

Advanced Algorithms (Fall 2025)

Extension Complexity of Polytopes

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- 1 Motivation and Definition
 - Example: Permutation Polytope
 - Extension Complexity of Spanning Tree Polytope
- 2 Connection Between Extension Complexity and Non-Negative Rank
- 3 Polytopes with Exponential Extension Complexity

Motivation

Typical Combinatorial Optimization Problem

Input: $[n]$: ground set

$\mathcal{S}$: feasible sets: a family of subsets of U , often
implicitly given

$w_i, i \in [n]$: values/costs of elements

Output: the set $S \in \mathcal{S}$ with the minimum/maximum
 $w(S) := \sum_{i \in S} w_i$

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$\mathcal{P} := \text{conv}(\{\chi^S : S \in \mathcal{S}\})$: convex hull of all valid solutions

LP to Solve Problem Exactly

$$\min / \max \quad \sum_{i=1}^n w_i x_i \quad \text{s.t.} \quad x \in \mathcal{P}$$

- inequality constraints needed to describe $x \in \mathcal{P}$ (or $\mathcal{P}$ in short) is $\text{facets}(\mathcal{P}) := \text{the number of facets of } \mathcal{P}$

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Def. An **extension** of a polytope $\mathcal{P} \in \mathbb{R}^n$ is a polyhedron $\mathcal{Q} \subseteq \mathbb{R}^{n+r}$ for some $r \geq 0$, such that $\mathcal{P}$ is the projection of $\mathcal{Q}$ to $\mathbb{R}^n$:

$$\mathcal{P} = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^r, (x, y) \in \mathcal{Q}\}$$

LP to Solve Problem Exactly with Auxiliary Variables

$$\min / \max \quad \sum_{i=1}^n c_i x_i \quad \text{s.t.} \quad (x, y) \in \mathcal{Q},$$

where $\mathcal{Q}$ is an extension of $\mathcal{P}$.

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Def. The **extension complexity** of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, denoted as $\text{xc}(\mathcal{P})$, is defined as follows:

$$\text{xc}(\mathcal{P}) := \min\{\text{facets}(\mathcal{Q}) : \mathcal{Q} \text{ is an extension of } \mathcal{P}\}.$$

Def. An **extended formulation** of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is a set of linear constraints:

$$(E, F) \begin{pmatrix} x \\ y \end{pmatrix} = g$$
$$y \geq 0$$

where $E \in \mathbb{R}^{N \times n}$, $F \in \mathbb{R}^{N \times r}$, $g \in \mathbb{R}^N$ are given, and $x \in \mathbb{R}^n$ is the vector of **main variables**, $y \in \mathbb{R}^r$ is the vector of **auxiliary variables**.

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$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \exists y \geq 0, (E, F) \begin{pmatrix} x \\ y \end{pmatrix} = g \right\}.$$

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Def. (An alternative definition) The **extension complexity** of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, denoted as $\text{xc}(\mathcal{P})$, is defined as the minimum complexity of an extended formulation of $\mathcal{P}$.

The Equivalence Between the Two Definitions

- $\text{xc}_1(\mathcal{P}), \text{xc}_2(\mathcal{P})$: $\text{xc}(\mathcal{P})$ according to the first/second definition

$$\text{xc}_2(\mathcal{P}) \leq \text{xc}_1(\mathcal{P})$$

- Given an extension $\mathcal{Q}$ of $\mathcal{P}$, we can use $\text{facets}(\mathcal{Q})$ inequalities (and some equalities, if the dimension of $\mathcal{Q}$ is smaller than the dimension of its host space) to describe $\mathcal{Q}$, one for each facet.
- For the i -th inequality $a_i x \geq b_i$, we introduce a variable y_i , and replace the inequality by $y_i = a_i x - b_i, y_i \geq 0$.
- This gives an extended formulation of $\mathcal{P}$ with $\text{facets}(\mathcal{Q})$ y -variables.
- Remark: there might be some auxiliary variables with no non-negativity constraints; but they can be removed.

$$\text{xc}_1(\mathcal{P}) \leq \text{xc}_2(\mathcal{P})$$

- An extended formulation with m y -variables defines a polyhedron with at most m facets.

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Lemma For any $S \subsetneq [n], S \neq \emptyset$, $\sum_{i \in S} x_i \geq \frac{|S|(|S|+1)}{2}$ is a facet of $\mathcal{P}$.

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Proof Sketch.

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- To show it's a **facet**, need to prove its dimension is $n - 2$
- We can find $x^0, x^1, \dots, x^{n-2}$ on the face such that $x^1 - x^0, x^2 - x^0, \dots, x^{n-2} - x^0$ are linearly independent. $\square$

Representation using Permutation Matrices

- Represent a permutation $x \in [n]^{[n]}$ by the **permutation matrix** $M \in \{0, 1\}^{n \times n}$ so that $M_{ij} = 1$ iff $x_i = j$.

Example : $(3, 1, 2) \iff \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

- **Crucial property:** x is a linear function of entries in M
- $\mathcal{P}' := \text{conv}(\{M : M \text{ is a permutation matrix}\})$

Lemma $\mathcal{P}' = \{y \in [0, 1]^{n \times n} : \sum_i y_{i,j} = 1, \forall j; \sum_j y_{i,j} = 1, \forall i\}.$

Proof.

- permutation $\iff$ perfect matching in complete bipartite graph over $2n$ vertices
- permutation matrix polytope $\iff$ perfect matching polytope



Extended Formulation of $\mathcal{P}$

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Lemma The permutation polytope $\mathcal{P}$ has extension complexity $O(n^2)$.

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Spanning Tree Polytope

Recall:

Spanning Tree Polytope

- Given a connected graph $G = (V, E)$
- $\mathcal{P}_{\text{ST}} := \text{conv}(\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\})$

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Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\text{ST}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\begin{aligned} \sum_{e \in E} x_e &= n - 1 \\ \sum_{e \in E[S]} x_e &\leq |S| - 1 & \forall S \subseteq V, 2 \leq |S| \leq n - 1 & \quad (*) \\ x_e &\geq 0 & \forall e \in E & \end{aligned}$$

- Choose a root $r \in V$ arbitrarily.
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$$\begin{array}{ll}
 \sum_{(u,v) \in E} y_{u \rightarrow v} = 1 & \forall v \in V \setminus \{r\} \\
 y_{v \rightarrow r} = 0 & \forall (v, r) \in E \\
 y_{u \rightarrow v} \geq 0 & \forall u, v \text{ with } (u, v) \in E \\
 x_{\{u,v\}} = y_{u \rightarrow v} + y_{v \rightarrow u} & \forall (u, v) \in E \\
 y \text{ supports 1 unit flow from } r \text{ to } v & \forall v \in V \setminus \{r\} \quad (\dagger)
 \end{array}$$

- $(\dagger)$ for every v can be captured using a maximum-flow LP, with $O(|E|)$ variables and constraints.

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- Remaining goal: prove that every valid (x, y) satisfies:

$$\sum_{e \in E} x_e = n - 1 \quad (1)$$

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- $\implies$ total fractional number of edges is $n - 1 \implies (1)$

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- y supports 1 unit $r \rightarrow v$ flow
 - $\implies \geq 1$ fractional edge from $V \setminus S$ to S
 - $\implies$ at most $|S| - 1$ fractional edges inside S

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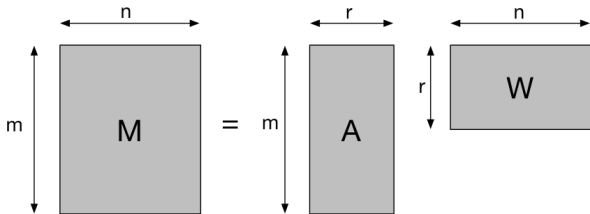
- When G is complete graph, $\mathcal{P}_{\text{ST}}$ has $O(n^3)$ extension complexity
- The lower bound is $\Omega(n^2)$
- Big open problem to close the gap.

Outline

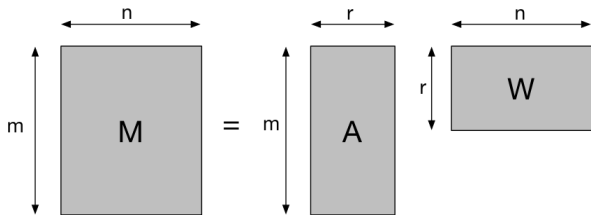
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Def. The **non-negative rank** of a matrix $M \in \mathbb{R}_{\geq 0}^{m \times n}$ is the minimum $r \geq 0$ such that there are matrices $L \in \mathbb{R}_{\geq 0}^{m \times r}$ and $R \in \mathbb{R}_{\geq 0}^{r \times n}$ such that $M = LR$. We use $\text{rank}_+(M)$ to denote the non-negative rank of M .

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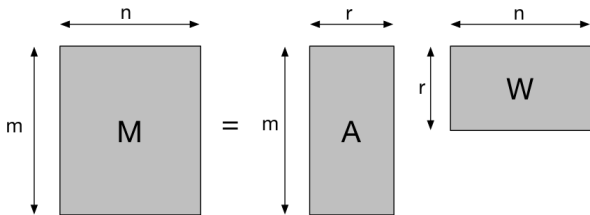


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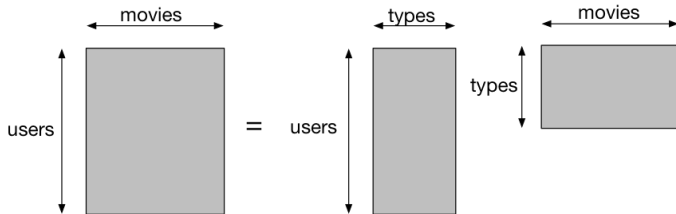
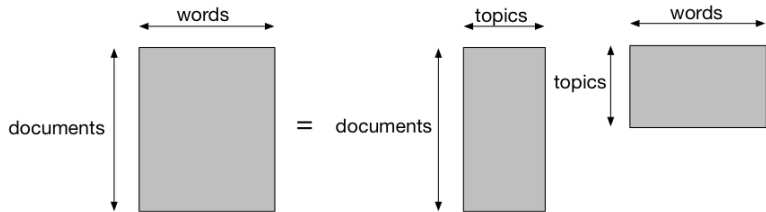
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- if we allow $L \in \mathbb{R}^{m \times r}$ and $R \in \mathbb{R}^{r \times n}$, then the non-negative rank becomes the **rank**
- the rank of a matrix can be computed efficiently
- it is **NP-hard** to compute the non-negative rank of a matrix

Application of Non-Negative Rank



Def. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be defined as

$$\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b; Ex = f\},$$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $E \in \mathbb{R}^{m' \times n}$, $f \in \mathbb{R}^{m'}$. Assume the equations $Ex = f$ are linearly independent, and there is a 1-1 correspondence between inequalities in $Ax \leq b$ and facets of $\mathcal{P}$.

Let $x^1, x^2, \dots, x^v$ be all the vertices of $\mathcal{P}$. The **slack matrix** $\text{SM}^{\mathcal{P}}$ of $\mathcal{P}$ w.r.t this description is a matrix in $\mathbb{R}_{\geq 0}^{m \times v}$ such that

$$\text{SM}_{i,j}^{\mathcal{P}} = b_i - a_i x^j, \quad \text{where } a_i \text{ is the } i\text{-th row vector of } A.$$

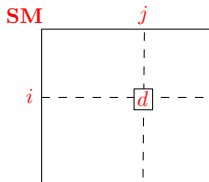
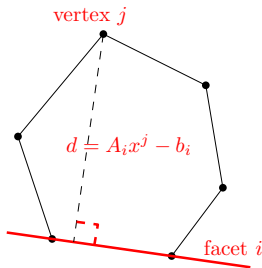
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Slack Matrix Theorem

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Notes

- Considering non-vertex points in $\mathcal{P}$ for the columns of $\text{SM}^{\mathcal{P}}$ does not increase its non-negative rank
- Considering non-facet faces of $\mathcal{P}$ for rows of $\text{SM}^{\mathcal{P}}$ does not increase its non-negative rank

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Proof of $\text{xc}(\mathcal{P}) \leq \text{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

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- Given non-negative decomposition $\text{SM}^{\mathcal{P}} = FV$ with $F \in \mathbb{R}_{\geq 0}^{m \times r}$ and $V \in \mathbb{R}_{\geq 0}^{r \times v}$
- we show the following is an extended formulation of $\mathcal{P}$ with complexity r :

$$Ax + Fy = b, y \geq 0 \quad \mathcal{P}' = \{x : \exists y \geq 0, Ax + Fy = b\}$$

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Proof of $\text{xc}(\mathcal{P}) \leq \text{rank}_+(\text{SM}^{\mathcal{P}})$.

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- so, $\exists y \geq 0$ with $b - Ax^j = Fy$ $\mathcal{P} \subseteq \mathcal{P}'$ □

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- This is a decomposition with rank r .



Outline

- 1 Motivation and Definition
 - Example: Permutation Polytope
 - Extension Complexity of Spanning Tree Polytope
- 2 Connection Between Extension Complexity and Non-Negative Rank
- 3 Polytopes with Exponential Extension Complexity

Travelling Salesman Problem (TSP) Polytope

- Given the complete graph $G = (V, \binom{V}{2})$
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- [Samuel Fiorini, Serge Massar, Sebastian Pokutta, Hans Raj Tiwary and Ronald de Wolf]: “Exponential Lower Bounds for Polytopes in Combinatorial Optimization”: All the above polytopes have exponential extension complexity.
- 2023 Godel Prize Winner Paper

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- [Rothvoss 2017]: “The Matching Polytope has Exponential Extension Complexity.” 2023 Godel Prize Winner Paper