Advanced Algorithms (Fall 2025)

Linear Programming Duality

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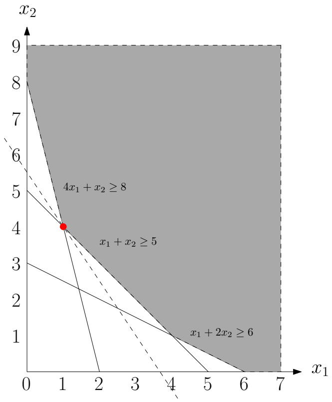
Outline

1. Duality of Linear Programming
   - Linear Programming Duality
2. Examples
   - Max-Flow Min-Cut Theorem Using LP Duality
   - 0-Sum Game and Nash Equilibrium

Outline

$$
\begin{aligned}
\min \quad & 7x_1 + 4x_2 \\
& x_1 + x_2 \geq 5 \\
& x_1 + 2x_2 \geq 6 \\
& 4x_1 + x_2 \geq 8 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



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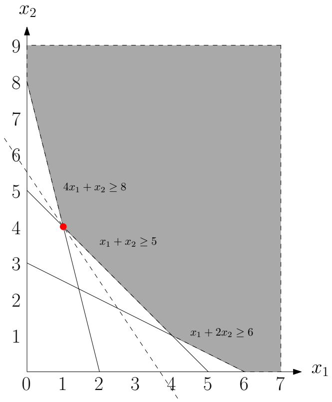

Q: How can we prove a lower bound for the value?

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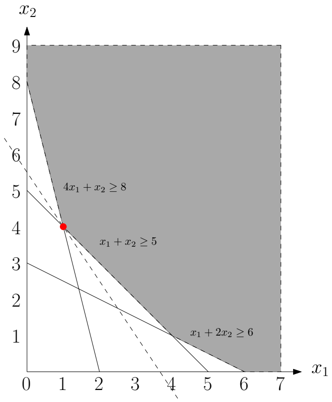

Q: How can we prove a lower bound for the value?

- $7x_1 + 4x_2 \geq 2(x_1 + x_2) + (x_1 + 2x_2) \geq 2 \times 5 + 6 = 16$
- $7x_1 + 4x_2 \geq (x_1 + x_2) + (x_1 + 2x_2) + (4x_1 + x_2) \geq 5 + 6 + 8 = 19$
- $7x_1 + 4x_2 \geq 4(x_1 + x_2) \geq 4 \times 5 = 20$
- $7x_1 + 4x_2 \geq 3(x_1 + x_2) + (4x_1 + x_2) \geq 3 \times 5 + 8 = 23$

Primal LP

$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

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A way to prove lower bound on the value of primal LP

$$\begin{aligned} & 7x_1 + 4x_2 && (\text{if } 7 \geq y_1 + y_2 + 4y_3 \text{ and } 4 \geq y_1 + 2y_2 + y_3) \\ \geq\ & y_1(x_1 + x_2) + y_2(x_1 + 2x_2) + y_3(4x_1 + x_2) && (\text{if } y_1, y_2, y_3 \geq 0) \\ \geq\ & 5y_1 + 6y_2 + 8y_3. \end{aligned}$$

- Goal: need to maximize $5y_1 + 6y_2 + 8y_3$

Primal LP

$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & 5y_1 + 6y_2 + 8y_3 \\ & y_1 + y_2 + 4y_3 \leq 7 \\ & y_1 + 2y_2 + y_3 \leq 4 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

A way to prove lower bound on the value of primal LP

$$\begin{aligned} & 7x_1 + 4x_2 && (\text{if } 7 \geq y_1 + y_2 + 4y_3 \text{ and } 4 \geq y_1 + 2y_2 + y_3) \\ \geq\ & y_1(x_1 + x_2) + y_2(x_1 + 2x_2) + y_3(4x_1 + x_2) && (\text{if } y_1, y_2, y_3 \geq 0) \\ \geq\ & 5y_1 + 6y_2 + 8y_3. \end{aligned}$$

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Dual LP

$$\max \quad 5y_1 + 6y_2 + 8y_3$$
$$y_1 + y_2 + 4y_3 \leq 7$$
$$y_1 + 2y_2 + y_3 \leq 4$$
$$y_1, y_2, y_3 \geq 0$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 4 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} \quad c = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$\min \quad c^T x$ s.t.

$$Ax \geq b$$
$$x \geq 0$$

$\max \quad b^T y$ s.t.

$$A^T y \leq c$$
$$y \geq 0$$

Primal LP

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Dual LP

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- $P =$ value of primal LP
- $D =$ value of dual LP

Theorem (weak duality theorem) $D \leq P$.

Theorem (strong duality theorem) $D = P$.

- Can always prove the optimality of the primal solution, by adding up primal constraints.

Primal LP

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$$Ax \geq b$$
$$x \geq 0$$

Dual LP

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$$A^T y \leq c$$
$$y \geq 0$$

- P = value of primal LP
- D = value of dual LP

Theorem (weak duality theorem) $D \leq P$.

Proof.

- x^*: optimal primal solution
- y^*: optimal dual solution

$$D = b^{\mathrm{T}} y^* \leq (Ax^*)^{\mathrm{T}} y^* = (x^*)^{\mathrm{T}} A^{\mathrm{T}} y^* \leq (x^*)^{\mathrm{T}} c = c^{\mathrm{T}} x^* = P. \qquad \square$$

Fact If a point x does not belong to a polytope $\mathcal{P}$, then there is a hyperplane separating x and $\mathcal{P}$.

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Lemma (Farkas Lemma) $Ax = b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}}b < 0$ is feasible.

Proof.

- b does not belong to $\{Ax : x \geq 0\}$, so $\exists$ some hyperplane separating b and $\{Ax : x \geq 0\}$.
- $y^{\mathrm{T}}b < g$ and $y^{\mathrm{T}}Ax > g$ for every $x \geq 0$
- $g < 0$ and $y^{\mathrm{T}}A \geq 0$
- $y^{\mathrm{T}}b < g < 0$ $\square$

Lemma (Farkas Lemma) $Ax = b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}}b < 0$ is feasible.

Lemma (Variant of Farkas Lemma) $Ax \leq b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}}b < 0, y \geq 0$ is feasible.

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Proof.

- system equivalent to $Ax + x' = b, x, x' \geq 0$

$$(A, I)\begin{pmatrix} x \\ x' \end{pmatrix} = b, \qquad \begin{pmatrix} x \\ x' \end{pmatrix} \geq 0$$

- By Farkas Lemma, $\exists y$ such that $y^{\mathrm{T}}(A, I) \geq 0, y^{\mathrm{T}}b < 0$
- $\iff y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}} \geq 0, y^{\mathrm{T}}b < 0$ □

Primal LP

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$$Ax \geq b$$
$$x \geq 0$$

Dual LP

$$\max \quad b^T y \qquad \text{s.t.}$$
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Proof of Strong Duality Theorem

- $\forall \epsilon > 0, \begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} x \leq \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix}, x \geq 0$ is infeasible

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- $\forall \epsilon > 0, \begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} x \leq \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix}, x \geq 0$ is infeasible
- There exists $y \in \mathbb{R}^m_{\geq 0}, \alpha \geq 0$, such that $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} \geq 0$, $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix} < 0$

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$$Ax \geq b$$
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- we can prove $\alpha > 0$, since the primal LP is feasible.

Proof of Strong Duality Theorem

- There exists $y \in \mathbb{R}^m_{\geq 0}, \alpha \geq 0$, such that $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} \geq 0$, $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix} < 0$

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- assume $\alpha = 1$

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- assume $\alpha = 1$
- $-y^{\mathrm{T}} A + c^{\mathrm{T}} \geq 0, -y^{\mathrm{T}} b + P - \epsilon < 0 \Longleftrightarrow A^{\mathrm{T}} y \leq c, b^{\mathrm{T}} y > P - \epsilon$

Proof of Strong Duality Theorem

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- assume $\alpha = 1$
- $-y^{\mathrm{T}}A + c^{\mathrm{T}} \geq 0, -y^{\mathrm{T}}b + P - \epsilon < 0 \iff A^{\mathrm{T}}y \leq c, b^{\mathrm{T}}y > P - \epsilon$
- $\forall \epsilon > 0, D > P - \epsilon \implies D = P$ (since $D \leq P$) $\square$

Primal LP

$$\min \quad c^{\mathrm{T}}x$$
$$Ax \geq b$$
$$x \geq 0$$

Dual LP

$$\max \quad b^{\mathrm{T}}y$$
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Relationships

Primal LP	dual LP
variables	constraints
constraints	variables
obj. coefficients	RHS constants
RHS constants	obj. coefficients

Primal LP

$$\begin{aligned} \min \quad & c^{\mathrm{T}} x \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual LP

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Relationships

Primal LP	dual LP
variables	constraints
constraints	variables
obj. coefficients	RHS constants
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More Relationships

Primal LP	Dual LP
variable in $\mathbb{R}$	equlities
equlities	variable in $\mathbb{R}$

- duality is mutual: the dual of the dual of an LP is the LP itself.

Primal LP

$$\begin{aligned} \max \quad & b^{\mathrm{T}} y \\ A^{\mathrm{T}} y & \leq c \\ y & \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & c^{\mathrm{T}} x \\ Ax & \geq b \\ x & \geq 0 \end{aligned}$$

- duality is mutual: the dual of the dual of an LP is the LP itself.

Primal LP

$$\max \quad b^{\mathrm{T}} y \qquad A^{\mathrm{T}} y \le c \qquad y \ge 0$$

Dual LP

$$\min \quad c^{\mathrm{T}} x \qquad Ax \ge b \qquad x \ge 0$$

- Duality theorem holds when one LP is infeasible:
- Minimization LP is infeasible $\implies$ value $= \infty$ $\iff$ dual LP value $= \infty$ $\implies$ feasible region of dual LP is unbounded

Complementary Slackness

Primal LP

$$\begin{aligned} \min \quad & c^{\mathrm{T}} x \\ Ax &\geq b \\ x &\geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & b^{\mathrm{T}} y \\ A^{\mathrm{T}} y &\leq c \\ y &\geq 0 \end{aligned}$$

- x^* and y^*: optimum primal and dual solutions
- $D = b^{\mathrm{T}} y^* \leq (Ax^*)^{\mathrm{T}} y^* = (x^*)^{\mathrm{T}} A^{\mathrm{T}} y^* \leq (x^*)^{\mathrm{T}} c = c^{\mathrm{T}} x^* = P$.
- $P = D$: all the inequlaities hold with equalities.

Complementary Slackness

- $y_i^* > 0 \implies \sum_j a_{ij} x_j^* = b_i$.
- $x_j^* > 0 \implies \sum_i a_{ij} y_i^* = c_j$.

Outline

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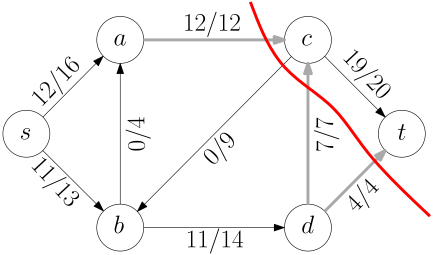
Maximum Flow Problem

Input: flow network $(G = (V, E), c, s, t)$

Output: maximum value of a s-t flow f



LP for Maximum Flow

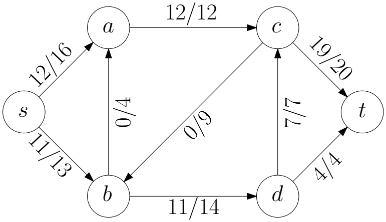
$$\max \quad \sum_{e \in \delta^{\text{in}}(t)} x_e$$

$$x_e \leq c_e \qquad \forall e \in E$$

$$\sum_{e \in \delta^{\text{out}}(v)} x_e - \sum_{e \in \delta^{\text{in}}(v)} x_e = 0 \qquad \forall v \in V \setminus \{s, t\}$$

$$x_e \geq 0 \qquad \forall e \in E$$

An Equivalent Packing LP



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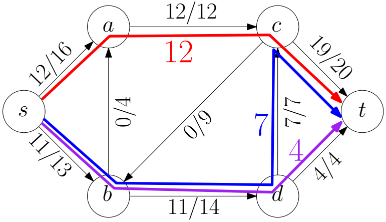

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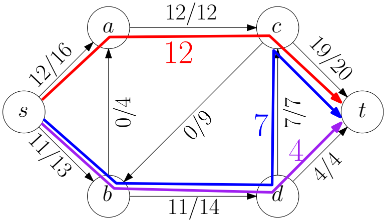
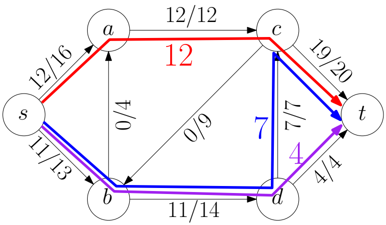


- $\mathcal{P}$: the set of all simple paths from s to t
- $f_P, P \in \mathcal{P}$: the flow on P

An Equivalent Packing LP



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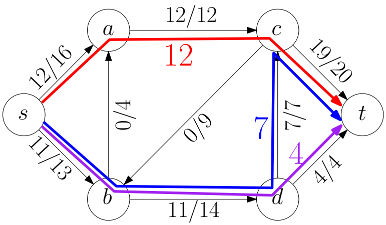
- $f_P, P \in \mathcal{P}$: the flow on P

$$\max \sum_{P \in \mathcal{P}} f_P$$

$$\sum_{P \in \mathcal{P}: e \in P} f_P \leq c_e \qquad \forall e \in E$$

$$f_P \geq 0 \qquad \forall P \in \mathcal{P}$$

An Equivalent Packing LP



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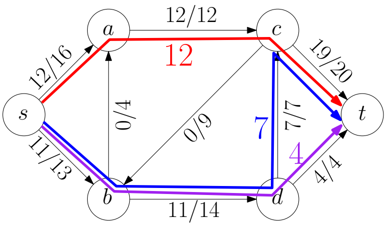
$$f_P \geq 0 \qquad \forall P \in \mathcal{P}$$

$$\min \quad \sum_{e \in E} c_e y_e$$

$$\sum_{e \in P} y_e \geq 1 \qquad \forall P \in \mathcal{P}$$

$$y_e \geq 0 \qquad \forall e \in E$$

An Equivalent Packing LP



- $\mathcal{P}$: the set of all simple paths from s to t
- $f_P, P \in \mathcal{P}$: the flow on P

$$\max \quad \sum_{P \in \mathcal{P}} f_P \qquad\qquad \min \quad \sum_{e \in E} c_e y_e$$

$$\sum_{P \in \mathcal{P}: e \in P} f_P \le c_e \quad \forall e \in E \qquad\qquad \sum_{e \in P} y_e \ge 1 \quad \forall P \in \mathcal{P}$$

$$f_P \ge 0 \quad \forall P \in \mathcal{P} \qquad\qquad y_e \ge 0 \quad \forall e \in E$$

- dual constraints: the shortest s-t path w.r.t weights y has length ≥ 1

Dual LP

$$\min \sum_{e \in E} c_e y_e$$

$$\sum_{e \in P} y_e \geq 1 \qquad \forall P \in \mathcal{P}$$

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Theorem The optimum value can be attained at an integral point y.

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Maximum Flow Minimum Cut Theorem The value of the maximum flow equals the value of the minimum cut.

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Proof of Theorem.

- Given any optimum y, let d_v be the length of shortest path from s to v, for every $v \in V$. $\qquad d_s = 0, d_t = 1$

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- Given any optimum y, let d_v be the length of shortest path from s to v, for every $v \in V$. $d_s = 0, d_t = 1$
- Randomly choose $\theta \in (0,1)$, and output cut $(S := \{v : d_v \leq \theta\}, T := \{v : d_v > \theta\})$

Dual LP

$$\min \quad \sum_{e \in E} c_e y_e$$

$$\sum_{e \in P} y_e \geq 1 \qquad \forall P \in \mathcal{P}$$

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- Lemma: $\mathbb{E}[\text{cut value of}(S, T)] \leq \sum_{e \in E} c_e y_e$

Dual LP

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- Randomly choose $\theta \in (0, 1)$, and output cut $(S := \{v : d_v \leq \theta\}, T := \{v : d_v > \theta\})$
- Lemma: $\mathbb{E}[\text{cut value of}(S, T)] \leq \sum_{e \in E} c_e y_e$
- Any cut (S, T) in the support is optimum $\square$

$$
\begin{aligned}
\max \quad & \sum_{P \in \mathcal{P}} f_P \\
\sum_{P \in \mathcal{P} : e \in P} & f_P \le c_e && \forall e \in E \\
& f_P \ge 0 && \forall P \in \mathcal{P}
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e y_e \\
\sum_{e \in P} & y_e \ge 1 && \forall P \in \mathcal{P} \\
& y_e \ge 0 && \forall e \in E
\end{aligned}
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\max \quad & \sum_{P \in \mathcal{P}} f_P \\
\sum_{P \in \mathcal{P}: e \in P} & f_P \leq c_e \qquad \forall e \in E \\
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$$
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& y_e \geq 0 \qquad \forall e \in E
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- pros of new LP: it is a packing LP, dual is a covering LP, easier to understand and analyze

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- pros of new LP: it is a packing LP, dual is a covering LP, easier to understand and analyze
- cons of new LP: exponential size, can not be solved directly
  - when we only need to do non-algorithmic analysis
  - ellipsoid method with separation oracle can solve some exponential size LP

Outline

0-Sum Game

Input: a payoff matrix $M \in \mathbb{R}^{m \times n}, m, n \geq 1$,
two players: row player R, column player C

Output: R plays a row $i \in [m]$, C plays a column $j \in [n]$
payoff of game is M_{ij}
R wants to minimize M_{ij}, C wants to maximize M_{ij}

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Rock-Scissor-Paper Game

payoff	R	S	P
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- game depends on who plays first

By allowing mixed strategies, each player has a best strategy, regardless of who plays first

	row player R	column player C
pure strategy	row $i \in [m]$	column $j \in [n]$
mixed strategy	distribution x over $[m]$ $x \in [0,1]^m, \sum_{i=1}^m x_i = 1$	distribution y over $[n]$ $y \in [0,1]^n, \sum_{j=1}^n y_j = 1$

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- If C plays a mixed strategy x first, then it is the best for R to play a pure strategy i. Value of game is $\sup_y \min_{i\in[m]} M(i,y)$.

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- $M(x^*, y^*) \leq V, M(x^*, y^*) \geq V \implies M(x^*, y^*) = V$
- $M(x^*, y) \leq V, \forall y$ and $M(x, y^*) \geq V, \forall x$. □

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- Can be proved by LP duality.

LP for Row Player

$$
\begin{aligned}
\min \quad & R \\
\textstyle\sum_{i=1}^{m} x_i &= 1 \\
R - \textstyle\sum_{i=1}^{m} M_{ij} x_i &\geq 0 \qquad \forall j \in [n] \\
x_i &\geq 0 \qquad \forall i \in [m]
\end{aligned}
$$

LP for Column Player

$$
\begin{aligned}
\max \quad & C \\
\textstyle\sum_{j=1}^{n} y_j &= 1 \\
C - \textstyle\sum_{j=1}^{n} M_{ij} y_j &\leq 0 \qquad \forall i \in [m] \\
y_j &\geq 0 \qquad \forall j \in [n]
\end{aligned}
$$

- The two LPs are dual to each other.

$$
\begin{array}{c|c|c}
1 & 0, 0, 0, \cdots, 0 & \\
\hline
0 & 1, 1, 1, \cdots, 1 & 1 \\
\hdashline
1 & & 0 \\
1 & & 0 \\
\vdots & -M & \vdots \\
\vdots & & \vdots \\
1 & & 0
\end{array}
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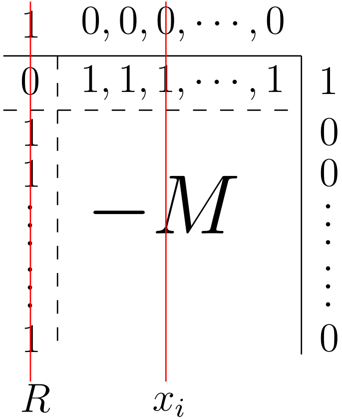

LP for Row Player

$$\begin{aligned} \min \quad & R \\ \textstyle\sum_{i=1}^m x_i &= 1 \\ R - \textstyle\sum_{i=1}^m M_{ij} x_i &\geq 0 \quad \forall j \in [n] \\ x_i &\geq 0 \quad \forall i \in [m] \end{aligned}$$

LP for Column Player

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- The two LPs are dual to each other.

$x_i, i \in [m]$	primal variable ($\in \mathbb{R}_{\geq 0}$)	dual constraint ($\leq$)
$y_j, j \in [n]$	dual variable ($\in \mathbb{R}_{\geq 0}$)	primal constraint ($\geq$)
R	primal variable ($\in \mathbb{R}$)	dual constraint ($=$)
C	dual variable ($\in \mathbb{R}$)	primal constraint ($=$)

- Let V be the value of the game, x^* and y^* be the two optimum strategies. Complementrary slackness implies:
  - If $x_i^* > 0$, then $M(i, y^*) = V$.
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  - If $x_i^* > 0$, then $M(i, y^*) = V$.
  - If $y_j^* > 0$, then $M(x^*, j) = V$.
- The game is called 0-sum game as the payoff for R is the negative of the payoff for C.