma, h^*)\mu(\gamma)\mathbf{d}\gamma = \int_1^\infty \alpha\left(\gamma, \sum_{i=0}^{m-1}(c_{i+1} - c_i)(1 - p_i)h_{p_i}\right)\mu(\gamma)\mathbf{d}\gamma$$

by the linearity of α

$$= \int_1^\infty \sum_{i=0}^{m-1}(c_{i+1} - c_i)(1 - p_i)\alpha\left(\gamma, h_{p_i}\right)\mu(\gamma)\mathbf{d}\gamma$$

$$= \sum_{i=0}^{m-1}(c_{i+1} - c_i)(1 - p_i)\int_1^\infty \alpha\left(\gamma, h_{p_i}\right)\mu(\gamma)\mathbf{d}\gamma.$$

Since $\sum_{i=0} m - 1(c_{i+1} - c_i)(1 - p_i) = 1$, for some $q = p_i, 0 \le i \le m - 1$,

$$\int_1^\infty \alpha(\gamma, h_q)\mu(\gamma)\mathbf{d}\gamma \ge \int_1^\infty \alpha(\gamma, h^*)\mu(\gamma)\mathbf{d}\gamma.$$

Thus,

$$\begin{aligned}\nu(\mu, \theta, h_q) &= \max\left\{\int_1^\infty \gamma\mu(\gamma)\mathbf{d}\gamma + 1.11\theta, \int_1^\infty \alpha(\gamma, h_q)\mu(\gamma)\mathbf{d}\gamma + 1.78\theta\right\} \\ &\ge \max\left\{\int_1^\infty \gamma\mu(\gamma)\mathbf{d}\gamma + 1.11\theta, \int_1^\infty \alpha(\gamma, h^*)\mu(\gamma)\mathbf{d}\gamma + 1.78\theta\right\} \\ &= \nu(\mu, \theta, h^*). \end{aligned} \quad (25)$$

This finishes the proof. $\square$

Now, our goal becomes finding a strategy (θ, μ) for A such that $\sup_{q\in[0,1)} \nu(\mu, \theta, h_q)$ is minimized. With the help of a computer program, we obtain a strategy for A. We first restrict the support of μ to [1,3). Then, we discretize the domain [1,3) into $2n$ small intervals divided by points $\{r_i = 1 + i/n : 0 \le i \le 2n\}$. The value of the game is approximated by the following LP.

$$\min \quad \beta \qquad \text{s.t}$$

$$\frac{1}{2n}\sum_{i=1}^{2n} x_i + \theta = 1 \tag{26}$$

$$\frac{1}{2n}\sum_{i=1}^{2n} \frac{r_{i-1}+r_i}{2} x_i + 1.11\theta \le \beta \tag{27}$$

$$\frac{1}{2n}\sum_{i=1}^{2n} \alpha\left(\frac{r_{i-1}+r_i}{2}, h_q\right) x_i + 1.78\theta \le \beta \quad \forall q \in [0,1) \tag{28}$$

$$x_1, x_2, \cdots, x_{2n}, \theta \ge 0$$

In the above LP, x_i is $2n$ times the probability that we run $A_1\left(\frac{r_{i-1}+r_i}{2}\right)$, θ is the probability that we run A_2 and β is approximation ratio we can get by using the strategy specified by $x_i, 1 \le i \le 2n$ and θ. Equation (26) requires that with probability 1 we run either A_1 or A_2. Inequality (27) and (28) together say that the value of the game is at most β, no matter what B plays : Inequality (27) bounds the scaling factor of the facility cost, while Inequality (28) bounds that of the connection cost.

We solve the above LP for $n = 500$ using Matlab. Since we can only handle finite number of constraints using Matlab, we only require constraint (28) holds for $q = i/n, i = 0, 1, \cdots, n-1$. The value of LP is at most 1.4879 and the correspondent strategy (μ, θ) for A is roughly the following. With probability $\theta \approx 0.2$, run A_2; with probability about 0.5, run $A_1(\gamma_1)$ for $\gamma_1 \approx 1.5$; with the remaining probability, run $A_1(\gamma)$ for γ uniformly selected between γ_1 and $\gamma_2 \approx 2$.

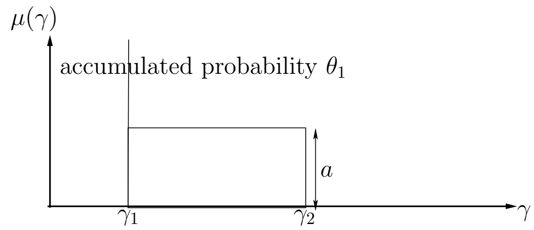

Figure 4: The distribution of γ. With probability θ_1, we run $A_1(\gamma_1)$; with probability $a(\gamma_2 - \gamma_1)$, we run $A_1(\gamma)$ with γ randomly selected from $[\gamma_1, \gamma_2]$; with probability $\theta_2 = 1 - \theta_1 - a(\gamma_2 - \gamma_1)$, we run A_2.

In light of the program generated solution, we give a pure analytical strategy for A and show that the value of the game is at most 1.488. The strategy (μ, θ) is defined as follows. With probability $\theta = \theta_2$, we run A_2; with probability θ_1, we run $A_1(\gamma)$ with $\gamma = \gamma_1$; with probability $1 - \theta_2 - \theta_1$, we run $A_1(\gamma)$ with γ randomly chosen between γ_1 and γ_2. Thus, the function μ is

$$\mu(\gamma) = aI_{\gamma_1,\gamma_2}(\gamma) + \theta_1\delta(\gamma - \gamma_1), \tag{29}$$

where δ is the Dirac-Delta function, $a = \dfrac{1 - \theta_1 - \theta_2}{\gamma_2 - \gamma_1}$, and $I_{\gamma_1,\gamma_2}(\gamma)$ is 1 if $\gamma_1 < \gamma < \gamma_2$ and 0 otherwise(See Figure 4). The values of $\theta_1, \theta_2, \gamma_1, \gamma_2, a$ are given later.

The remaining part of the paper is devoted to prove the following lemma.

Lemma 24. *There exists some $\theta_1 \geq 0, \theta_2 \geq 0, 1 \leq \gamma_1 < \gamma_2$ and $a \geq 0$ such that $\theta_1 + \theta_2 + a(\gamma_2 - \gamma_1) = 1$ and $\sup_{q \in [0,1)} \nu(\mu, \theta, h_q) \leq 1.488$.*

Proof. The scaling factor for the facility cost is

$$\gamma_f = \theta_1\gamma_1 + a(\gamma_2 - \gamma_1)\frac{\gamma_1 + \gamma_2}{2} + 1.11\theta_2. \tag{30}$$

Now, we consider the scaling factor γ_c when $h = h_q$. By Lemma 20,

$$\gamma_c(q) \leq \int_1^\infty \left(\int_0^1 e^{-\gamma p}\gamma h_q(p)\mathbf{d}p + e^{-\gamma}(\gamma + (3 - \gamma)h_q(1/\gamma))\right)\mu(\gamma)\mathbf{d}\gamma + 1.78\theta_2$$

replacing μ with $aI_{\gamma_1,\gamma_2}(\gamma)+\theta_1\delta(\gamma-\gamma_1)$ and $h_q(p)$ with 0 or $1/(1-q)$ depending on whether $p \le q$,

$$
\begin{aligned}
&= \int_{\gamma_1}^{\gamma_2} \left(\int_q^1 e^{-\gamma p}\gamma \frac{1}{1-q}\mathbf{d}p + e^{-\gamma}\gamma + (3-\gamma)h_q(1/\gamma) \right) a\mathbf{d}\gamma \\
&\quad + \theta_1 \left(\int_q^1 e^{-\gamma_1 p}\gamma_1 \frac{1}{1-q}\mathbf{d}p + e^{-\gamma_1}(\gamma_1 + (3-\gamma_1)h_q(1/\gamma_1)) \right) + 1.78\theta_2 \\
&= B_1(q) + B_2(q) + B_3(q) + 1.78\theta_2,
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
B_1(q) &= \int_{\gamma_1}^{\gamma_2} \int_q^1 e^{-\gamma p}\gamma \frac{1}{1-q}\mathbf{d}pa\mathbf{d}\gamma + \int_{\gamma_1}^{\gamma_2} e^{-\gamma}\gamma a\mathbf{d}\gamma, \\
B_2(q) &= \int_{\gamma_1}^{\gamma_2} (3-\gamma)h_q(1/\gamma)a\mathbf{d}\gamma,
\end{aligned}
$$

and

$$
B_3(q) = \theta_1 \int_q^1 e^{-\gamma_1 p}\gamma_1 \frac{1}{1-q}\mathbf{d}p + \theta_1 e^{-\gamma_1}(\gamma_1 + (3-\gamma_1)h_q(1/\gamma_1)).
$$

Then, we calculate $B_1(q), B_2(q)$ and $B_3(q)$ separately.

$$
\begin{aligned}
B_1(q) &= \int_{\gamma_1}^{\gamma_2} \int_q^1 e^{-\gamma p}\gamma \frac{1}{1-q}\mathbf{d}pa\mathbf{d}\gamma + \int_{\gamma_1}^{\gamma_2} e^{-\gamma}\gamma a\mathbf{d}\gamma \\
&= \frac{a}{1-q}\int_{\gamma_1}^{\gamma_2} (e^{-\gamma q} - e^{-\gamma})\mathbf{d}\gamma - a(\gamma+1)e^{-\gamma}\Big|_{\gamma_1}^{\gamma_2} \\
&= \frac{a}{(1-q)q}(e^{-\gamma_1 q} - e^{-\gamma_2 q}) - \frac{a}{1-q}(e^{-\gamma_1} - e^{-\gamma_2}) \\
&\quad + a((\gamma_1+1)e^{-\gamma_1} - (\gamma_2+1)e^{-\gamma_2}).
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
B_2(q) &= \int_{\gamma_1}^{\gamma_2} (3-\gamma)h_q(1/\gamma)a\mathbf{d}\gamma \\
&= \begin{cases}
\frac{a}{1-q}\left((2-\gamma_1)e^{-\gamma_1} - (2-\gamma_2)e^{-\gamma_2}\right) & 0 \le q < 1/\gamma_2 \\
\frac{a}{1-q}\left((2-\gamma_1)e^{-\gamma_1} - (2-1/q)e^{-1/q}\right) & 1/\gamma_2 \le q \le 1/\gamma_1 \\
0 & 1/\gamma_1 < q < 1
\end{cases}.
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
B_3(q) &= \theta_1 \int_q^1 e^{-\gamma_1 p}\gamma_1 \frac{1}{1-q}\mathbf{d}p + \theta_1 e^{-\gamma_1}(\gamma_1 + (3-\gamma_1)h_q(1/\gamma_1)) \\
&= \theta_1 \left(\frac{1}{1-q}(e^{-\gamma_1 q} - e^{-\gamma_1}) + e^{-\gamma_1}\gamma_1 + e^{-\gamma_1}(3-\gamma_1)h_q(1/\gamma_1) \right) \\
&= \begin{cases} \theta_1 \left(\frac{1}{1-q}(e^{-\gamma_1 q} - e^{-\gamma_1}) + e^{-\gamma_1}\gamma_1 + \frac{e^{-\gamma_1}(3-\gamma_1)}{1-q} \right) & 0 \le q \le 1/\gamma_1 \\ \theta_1 \left(\frac{1}{1-q}(e^{-\gamma_1 q} - e^{-\gamma_1}) + e^{-\gamma_1}\gamma_1 \right) & 1/\gamma_1 < q < 1 \end{cases}.
\end{aligned} \tag{34}
$$

So, we have 3 cases :

1. $0 \le q < 1/\gamma_2$.

$$
\begin{aligned}
\gamma_c(q) &\le B_1(q) + B_2(q) + B_3(q) \\
&= \frac{a}{(1-q)q}(e^{-\gamma_1 q} - e^{-\gamma_2 q}) - \frac{a}{1-q}(e^{-\gamma_1} - e^{-\gamma_2}) \\
&\quad + a((\gamma_1+1)e^{-\gamma_1} - (\gamma_2+1)e^{-\gamma_2}) + \frac{a}{1-q}\left((2-\gamma_1)e^{-\gamma_1} - (2-\gamma_2)e^{-\gamma_2}\right) \\
&\quad + \theta_1\left(\frac{1}{1-q}(e^{-\gamma_1 q} - e^{-\gamma_1}) + e^{-\gamma_1}\gamma_1 + \frac{1}{1-q}e^{-\gamma_1}(3-\gamma_1)\right) + 1.78\theta_2 \\
&= \frac{a}{(1-q)q}(e^{-\gamma_1 q} - e^{-\gamma_2 q}) + \frac{A_1}{1-q} + \theta_1 \frac{e^{-\gamma_1 q}}{1-q} + A_2,
\end{aligned}
$$

where $A_1 = a(e^{-\gamma_1} - \gamma_1 e^{-\gamma_1} - e^{-\gamma_2} + \gamma_2 e^{-\gamma_2}) + 2\theta_1 e^{-\gamma_1} - \theta_1 e^{-\gamma_1}\gamma_1$

and $A_2 = a((\gamma_1+1)e^{-\gamma_1} - (\gamma_2+1)e^{-\gamma_2}) + \theta_1 e^{-\gamma_1}\gamma_1 + 1.78\theta_2$.

2. $1/\gamma_2 \le q \le 1/\gamma_1$.

The only difference between this case and the first case is the definition of $B_2(q)$. Comparing the definition of $B_2(q)$ for the case $0 \le q < 1/\gamma_2$ and the case $1/\gamma_2 \le q \le 1/\gamma_1$, we can get

$$
\begin{aligned}
\gamma_c(q) = \frac{a}{(1-q)q}(e^{-\gamma_1 q} - e^{-\gamma_2 q}) + \frac{A_1}{1-q} + \theta_1 \frac{e^{-\gamma_1 q}}{1-q} + A_2 \\
+ \frac{a}{1-q}\left((2-\gamma_2)e^{-\gamma_2} - (2-1/q)e^{-1/q}\right).
\end{aligned} \tag{35}
$$

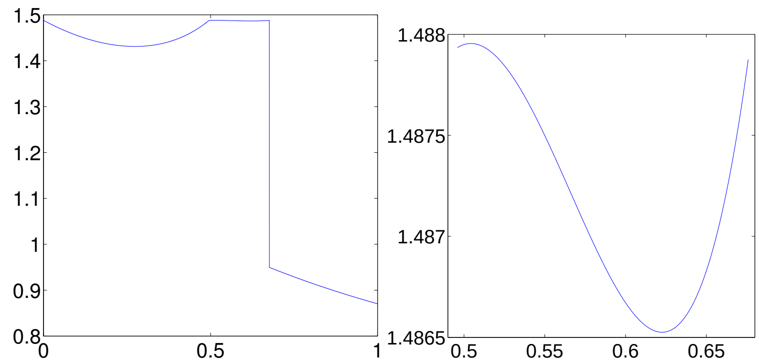

Figure 5: The function $\gamma_c(q)$. The curve on the right-hand side is the function restricted to the interval $(1/\gamma_2, 1/\gamma_1)$.

3. $1/\gamma_1 < q < 1$

$$\begin{aligned}\gamma_c(q) \leq & \frac{a}{(1-q)q}(e^{-\gamma_1 q} - e^{-\gamma_2 q}) - \frac{a}{1-q}(e^{-\gamma_1} - e^{-\gamma_2}) \\ & + a((\gamma_1+1)e^{-\gamma_1} - (\gamma_2+1)e^{-\gamma_2}) \\ & + \theta_1\left(\frac{1}{1-q}(e^{-\gamma_1 q} - e^{-\gamma_1}) + e^{-\gamma_1}\gamma_1\right) + 1.78\theta_2 \\ = & \frac{a}{(1-q)q}(e^{-\gamma_1 q} - e^{-\gamma_2 q}) + \frac{A_3}{1-q} + \theta_1 \frac{e^{-\gamma_1 q}}{1-q} + A_2, \end{aligned} \tag{36}$$

where $A_3 = a(-e^{-\gamma_1} + e^{-\gamma_2}) - \theta_1 e^{-\gamma_1}$.

We set $\gamma_1 = 1.479311, \gamma_2 = 2.016569, \theta_1 = 0.503357, a = 0.560365$ and $\theta_2 = 1 - \theta_1 - a(\gamma_2 - \gamma_1) \approx 0.195583$. Then,

$$\gamma_f = \theta_1\gamma_1 + a(\gamma_2 - \gamma_1)\frac{\gamma_1 + \gamma_2}{2} + 1.11\theta_2 \approx 1.487954. \tag{37}$$

$A_1 \approx 0.074347, A_2 \approx 0.609228$ and $A_3 \approx -0.167720$. $\gamma_c(q)$ has the maximum value about 1.487989, achieved at $q = 0$(see Figure 5). This finishes the proof of Lemma 24.

□

Thus, Theorem 1 follows immediately from lemma Lemma 23 and Lemma 24.

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