Advanced Algorithms (Fall 2025)

# Greedy Algorithms

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Nanjing University

# Outline

1. Greedy Algorithms and Matroids
   - Recap: Maximum-Weight Spanning Tree Problem
   - Maximum-Weight Independent Set in Matroids
   - Examples of Matroids

2. Greedy Approximation Algorithms
   - $(\ln n + 1)$-Approximation for Set-Cover
   - $\left(1 - \frac{1}{e}\right)$-Approximation for Maximum Coverage
   - Submodular Functions
   - $\left(1 - \frac{1}{e}\right)$-Approximation for Cardinality-Constrained Submodular Maximization

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**Input:** Graph $G = (V, E)$ and edge weights $w \in \mathbb{Z}_{>0}^{E}$

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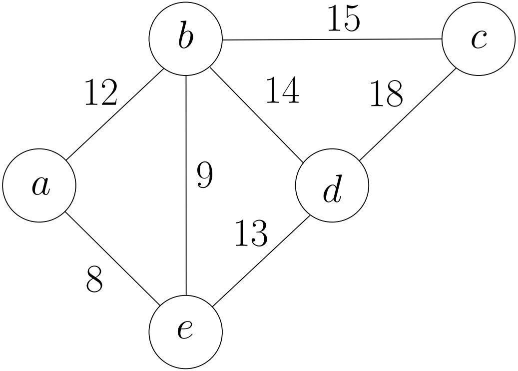

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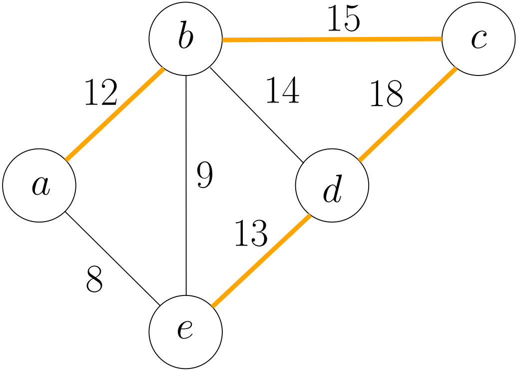

**Kruskal's Algorithm for Maximum-Weight Spanning Tree**

$F \leftarrow \emptyset$
sort edges in $E$ in non-increasing order of weights $w$
**for** each edge $(u, v)$ in the order **do**
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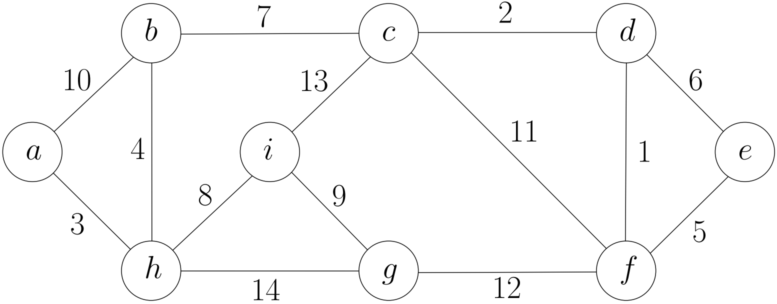

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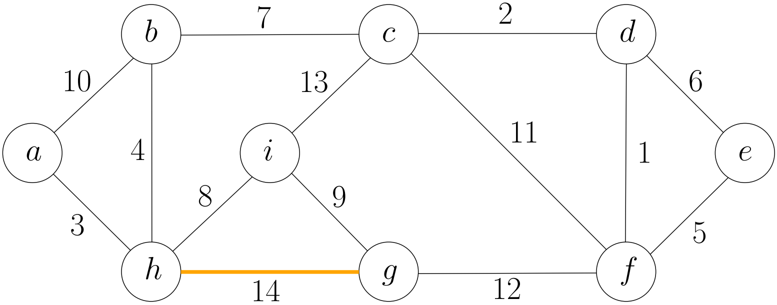

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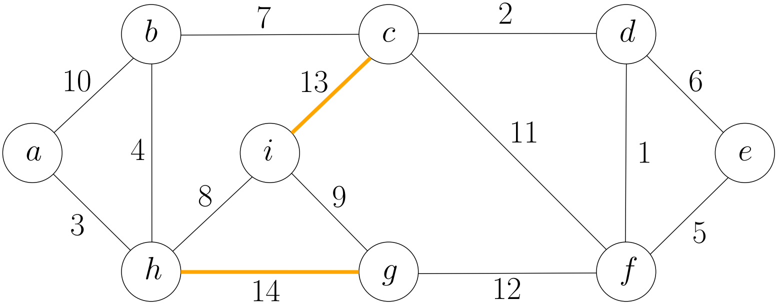

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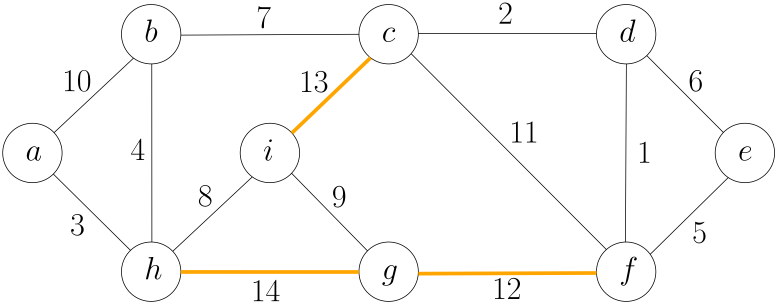

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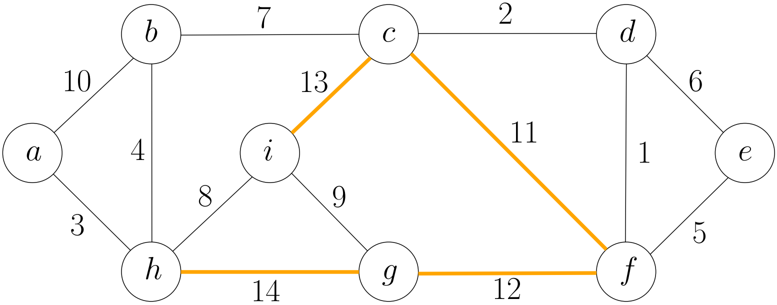

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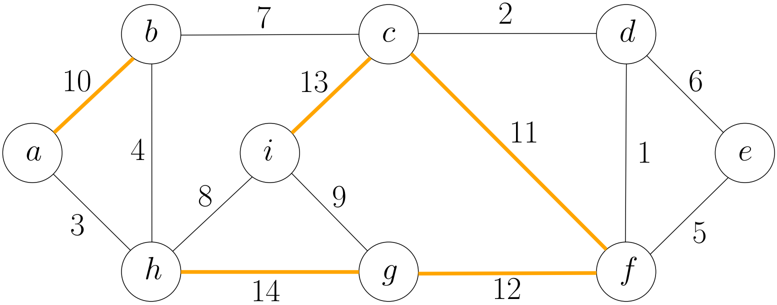

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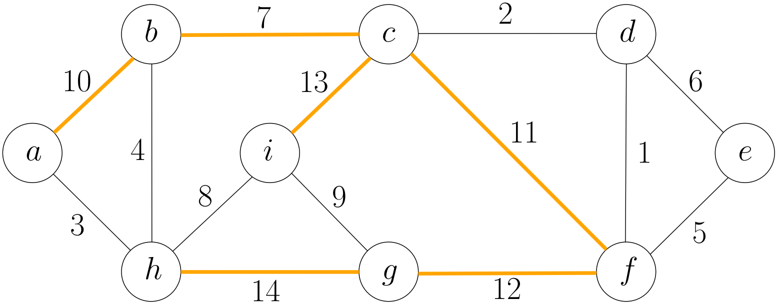

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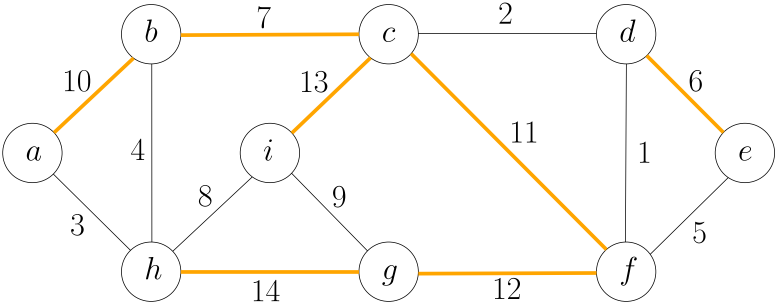

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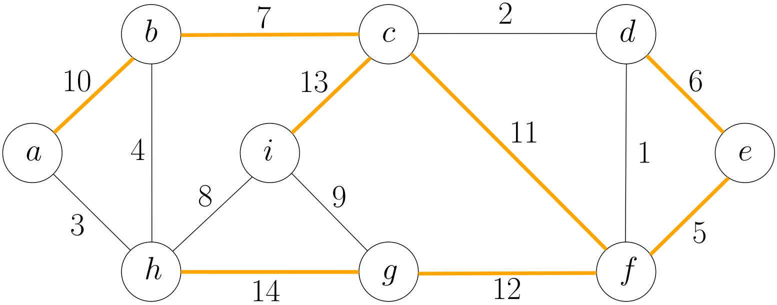

# Proof of Correctness of Kruskal's Algorithm

**Maximum-Weight Spanning Tree (MST) with Pre-Selected Edges**

**Input:** Graph $G = (V, E)$ and edge weights $w \in \mathbb{Z}_{>0}^{E}$
a set $F_0 \subseteq E$ of edges, that does not contain a cycle

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**Output:** the maximum-weight spanning tree $T = (V, E_T)$ of $G$ satisfying $F_0 \subseteq E_T$

**Lemma** (Key Lemma) Given an instance $(G = (V, E), w, F_0)$ of the MST with pre-selected edges problem, let $e^*$ be the maximum weight edge in $E \setminus F_0$ such that $F_0 \cup \{e^*\}$ does not contain a cycle. Then there is an optimum solution $T = (V, E_T)$ to the instance with $e^* \in E_T$.

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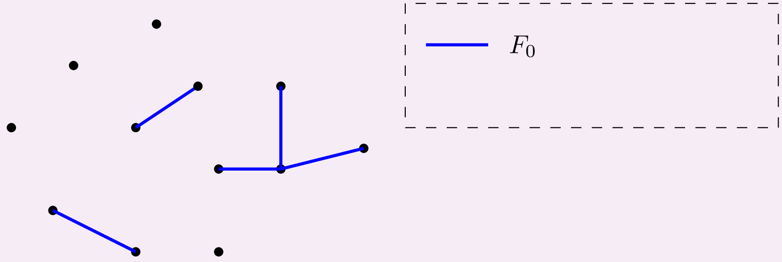
**Proof of Key Lemma.**

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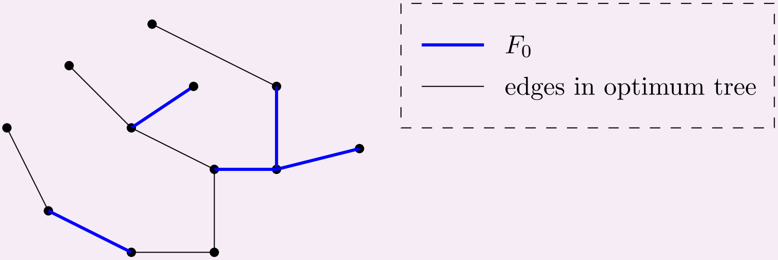


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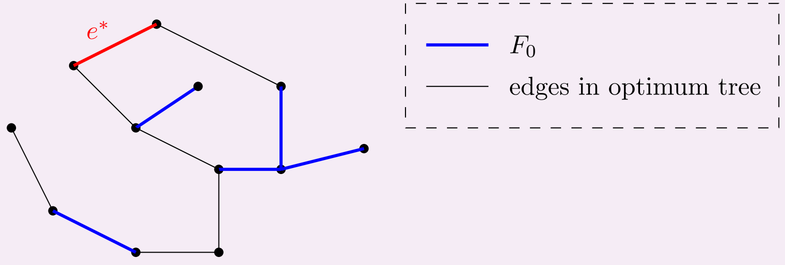


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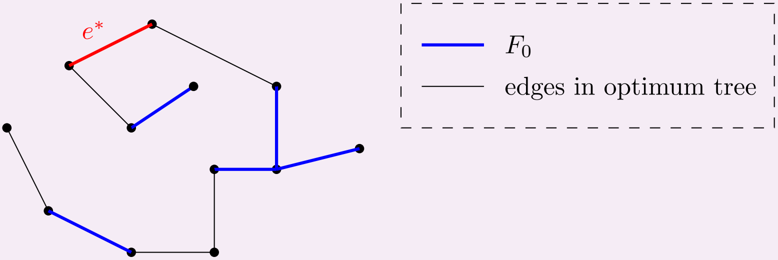


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### A General Maximization Problem

**Input:** $E$: the ground set of elements

$w \in \mathbb{Z}_{>0}^E$: weight vector on elements

$\mathcal{S}$: an (implicitly given) family of subsets of $E$

- $\emptyset \in \mathcal{S}$
- $\mathcal{S}$ is downward closed: if $A \in \mathcal{S}, B \subsetneq A$, then $B \in \mathcal{S}$.

**Output:** $A \in \mathcal{S}$ that maximizes $\sum_{e \in A} w_e$

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- maximum-weight spanning tree: $\mathcal{S} =$ family of forests

## Greedy Algorithm

$A \leftarrow \emptyset$

sort elements in $E$ in non-decreasing order of weights $w$

**for** each element $e$ in the order **do**

  **if** $A \cup \{e\} \in \mathcal{S}$ **then** $A \leftarrow A \cup \{e\}$

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- Knapsack Packing: given elements $E$, where every element has a value and a cost, and a cost budget $C$, the goal is to find a maximum value subset of items with cost at most $C$

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A ← ∅
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for each element e in the order do
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- Matroids: cases where greedy algorithm is optimum

**Def.** A (finite) matroid $\mathcal{M}$ is a pair $(E, \mathcal{I})$, where $E$ is a finite set (called the ground set) and $\mathcal{I}$ is a family of subsets of $E$ (called independent sets) with the following properties:

1. $\emptyset \in \mathcal{I}$.
2. (downward-closed property) If $B \subsetneq A \in \mathcal{I}$, then $B \in \mathcal{I}$.
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Proof of Exchange Property.

- $|B| < |A| \Rightarrow (V, B)$ has more CC than $(V, A)$.
- Some edge in $A$ connects two different CC of $(V, B)$. $\square$

## Feasible Family for Knapsack Packing Does Not Satisfy Augmentation Property

- $c_1 = c_2 = 10, c_3 = 20, C = 20$.
- $\{1, 2\}, \{3\} \in \mathcal{I}$, but $\{1, 3\}, \{2, 3\} \notin \mathcal{I}$.

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### Feasible Family for Bipartite Matching Does Not Satisfy Augmentation Property

- Complete bipartite graph between $\{a_1, a_2\}$ and $\{b_1, b_2\}$.
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**Theorem** The greedy algorithm gives optimum solution for the maximum-weight independent set problem in a matroid.

**Lemma** (Key Lemma)

- given: matroid $\mathcal{M} = (E, \mathcal{I})$, weights $w \in \mathbb{Z}_{>0}^{E}$, $A \in \mathcal{I}$,
- goal: find a maximum weight independent set containing $A$
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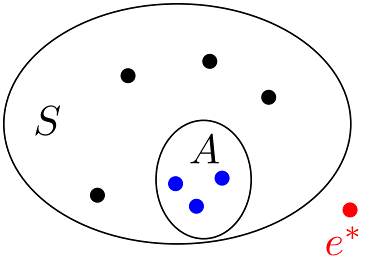

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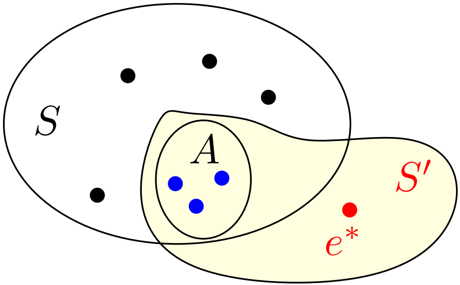

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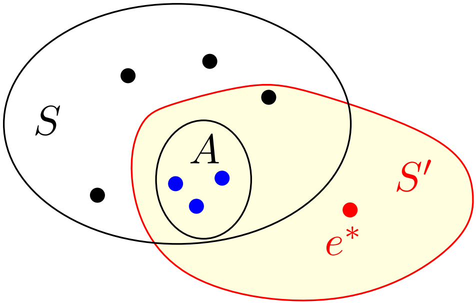

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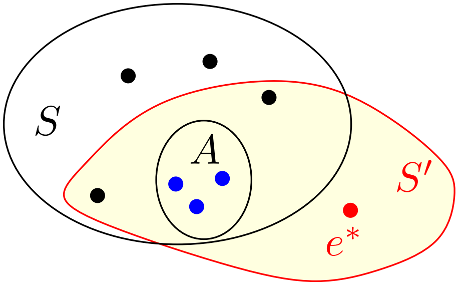

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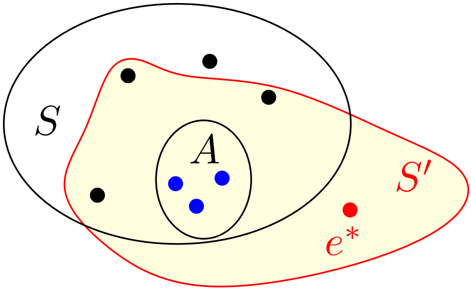

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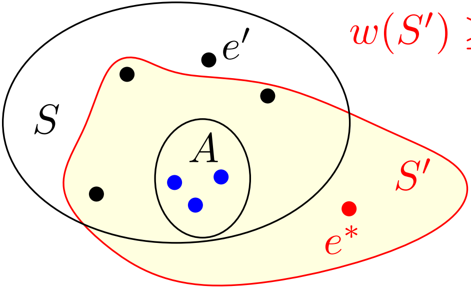
$w_{e^*} \geq w_{e'}$

$w(S') \geq w(S)$



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$S' \leftarrow A \cup \{e^*\}$
**while** $|S'| < |S|$ **do**
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**Lemma** (Key Lemma)

- given: matroid $\mathcal{M} = (E, \mathcal{I})$, weights $w \in \mathbb{Z}_{>0}^E$, $A \in \mathcal{I}$,
- goal: find a maximum weight independent set containing $A$
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- $w(S') := \sum_{e \in S'} w_e \geq w(S) \implies S'$ is also optimum $\square$

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- Laminar Matroid: laminar family of subsets of $E$ $\{E_1, E_2, \cdots, E_t\}$, positive integers $k_1, k_2, \cdots, k_t$
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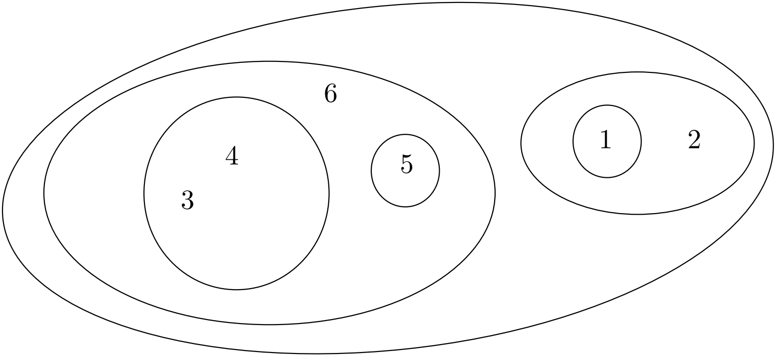
**Def.** A family $\{E_1, E_2, \cdots, E_t\}$ of subsets of $E$ is said to be laminar if for every two distinct subsets $E_i, E_j$ in the family, we have $E_i \cap E_j = \emptyset$ or $E_i \subsetneq E_j$ or $E_j \subsetneq E_i$.

- $\Big\{\{1\}, \{1,2\}, \{3,4\}, \{5\}, \{3,4,5,6\}, \{1,2,3,4,5,6\}\Big\}$ is a laminar family.



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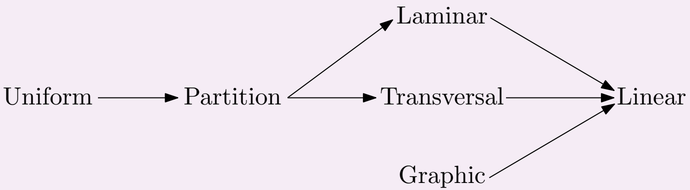
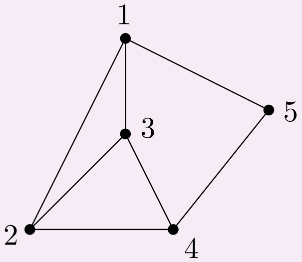
**Relationship between matroids**



## A Graphic Matroid is A Linear Matroid



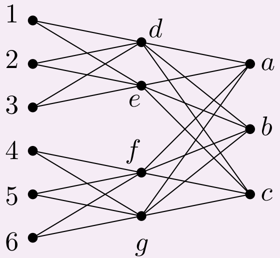

| edges | vectors |
|---|---|
| $(1,2)$ | $(1,-1,0,0,0)$ |
| $(1,3)$ | $(1,0,-1,0,0)$ |
| $(1,5)$ | $(1,0,0,0,-1)$ |
| $(2,3)$ | $(0,1,-1,0,0)$ |
| $(2,4)$ | $(0,1,0,-1,0)$ |
| $(3,4)$ | $(0,0,1,-1,0)$ |
| $(4,5)$ | $(0,0,0,1,-1)$ |

# A Laminar Matroid is A Linear Matroid

## Example

| sets | upper bounds |
|---:|:---:|
| $\{1, 2, 3\}$ | 2 |
| $\{4, 5, 6\}$ | 2 |
| $\{1, 2, 3, 4, 5, 6\}$ | 3 |

## A DAG (left to right)



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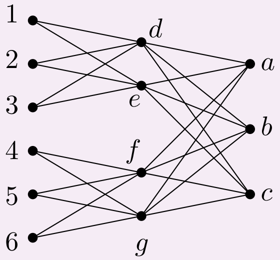


- $x^a, x^b, x^c \in \mathbb{R}^3$ are linearly independent rational vectors
- $x^d, x^e, x^f, x^g$: $\text{rand}(0,1) \cdot x^a + \text{rand}(0,1) \cdot x^b + \text{rand}(0,1) \cdot x^c$
- $x^1, x^2, x^3$: $\text{rand}(0,1) \cdot x^d + \text{rand}(0,1) \cdot x^e$
- $x^4, x^5, x^6$: $\text{rand}(0,1) \cdot x^f + \text{rand}(0,1) \cdot x^g$
- each $\text{rand}(0,1)$ gives an independent random real in $[0,1]$
- almost surely, all the random numbers are algebraically independent

# Outline

# Recap: Approximation Algorithms

- For minimization problems:

$$\text{approximation ratio} := \frac{\text{cost of our solution}}{\text{cost of optimum solution}} \geq 1$$

- For maximization problems:

$$\text{approximation ratio} := \frac{\text{value of our solution}}{\text{value of optimum solution}} \leq 1$$

or

$$\text{approximation ratio} := \frac{\text{value of optimum solution}}{\text{value of our solution}} \geq 1$$

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**Set Cover**

**Input:** $U, |U| = n$: ground set

$S_1, S_2, \cdots, S_m \subseteq U$

**Output:** minimum size set $C \subseteq [m]$ such that $\bigcup_{i \in C} S_i = U$

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## Greedy Algorithm for Set Cover

1: $C \leftarrow \emptyset, U' \leftarrow U$
2: **while** $U' \neq \emptyset$ **do**
3: choose the $i$ that maximizes $|U' \cap S_i|$
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5: **return** $C$

- $g$: minimum number of sets needed to cover $U$

**Lemma** Let $u_t, t \in \mathbb{Z}_{\geq 0}$ be the number of uncovered elements after $t$ steps. Then for every $t \geq 1$, we have

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- at beginning of step $t$, some set in $S_1^*, S_2^*, \cdots, S_g^*$ must contain $\geq \frac{u_{t-1}}{g}$ uncovered elements
- $u_t \leq u_{t-1} - \frac{u_{t-1}}{g} = \left(1 - \frac{1}{g}\right) u_{t-1}$. $\square$

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- Let $t = \lceil g \cdot \ln n \rceil$. $u_0 = n$. Then

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- A more careful analysis gives a $H_n$-approximation, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the $n$-th harmonic number.
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### $(1 - c)\ln n$-hardness for any $c = \Omega(1)$

Let $c > 0$ be any constant. There is no polynomial-time $(1 - c)\ln n$-approximation algorithm for set-cover, unless

- NP $\subseteq$ quasi-poly-time, [Lund, Yannakakis 1994; Feige 1998]
- P $=$ NP. [Dinur, Steuer 2014]

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- The $(1 - \frac{1}{e})$-approximation extends to a more general problem.

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**Def.** Let $n \in \mathbb{Z}_{>0}$. A set function $f : 2^{[n]} \to \mathbb{R}$ is called submodular if it satisfies one of the following three equivalent conditions:

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**Def.** Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the rank of any $A \subseteq E$ is defined as

$$r_{\mathcal{M}}(A) = \max \left\{ |A'| : A' \subseteq A, A' \in \mathcal{I} \right\}.$$

The function $r_{\mathcal{M}} : 2^E \to \mathbb{Z}_{\geq 0}$ is called the rank function of $\mathcal{M}$.

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- cut function: given graph $G = ([n], E)$

$$f(A) = \left| E(A, [n] \setminus A) \right|, \forall A \subseteq [n]$$

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# Outline

# $(1-\frac{1}{e})$-Approximation for Submodular Maximization with Cardinality Constraint

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**Input:** An oracle to a non-negative monotone submodular function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$, $\qquad k \in [n]$

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**Greedy Algorithm for the Problem**

```
S ← ∅
for t ← 1 to k do
    choose the i that maximizes f(S ∪ {i})
    S ← S ∪ {i}
return S
```

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**Theorem** Greedy algorithm gives $(1-\frac{1}{e})$-approximation for submodular-maximization under a cardinality constraint.

Proof.

- $o$: optimum value
- $p_t$: value obtained by greedy algorithm after step $t$
- need to prove: $p_t \geq p_{t-1} + \dfrac{o - p_{t-1}}{k}$
- $o - p_t \leq o - p_{t-1} - \frac{o-p_{t-1}}{k} = \left(1 - \frac{1}{k}\right)(o - p_{t-1})$
- $o - p_k \leq \left(1 - \frac{1}{k}\right)^k (o - p_0) \leq \frac{1}{e} \cdot o$
- $p_k \geq \left(1 - \frac{1}{e}\right) \cdot o$ $\square$

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**Lemma** A non-negative submodular set function $f: 2^{[n]} \to \mathbb{R}_{\geq 0}$ is sub-additive.

*Proof.*

For $A, B \subseteq [n]$, we have $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$.
So, $f(A \cup B) \leq f(A) + f(B)$ as $f(A \cap B) \geq 0$. $\square$

**Lemma** Let $f : 2^{[n]} \to \mathbb{R}$ be submodular. Let $S \subseteq [n]$, and $f_S(A) = f(S \cup A) - f(S)$ for every $A \subseteq [n]$. ($f_S$ is the marginal value function for set $S$.) Then $f_S$ is also submodular.

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Proof.

- Let $A, B \subseteq [n] \setminus S$; it suffices to consider ground set $[n] \setminus S$.

$$
\begin{aligned}
& f_S(A \cup B) + f_S(A \cap B) - (f_S(A) + f_S(B)) \\
&= f(S \cup A \cup B) - f(S) + f(S \cup (A \cap B)) - f(S) \\
&\quad - \Big( f(S \cup A) - f(S) + f(S \cup B) - f(S) \Big) \\
&= f(S \cup A \cup B) + f(S \cup (A \cap B)) - f(S \cup A) - f(S \cup B) \\
&\leq 0
\end{aligned}
$$

- The last inequality is by $S \cup A \cup B = (S \cup A) \cup (S \cup B)$, $S \cup (A \cap B) = (S \cup A) \cap (S \cup B)$ and submodularity of $f$. $\square$

## Proof of $p_t \geq p_{t-1} + \frac{o - p_{t-1}}{k}$.

- $S^* \subseteq [n]$: optimum set, $|S^*| = k$, $o = f(S^*)$
- $S$: set chosen by the algorithm at beginning of time step $t$
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- for the $i$, we have

$$f(S \cup \{i\}) - f(S) \geq \frac{1}{k}(f(S^*) - f(S))$$

$$p_t \geq f(S \cup \{i\}) \geq p_{t-1} + \frac{1}{k}(o - p_{t-1}) \qquad \square$$

Submodular Maximization for Monotone Functions:

| Constraint | Approx. | Hardness | Technique |
|---|---|---|---|
| $\|S\| \leq k$ | $1 - 1/e$ | $1 - 1/e$ | greedy |
| matroid | $1 - 1/e$ | $1 - 1/e$ | multilinear ext. |
| $O(1)$ knapsacks | $1 - 1/e$ | $1 - 1/e$ | multilinear ext. |
| $k$ matroids | $k + \epsilon$ | $\Omega(k/\log k)$ | local search |
| $k$ matroids<br>$O(1)$ knapsacks | $O(k)$ | $\Omega(k/\log k)$ | multilinear ext. |

Submodular Maximization for Non-Monotone Functions:

| Constraint | Approx. | Hardness | Technique |
|---|---|---|---|
| Unconstrained | $1/2$ | $1/2$ | combinatorial |
| matroid | $1/e$ | $0.48$ | multilinear ext. |
| $O(1)$ knapsacks | $1/e$ | $0.49$ | multilinear ext. |
| $k$ matroids | $k + O(1)$ | $\Omega(k/\log k)$ | local search |
| $k$ matroids<br>$O(1)$ knapsacks | $O(k)$ | $\Omega(k/\log k)$ | multilinear ext. |

From Prof. Jan Vodrak's slides "Optimization of Submodular Functions"

# Submodular Minimization

| Constraint | Approx. | Hardness | Technique |
| --- | --- | --- | --- |
| Unconstrained | $1$ | $1$ | combinatorial |
| $\|S\| \geq k$, Monotone | $\tilde{O}(\sqrt{n})$ * | $\Omega(\sqrt{n})$ * | combinatorial |

- * bounds are for query complexity under oracle model.