Advanced Algorithms (Fall 2025)

Extension Complexity of Polytopes

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Outline

Motivation

Typical Combinatorial Optimization Problem

Input: $[n]$: ground set

$\mathcal{S}$: feasible sets: a family of subsets of U, often implicitly given

$w_i, i \in [n]$: values/costs of elements

Output: the set $S \in \mathcal{S}$ with the minimum/maximum $w(S) := \sum_{i \in S} w_i$

$\mathcal{P} :=$ conv$(\{\chi^S : S \in \mathcal{S}\})$: convex hull of all valid solutions

LP to Solve Problem Exactly

$$\min / \max \quad \sum_{i=1}^{n} w_i x_i \quad \text{s.t.} \qquad x \in \mathcal{P}$$

- inequality constraints needed to describe $x \in \mathcal{P}$ (or $\mathcal{P}$ in short) is facets($\mathcal{P}$) := the number of facets of $\mathcal{P}$

Q: Can we do better?

A: Yes in some cases, by introducing new variables that we call auxiliary variables.

Def. An extension of a polytope $\mathcal{P} \in \mathbb{R}^n$ is a polyhedron $\mathcal{Q} \subseteq \mathbb{R}^{n+r}$ for some $r \geq 0$, such that $\mathcal{P}$ is the projection of $\mathcal{Q}$ to $\mathbb{R}^n$:

$$\mathcal{P} = \left\{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^r, (x, y) \in \mathcal{Q}\right\}$$

LP to Solve Problem Exactly with Auxiliary Variables

$$\min / \max \quad \sum_{i=1}^{n} c_i x_i \quad \text{s.t.} \qquad (x, y) \in \mathcal{Q},$$

where $\mathcal{Q}$ is an extension of $\mathcal{P}$.

- To require $(x, y) \in \mathcal{Q}$, the number of inequalities we need is facets$(\mathcal{Q})$
- It may be possible that facets$(\mathcal{Q}) \ll$ facets$(\mathcal{P})$

Def. The extension complexity of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, denoted as xc$(\mathcal{P})$, is defined as follows:

$$\text{xc}(\mathcal{P}) := \min\{\text{facets}(\mathcal{Q}) : \mathcal{Q} \text{ is an extension of } \mathcal{P}\}.$$

Def. An extended formulation of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is a set of linear constraints:

$$(E, F) \begin{pmatrix} x \\ y \end{pmatrix} = g$$
$$y \geq 0$$

where $E \in \mathbb{R}^{N \times n}, F \in \mathbb{R}^{N \times r}, g \in \mathbb{R}^N$ are given, and $x \in \mathbb{R}^n$ is the vector of main variables, $y \in \mathbb{R}^r$ is the vector of auxiliary variables. The following property needs to be satisfied:

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \exists y \geq 0, (E, F) \begin{pmatrix} x \\ y \end{pmatrix} = g \right\}.$$

The complexity of the extended formulation is defined as r.

Def. (An alternative definition) The extension complexity of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, denoted as $\mathrm{xc}(\mathcal{P})$, is defined as the minimum complexity of an extended formulation of $\mathcal{P}$.

The Equivalence Between the Two Definitions

- $\mathsf{xc}_1(\mathcal{P}), \mathsf{xc}_2(\mathcal{P})$: $\mathsf{xc}(\mathcal{P})$ according to the first/second definition

$\mathsf{xc}_2(\mathcal{P}) \leq \mathsf{xc}_1(\mathcal{P})$

- Given an extension $\mathcal{Q}$ of $\mathcal{P}$, we can use facets$(\mathcal{Q})$ inequalities (and some equalities, if the dimension of $\mathcal{Q}$ is smaller than the dimension of its host space) to describe $\mathcal{Q}$, one for each facet.
- For the i-th inequality $a_i x \geq b_i$, we introduce a variable y_i, and replace the inequality by $y_i = a_i x - b_i, y_i \geq 0$.
- This gives an extended formulation of $\mathcal{P}$ with facets$(\mathcal{Q})$ y-variables.
- Remark: there might be some auxiliary variables with no non-negativity constraints; but they can be removed.

$\mathsf{xc}_1(\mathcal{P}) \leq \mathsf{xc}_2(\mathcal{P})$

- An extended formulation with m y-variables defines a polyhedron with at most m facets.

Outline

1. Motivation and Definition
   - Example: Permutation Polytope
   - Extension Complexity of Spanning Tree Polytope
2. Connection Between Extension Complexity and Non-Negative Rank
3. Polytopes with Exponential Extension Complexity

Example: Permutation Polytope

- $\mathcal{S} := \left\{x \in [n]^{[n]} : x \text{ is a permutation of } [n]\right\}$
- $\mathcal{P} := \text{conv}(\mathcal{S})$
- note: $\mathcal{P}$ has dimension $n-1$, as $\sum_{i\in[n]} x_i = \frac{n(n+1)}{2}$ is valid.

Lemma For any $S \subsetneq [n], S \neq \emptyset$, $\sum_{i\in S} x_i \geq \frac{|S|(|S|+1)}{2}$ is a facet of $\mathcal{P}$.

- so, facets$(\mathcal{P}) = 2^{\Omega(n)}$

Proof Sketch.

- The constraint $\sum_{i\in S} x_i \geq \frac{|S|(|S|+1)}{2}$ gives a face
- To show it's a facet, need to prove its dimension is $n-2$
- We can find $x^0, x^1, \cdots, x^{n-2}$ on the face such that $x^1 - x^0, x^2 - x^0, \cdots, x^{n-2} - x^0$ are linearly independent. □

Representation using Permutation Matrices

- Represent a permutation $x \in [n]^{[n]}$ by the permutation matrix $M \in \{0,1\}^{n\times n}$ so that $M_{ij} = 1$ iff $x_i = j$.

$$\text{Example :} \qquad (3,1,2) \iff \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- Crucial property: x is a linear function of entries in M
- $\mathcal{P}' := \text{conv}\big(\{M : M \text{ is a permutation matrix}\}\big)$

Lemma $\mathcal{P}' = \left\{ y \in [0,1]^{n\times n} : \sum_i y_{i,j} = 1, \forall j; \sum_j y_{i,j} = 1, \forall i \right\}$.

Proof.

- permutation $\iff$ perfect matching in complete bipartite graph over $2n$ vertices
- permutation matrix polytope $\iff$ perfect matching polytope $\square$

Extended Formulation of $\mathcal{P}$

$$\sum_{i\in[n]} y_{i,j} = 1 \qquad \forall j \in [n]$$

$$\sum_{j\in[n]} y_{i,j} = 1 \qquad \forall i \in [n]$$

$$y_{ij} \geq 0 \qquad \forall i, j \in [n]$$

$$x_i = \sum_{j=1}^{n} j \cdot y_{ij} \qquad \forall i \in [n]$$

Lemma The permutation polytope $\mathcal{P}$ has extension complexity $O(n^2)$.

Outline

Spanning Tree Polytope

Recall:

Spanning Tree Polytope

- Given a connected graph $G = (V, E)$
- $\mathcal{P}_{\mathrm{ST}} :=$ conv $\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\mathrm{ST}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$
\begin{aligned}
\sum_{e \in E} x_e &= n - 1 && \\
\sum_{e \in E[S]} x_e &\leq |S| - 1 && \forall S \subseteq V, 2 \leq |S| \leq n - 1 \qquad (*) \\
x_e &\geq 0 && \forall e \in E
\end{aligned}
$$

- Choose a root $r \in V$ arbitrarily.
- For any spanning tree, we direct the edges from r to leaves: the tree becomes an out-arborescence rooted at r
- $y_{u \to v}$: whether (u, v) is a directed edge in the arborenscence.

$$
\begin{aligned}
\sum_{(u,v) \in E} y_{u \to v} &= 1 && \forall v \in V \setminus \{r\} \\
y_{v \to r} &= 0 && \forall (v, r) \in E \\
y_{u \to v} &\geq 0 && \forall u, v \text{ with } (u, v) \in E \\
x_{\{u,v\}} &= y_{u \to v} + y_{v \to u} && \forall (u, v) \in E \\
y \text{ supports 1 unit flow from } r \text{ to } v & && \forall v \in V \setminus \{r\} \qquad (\dagger)
\end{aligned}
$$

- $(\dagger)$ for every v can be captured using a maximum-flow LP, with $O(|E|)$ variables and constraints.

Theorem The formulation is an extended formulation of $\mathcal{P}_{\mathrm{ST}}$.

Proof.

- For any ST T of G, χ^T (with extension) is a valid solution
- Remaining goal: prove that every valid (x, y) satisfies:

$$\sum_{e \in E} x_e = n - 1 \tag{1}$$

$$\sum_{e \in E[S]} x_e \leq |S| - 1 \qquad \forall S \subseteq V, 2 \leq |S| \leq n - 1 \tag{2}$$

- every $v \in V \setminus \{r\}$ has 1 fractional incoming edge
- $\implies$ total fractional number of edges is $n - 1 \implies$ (1)

Theorem The formulation is an extended formulation of $\mathcal{P}_{\mathrm{ST}}$.

Proof.

- Remaining goal: prove that every valid (x, y) satisfies:

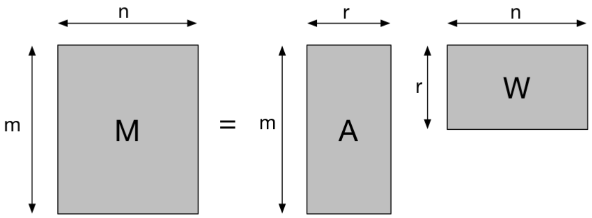
$$\sum_{e \in E[S]} x_e \leq |S| - 1 \qquad \forall S \subseteq V, 2 \leq |S| \leq n-1 \tag{2}$$

- Focus on $S \ni r$: $|S| - 1$ fractional edges with head in S
- Focus on $S \not\ni r, |S| \geq 2$. Let $v \in S$ be arbitrary.
- y supports 1 unit $r \to v$ flow
  $\implies$ ≥ 1 fractional edge from $V \setminus S$ to S
  $\implies$ at most $|S| - 1$ fractional edges inside S $\square$

- When G is complete graph, $\mathcal{P}_{\mathrm{ST}}$ has $O(n^3)$ extension complexity
- The lower bound is $\Omega(n^2)$
- Big open problem to close the gap.

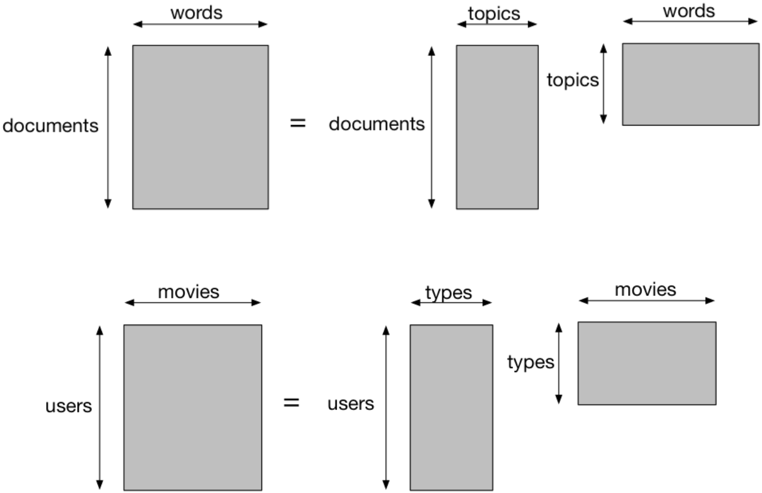
Outline

Def. The non-negative rank of a matrix $M \in \mathbb{R}_{\geq 0}^{m \times n}$ is the minimum $r \geq 0$ such that there are matrices $L \in \mathbb{R}_{\geq 0}^{m \times r}$ and $R \in \mathbb{R}_{\geq 0}^{r \times n}$ such that $M = LR$. We use $\text{rank}_+(M)$ to denote the non-negative rank of M.



- if we allow $L \in \mathbb{R}^{m \times r}$ and $R \in \mathbb{R}^{r \times n}$, then the non-negative rank becomes the rank
- the rank of a matrix can be computed efficiently
- it is NP-hard to compute the non-negative rank of a matrix

Application of Non-Negative Rank

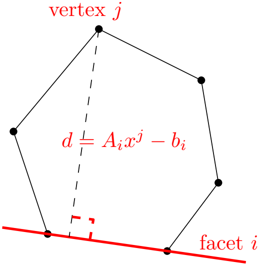


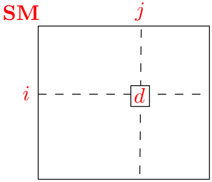
Def. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be defined as

$$\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b; Ex = f\},$$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $E \in \mathbb{R}^{m' \times n}$, $f \in \mathbb{R}^{m'}$. Assume the equations $Ex = f$ are linearly independent, and there is a 1-1 correspondence between inequalities in $Ax \leq b$ and facets of $\mathcal{P}$. Let $x^1, x^2, \cdots, x^v$ be all the vertices of $\mathcal{P}$. The slack matrix $\mathbf{SM}^{\mathcal{P}}$ of $\mathcal{P}$ w.r.t this description is a matrix in $\mathbb{R}^{m \times v}_{\geq 0}$ such that

$$\mathbf{SM}^{\mathcal{P}}_{i,j} = b_i - a_i x^j, \quad \text{where } a_i \text{ is the } i\text{-th row vector of } A.$$





Slack Matrix Theorem

Theorem [Yannakakis 91] For any polytope $\mathcal{P}$, we have $\text{xc}(\mathcal{P}) = \text{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

Notes

- Considering non-vertex points in $\mathcal{P}$ for the columns of $\mathbf{SM}^{\mathcal{P}}$ does not increase is non-negative rank
- Considering non-facet faces of $\mathcal{P}$ for rows of $\mathbf{SM}^{\mathcal{P}}$ does not increase its non-negative rank

Theorem [Yannakakis 91] For any polytope $\mathcal{P}$, we have $\text{xc}(\mathcal{P}) = \text{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

Proof of $\text{xc}(\mathcal{P}) \leq \text{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

- Given non-negative decomposition $\mathbf{SM}^{\mathcal{P}} = FV$ with $F \in \mathbb{R}_{\geq 0}^{m \times r}$ and $V \in \mathbb{R}_{\geq 0}^{r \times v}$
- we show the following is an extended formulation of $\mathcal{P}$ with complexity r:

$$Ax + Fy = b, y \geq 0 \qquad \mathcal{P}' = \{x : \exists y \geq 0, Ax + Fy = b\}$$

- if $\exists y \geq 0$ with $Ax + Fy = b$, then $Ax \leq b$ $\quad \mathcal{P}' \subseteq \mathcal{P}$
- fix vertex x^j: $b - Ax^j$ is the j-th column of $\mathbf{SM}^{\mathcal{P}}$
- it is a non-negative combination of columns of F
- so, $\exists y \geq 0$ with $b - Ax^j = Fy$ $\quad \mathcal{P} \subseteq \mathcal{P}'$ $\quad \square$

Slack Matrix Theorem

Theorem [Yannakakis 91] For any polytope $\mathcal{P}$, we have $\mathrm{xc}(\mathcal{P}) = \mathrm{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

Proof of $\mathrm{xc}(\mathcal{P}) \geq \mathrm{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

- Assume $\mathcal{P} = \{x : Ex + Fy = g, y \geq 0\}$, $E \in \mathbb{R}^{m \times n}, F \in \mathbb{R}^{m \times r}$ and $g \in \mathbb{R}^m$:
- For every i, $a_i x \leq b_i$ is implied by $Ex + Fy = g, y \geq 0$, and it is tight for some point in $\mathcal{P}$:

$$\exists \text{ row vector } \mu^i \in \mathbb{R}^m : \mu^i(E, g) = (a_i, b_i), \nu^i := \mu^i F \geq 0.$$

- Then, $b_i - a_i x^j = \mu^i g - \mu^i E x^j = \mu^i F y^j = \nu^i y^j$.

Slack Matrix Theorem

Theorem [Yannakakis 91] For any polytope $\mathcal{P}$, we have $\mathsf{xc}(\mathcal{P}) = \mathsf{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

Proof of $\mathsf{xc}(\mathcal{P}) \geq \mathsf{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

- $b_i - a_i x^j = \nu^i y^j$
- Then,

$$\mathbf{SM}^{\mathcal{P}} = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \vdots \\ \nu^m \end{pmatrix} (y^1, y^2, \cdots, y^v)$$

- This is a decomposition with rank r. $\square$

Outline

Travelling Salesman Problem (TSP) Polytope

- Given the complete graph $G = (V, \binom{V}{2})$
- $\mathcal{P}_{\text{TSP}} :=$ conv$(\{\chi^S, S \subseteq \binom{V}{2}$ is a TSP tour of V$\})$

Cut Polytope

- $G = (V, E)$: a connected graph
- $\mathcal{P}_{\text{cut}} :=$ conv $\left(\left\{\chi^{E(S,V \setminus S)} : S \subsetneq V, S \neq \emptyset\right\}\right)$

Correlation Polytope

- $\mathcal{P}_{\text{corr}} =$ conv $\left(\{bb^{\text{T}} : b \in \{0,1\}^n\}\right)$.

- [Samuel Fiorini, Serge Massar, Sebastian Pokutta, Hans Raj Tiwary and Ronald de Wolf]: "Exponential Lower Bounds for Polytopes in Combinatorial Optimization": All the above polytopes have exponential extension complexity.
- 2023 Godel Prize Winner Paper

General Matching Polytope

- Given a graph $G = (V, E)$
- $\mathcal{P}_{\mathrm{GM}} := \text{conv}\left(\left\{\chi^M : M \subseteq E \text{ is a matching in } G\right\}\right)$

Theorem (General Matching Polytope Theorem) $\mathcal{P}_{\mathrm{GM}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\begin{aligned} \sum_{e \in \delta(v)} x_e &\leq 1 && \forall v \in V \\ \sum_{e \in E(S)} x_e &\leq \frac{|S|-1}{2} && \forall S \subseteq V, |S| \text{ is odd} \\ x_e &\geq 0 && \forall e \in E \end{aligned} \tag{3}$$

- [Rothvoss 2017]: "The Matching Polytope has Exponential Extension Complexity." 2023 Godel Prize Winner Paper