Advanced Algorithms (Fall 2025)

Linear Programming

Lecturers: 尹一通，刘景铖，栗师

Nanjing University

Outline

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Typical Combinatorial Optimization Problem

Input: $[n]$: ground set

$\mathcal{S}$: feasible sets: a family of subsets of U, often implicitly given

$w_i, i \in [n]$: values/costs of elements

Output: the set $S \in \mathcal{S}$ with the minimum/maximum $w(S) := \sum_{i \in S} w_i$

Example:

- Shortest Path, Minimum Spanning Tree
- Maximum Independent Set, Maximum Matching, Knapsack Packing

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- CO problem $\Longleftrightarrow$ Integer Program (IP) $\xrightarrow{\text{relax?}}$ Linear Program (LP)
- In general: Integer programming is NP-hard; linear programming is in P

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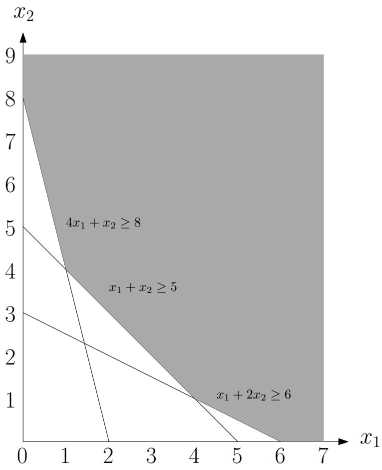
\begin{aligned}
\min \quad & 7x_1 + 4x_2 \\
& x_1 + x_2 \geq 5 \\
& x_1 + 2x_2 \geq 6 \\
& 4x_1 + x_2 \geq 8 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



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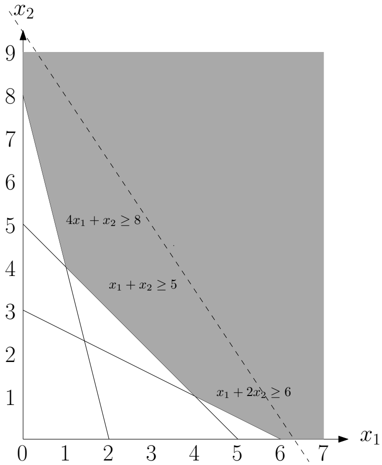

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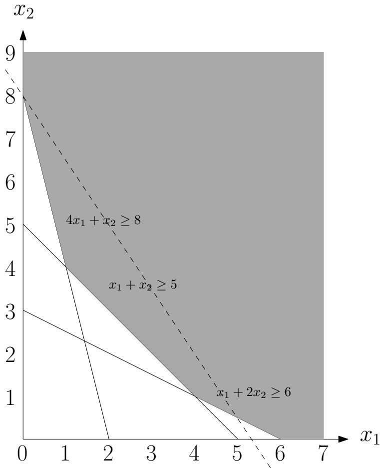

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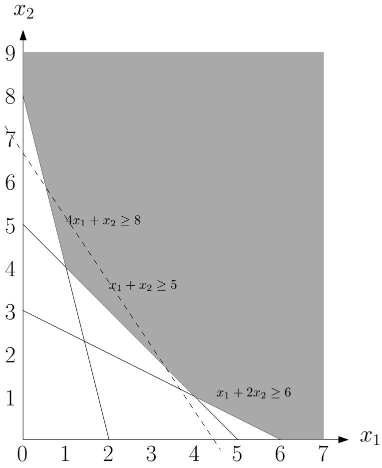

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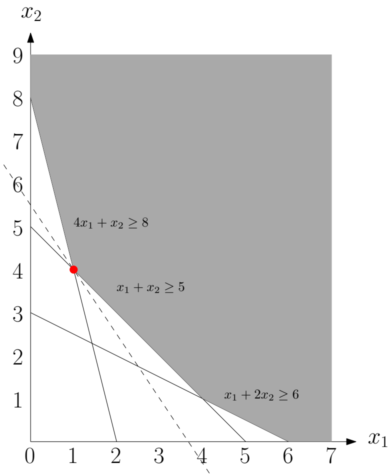

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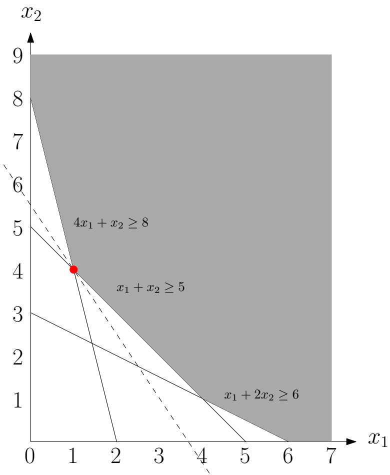
- optimum solution: $x_1 = 1, x_2 = 4$
- optimum value $=$ $7 \times 1 + 4 \times 4 = 23$



Standard Form of Linear Programs

$$
\begin{aligned}
\min \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &\geq b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &\geq b_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots & \\
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- n: number of variables $\qquad$ m: number of constraints
- Other considerations: $\leq$ constraints? equlities?
- variables can be negative? maximization problem?

Standard Form of Linear Programs

$$x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \qquad c := \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n,$$

$$A := \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \in \mathbb{R}^{n \times m}, \quad b := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m.$$

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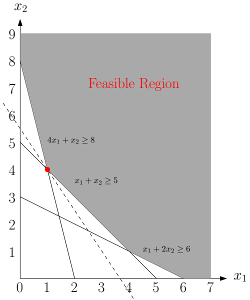
Standard Form of Linear Program

$$
\begin{aligned}
\min \quad & c^{\mathrm{T}}x \\
Ax &\ge b \\
x &\ge 0
\end{aligned}
$$

- $\geq$: coordinate-wise less than or equal to

Preliminaries

- feasible region: the set of x's satisfying $Ax \geq b, x \geq 0$



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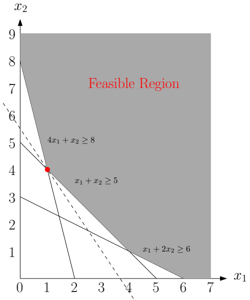
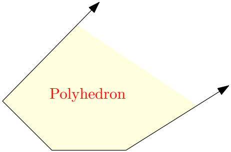
- a polyhedron is the intersection of finite number of closed half-spaces
- so, feasible region is a polyhedron





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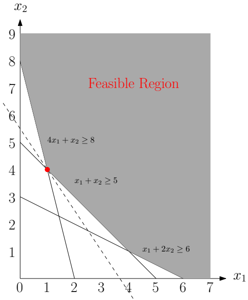
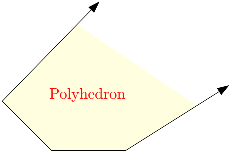
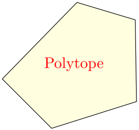
- if every coordinate has an upper and lower bound in the polyhedron, then the polyhedron is a polytope







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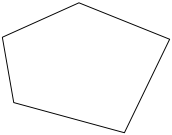
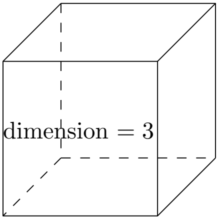

dimension $= 2$



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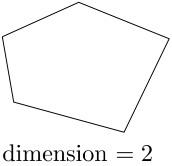
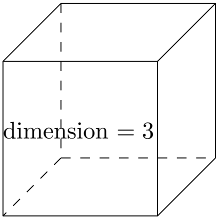
- Assume the linear inequality $a^{\mathrm{T}}x \leq b$ holds for every $x \in \mathcal{P}$, and some $x \in \mathcal{P}$ satisfies $a^{\mathrm{T}}x = b$. Then $\{x \in \mathcal{P} : a^{\mathrm{T}}x = b\}$ is said to be a face of $\mathcal{P}$.





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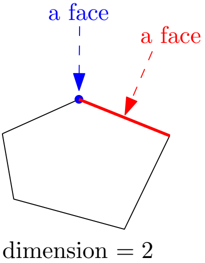

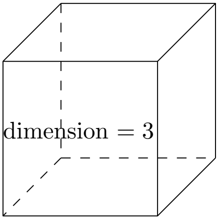

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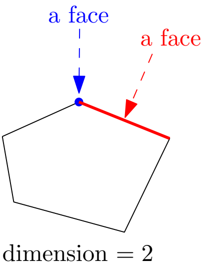

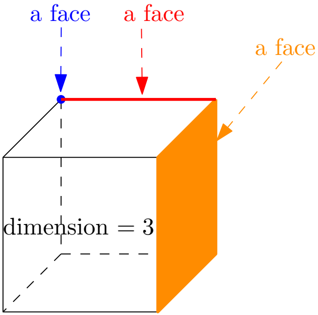

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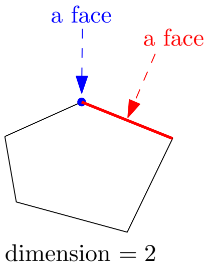

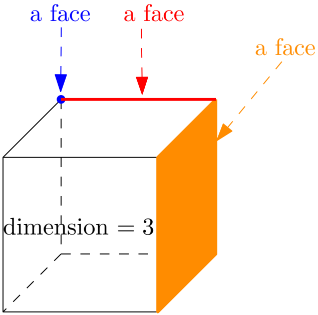

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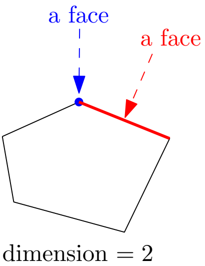

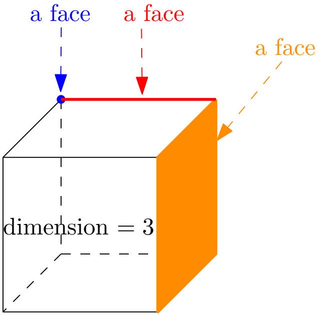

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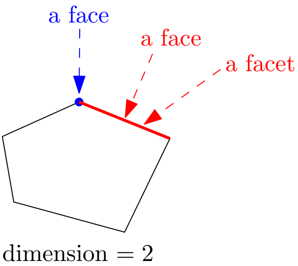

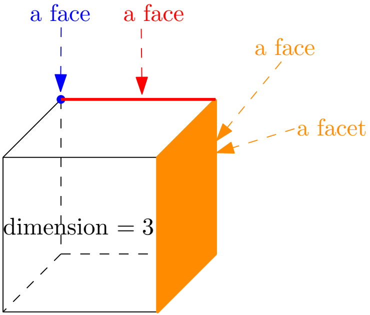

Preliminaries

- x is a convex combination of $\{x^{(1)}, x^{(2)}, \cdots, x^{(t)}\}$ if the following condition holds: there exist $\lambda_1, \lambda_2, \cdots, \lambda_t \in [0, 1]$ such that

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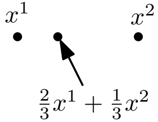

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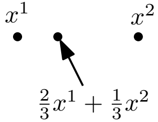

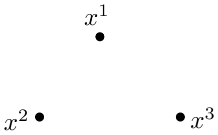

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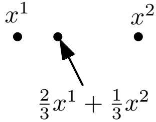

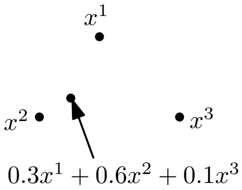

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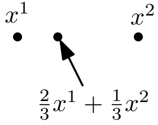

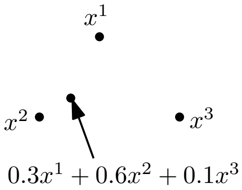

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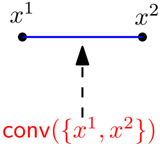

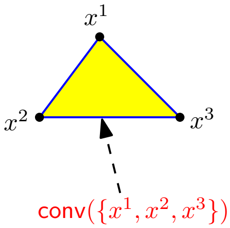

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- x is a convex combination of $\{x^{(1)}, x^{(2)}, \cdots, x^{(t)}\}$ if the following condition holds: there exist $\lambda_1, \lambda_2, \cdots, \lambda_t \in [0, 1]$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_t = 1, \qquad \lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots + \lambda_t x^{(t)} = x$$

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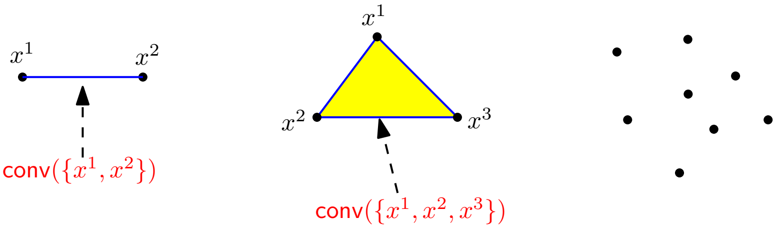

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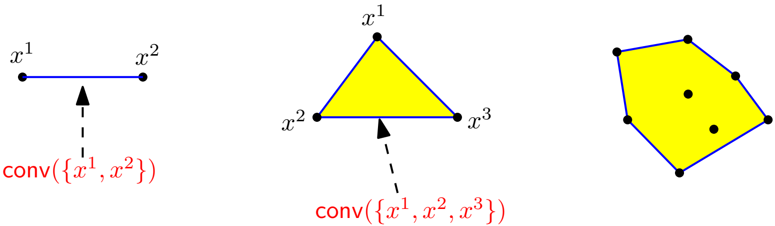
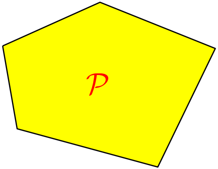


Terminology and Preliminaries

- let $\mathcal{P}$ be polytope, $x \in \mathcal{P}$. If there are no other points $x', x'' \in \mathcal{P}$ such that x is a convex combination of x' and x'', then x is called a vertex/extreme point of $\mathcal{P}$



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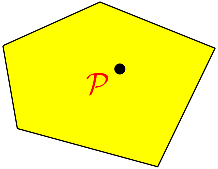

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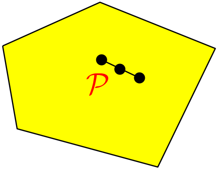

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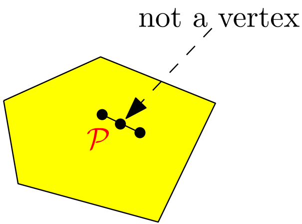

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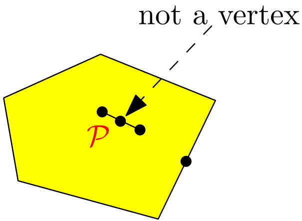

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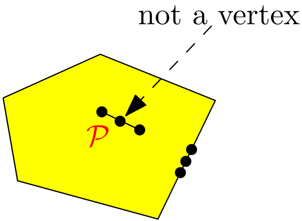

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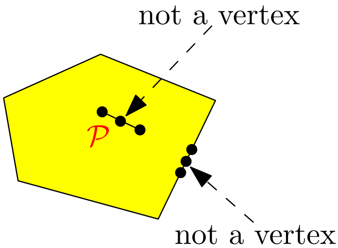

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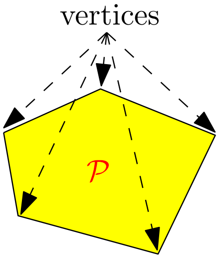

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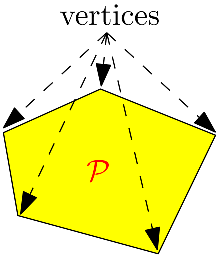
Lemma A polytope has finite number of vertices, and it is the convex hull of the vertices.



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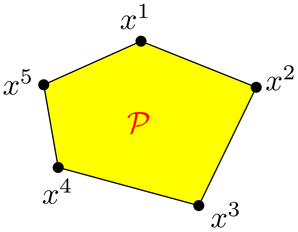
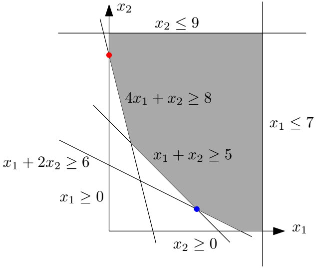


$$\mathcal{P} = \text{conv}(\{x^1, x^2, x^3, x^4, x^5\})$$

Lemma Let $x \in \mathbb{R}^n$ be a vertex of a polytope. Then, there are n constraints in the definition of the polytope, such that x is the unique solution to the linear system obtained from the n constraints by replacing inequalities to equalities.



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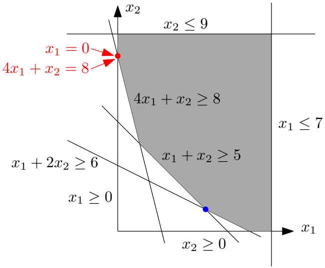

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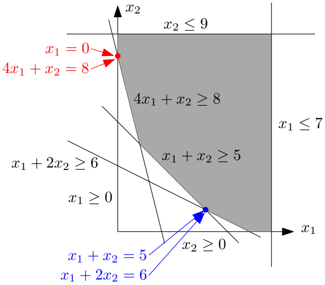

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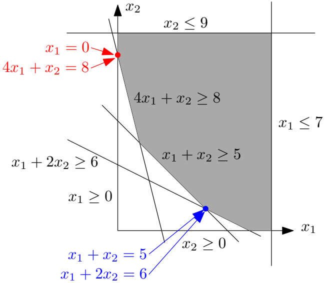

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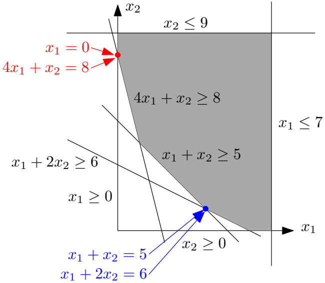

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Special cases (for minimization linear programs):

- if feasible region is empty, then its value is ∞
- if the feasible region is unbounded, then its value can be $-\infty$

Outline

Algorithms for Linear Programming

algorithm	running time	practice
Simplex Method	exponential time	fast
Ellipsoid Method	polynomial time	slow
Interior Point Method	polynomial time	fast

Simplex Method

- [Dantzig, 1946]

- move from one vertex to another, so as to improve the objective
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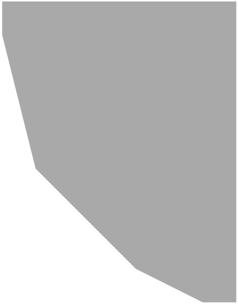

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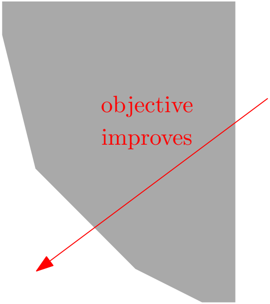

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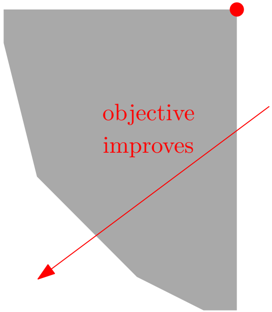

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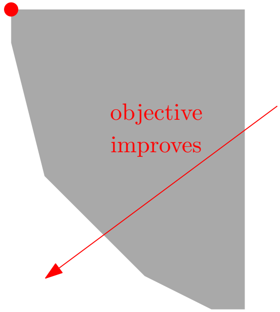

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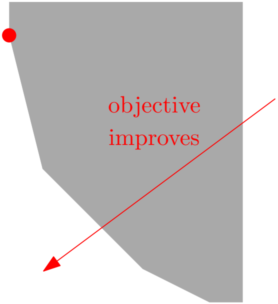

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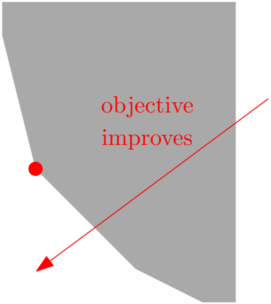

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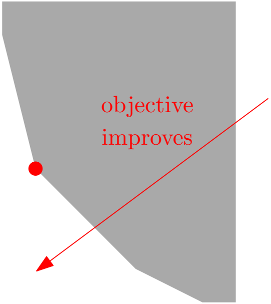

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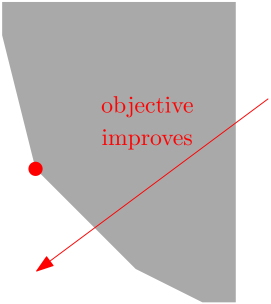

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- [Spielman-Teng,2002]: smoothed analysis

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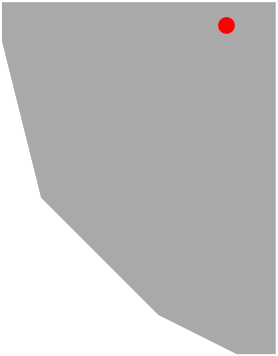

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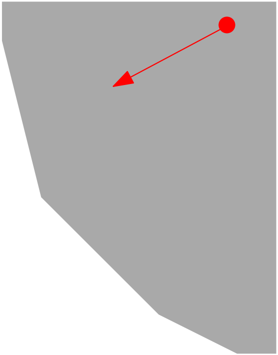

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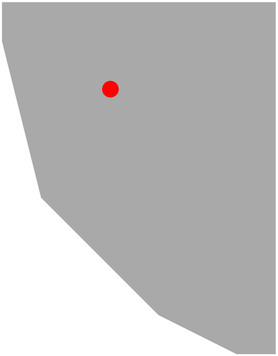

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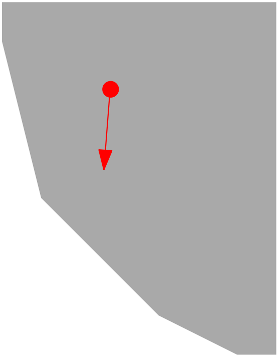

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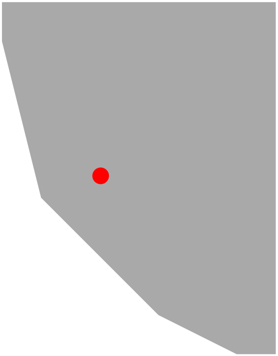

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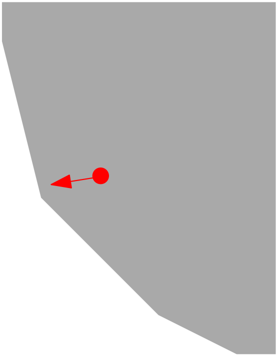

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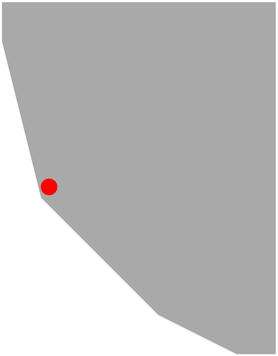

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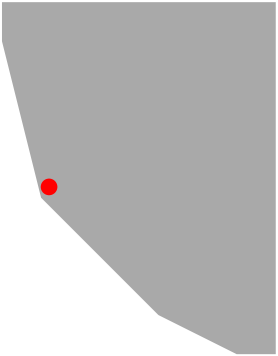

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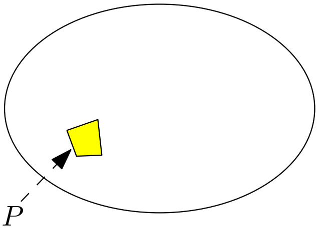

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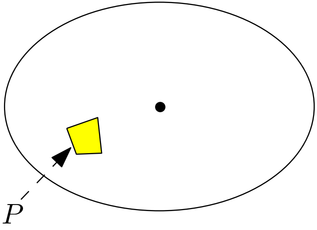

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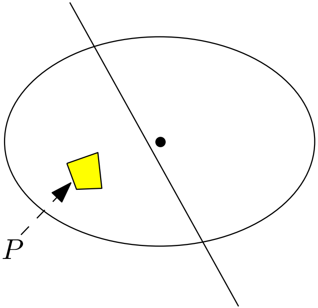

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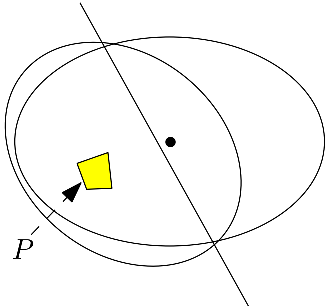

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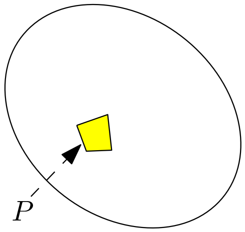

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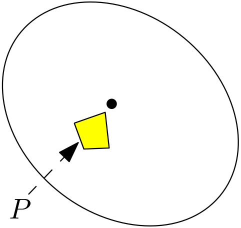

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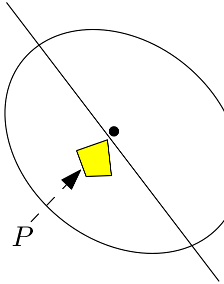

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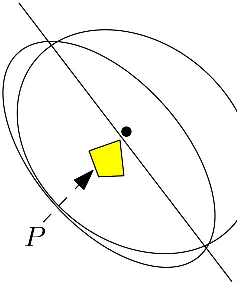

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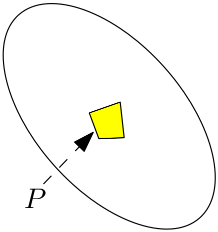

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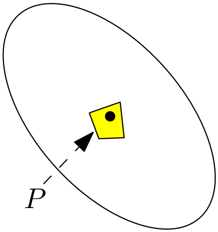

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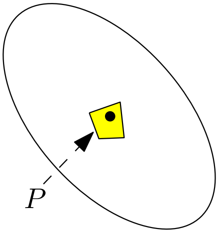

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Open Problem

Can linear programming be solved in strongly polynomial time algorithm?

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- domain: computer science, mathematics, operations research, economics
- types of problems: transportation, scheduling, clustering, network routing, resource allocation, facility location

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Research Directions

- polynomial time exact algorithm
- polynomial time approximation algorithm
- sub-routines for the branch-and-bound metheod for integer programming
- other algorithmic models: online algorithm, distributed algorithms, dynamic algorithms, fast algorithms

Typical Combinatorial Optimization Problem

Input: $[n]$: ground set

$\mathcal{S}$: feasible sets: a family of subsets of U, often implicitly given

$w_i, i \in [n]$: values/costs of elements

Output: the set $S \in \mathcal{S}$ with the minimum/maximum $w(S) := \sum_{i \in S} w_i$

Def. For any $S \subseteq [n]$, we use $\chi^S \in \{0,1\}^{[n]}$ to denote the indicator vector for S:

$$\chi_i^S = \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S \end{cases}$$

polytope of interest: $\mathcal{P} = \text{conv}\big(\{\chi^S : S \in \mathcal{S}\}\big)$

Examples

Bipartite Matching Polytope

- Given bipartite graph $G = (L \cup R, E)$
- $\mathcal{P}_{\mathrm{BM}} := \mathrm{conv}\big(\{\chi^M : M \text{ is a matching in } G\}\big)$

General Matching Polytope

- Given a graph $G = (V, E)$
- $\mathcal{P}_{\mathrm{GM}} := \mathrm{conv}\left(\left\{\chi^M : M \subseteq E \text{ is a matching in } G\right\}\right)$

Spanning Tree Polytope

- Given a connected graph $G = (V, E)$
- $\mathcal{P}_{\mathrm{ST}} := \mathrm{conv}\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

Examples

Travelling Salesman Problem (TSP) Polytope

- Given the complete graph $G = (V, \binom{V}{2})$
- $\mathcal{P}_{\text{TSP}} :=$ conv$(\{\chi^S, S \subseteq \binom{V}{2}$ is a TSP tour of V$\})$

polytope of interest: $\mathcal{P} = \mathsf{conv}\big(\{\chi^S : S \in \mathcal{S}\}\big)$

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1. In some cases, $\mathcal{P}$ has polynomial number of facets
2. In some cases, $\mathcal{P}$ has exponential number of facets, but has an efficient separation oracle.
3. In some cases, $\mathcal{P}$ does not have an efficient separation oracle, unless P $=$ NP.

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Def. A polytope $\mathcal{P} \subseteq [0,1]^n$ is said to be integral, if all vertices of $\mathcal{P}$ are in $\{0,1\}^n$.

Lemma For a $\mathcal{Q} \subseteq [0,1]^n$, if $\mathcal{Q} \cap \{0,1\}^n = \{\chi^S : S \in \mathcal{S}\}$ and $\mathcal{Q}$ is integral, then $\mathcal{Q} = \mathcal{P}$.

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Proof.

- $\mathcal{P} \subseteq \mathcal{Q}$, as every vertex of $\mathcal{P}$ is χ^S for some $S \in \mathcal{S}$, and $\chi^S \in \mathcal{Q}$.
- $\mathcal{Q} \subseteq \mathcal{P}$: take some vertex x of $\mathcal{Q}$
- $\mathcal{Q}$ is integral $\implies$ x is integral $\implies$ $x = \chi^S$ for some $S \subseteq [n]$
- As $\mathcal{Q} \cap \{0,1\}^n = \{\chi^S : S \in \mathcal{S}\}$, $x = \chi^S$ for some $S \in \mathcal{S}$
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Outline

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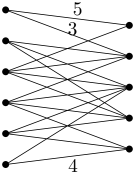
Bipartite Matching Polytope

Maximum Weight Bipartite Matching

Input: bipartite graph $G = (L \uplus R, E)$

edge weights $w \in \mathbb{Z}^{E}_{>0}$

Output: a matching $M \subseteq E$ so as to maximize $\sum_{e \in M} w_e$



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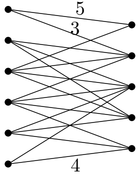

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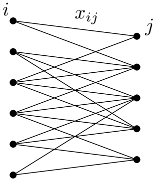

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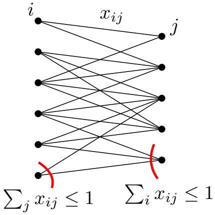

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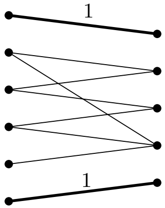

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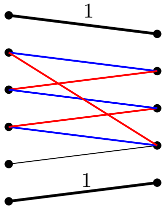

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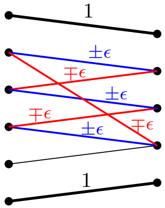

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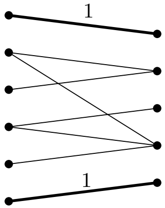

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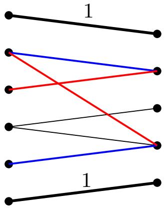

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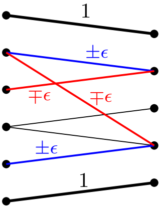

Outline

Def. A matrix $A \in \mathbb{R}^{m \times n}$ is said to be totally unimodular (TUM), if every sub-square of A has determinant in $\{-1, 0, 1\}$.

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- Let $x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$, so that $A'x^1 = b'$ and $x^2 = 0$.
- Cramer's rule: $x_i^1 = \frac{\det(A_i'|b)}{\det(A')}$ for every $i \implies x_i^1$ is integer

  $A_i'|b$: the matrix of A' with the i-th column replaced by b □

Example for the Proof

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \geq \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

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The following equation system may give a vertex:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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Equivalently, the vertex satisfies

$$\begin{pmatrix} a_{1,2} & a_{1,3} & 0 & 0 & 0 \\ a_{3,2} & a_{3,3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_1 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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Coro. In the LP for s-t network flow problem with integer capacities, every vertex solution to the LP is integral.

Example for the Proof

$$\begin{pmatrix}1 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 1\\ 1 & 0 & 0 & 0 & -1 & 0 & 0\end{pmatrix}$$

Example for the Proof

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

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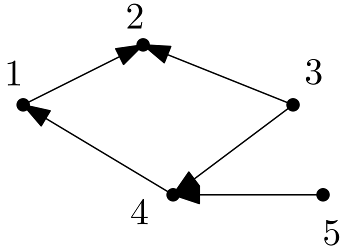

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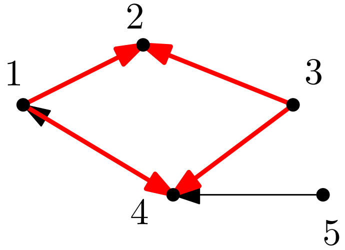

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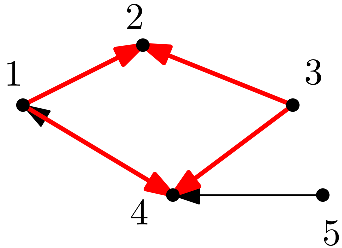

$$\begin{array}{cc} + & (1 \quad -1 \quad 0 \quad 0 \quad 0) \\ - & (0 \quad -1 \quad 1 \quad 0 \quad 0) \\ + & (0 \quad 0 \quad 1 \quad -1 \quad 0) \\ - & (1 \quad 0 \quad 0 \quad -1 \quad 0) \\ \\ = & (0 \quad 0 \quad 0 \quad 0 \quad 0) \end{array}$$

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$$M = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$ det$(M) = 1$.

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- $\det(A'M) \in \{0, \pm 1\} \implies \det(A') \in \{0, \pm 1\}$. □

Example for the Proof

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

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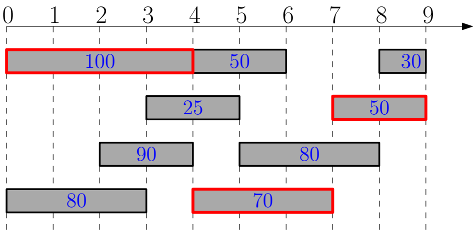
Weighted Interval Scheduling Problem

Input: n activities, activity i starts at time s_i, finishes at time f_i, and has weight $w_i > 0$

i and j can be scheduled together iff $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: maximum weight subset of jobs that can be scheduled



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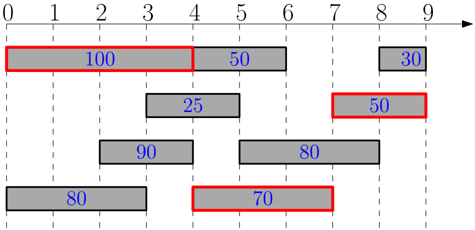

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Weighted Interval Scheduling Problem

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$$\max \quad \sum_{j\in[n]} x_j w_j$$

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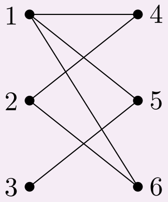

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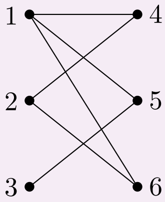

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

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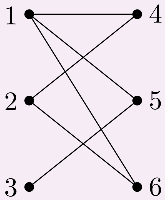

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A different proof for the theorem we proved:

Theorem $\mathcal{P}_{\mathrm{BM}}$ is the set of $x \in \mathbb{R}^E$ satisfying the following constraints:

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- remark: bipartiteness is needed. The edge-vertex incidence matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ of a triangle has determinent 2.

Outline

Def. A separation oracle for a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is an algorithm that, given some $x^* \in \mathbb{R}^n$,

- either correctly claims that $x \in \mathcal{P}$,
- or outputs a linear constraint $a^{\mathrm{T}}x \leq b$ that separating x^* from $\mathcal{P}$: every $x \in \mathcal{P}$ satisfies $a^{\mathrm{T}}x \leq b$, but $a^{\mathrm{T}}x^* > b$. We say $a^{\mathrm{T}}x \leq b$ is a separation plane for x^*.

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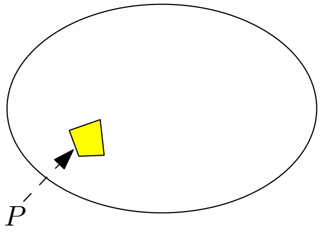
- Clearly, if $\mathcal{P} \subseteq \mathbb{R}^n$ can be described using a polynomial-size LP, then it has an efficient separation oracle.
- However, there are cases where $\mathcal{P} \subseteq \mathbb{R}^n$ has exponential number of facets, but still admits an efficient separation oracle.

- We can use ellipsoid method to solve the LP $\min / \max w^{\mathrm{T}} x, x \in \mathcal{P}$, when $\mathcal{P}$ has an efficient separation oracle, using the ellipsoid method.

Ellipsoid Method

- maintain an ellipsoid that contains the feasible region
- query a separation oracle if the center of ellipsid is in the feasible region:
  - yes: then the feasible region is not empty
  - no: cut the elliposid in half, find smaller ellipsoid to enclose the half-ellipsoid, and repeat



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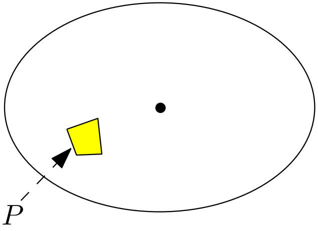

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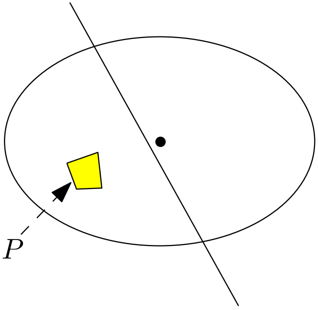

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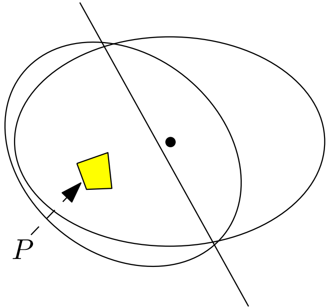

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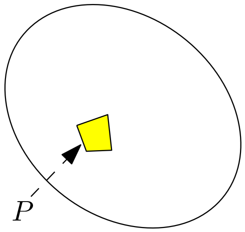

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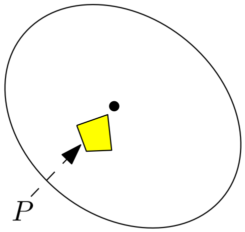

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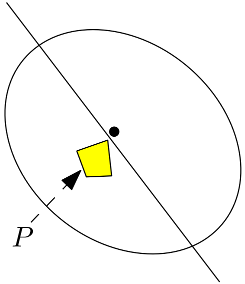

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Ellipsoid Method

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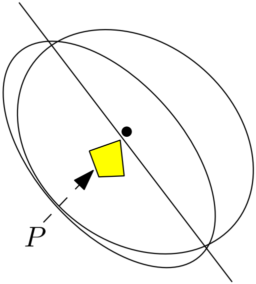

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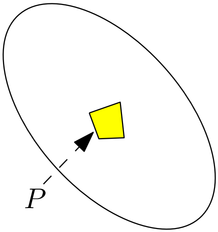

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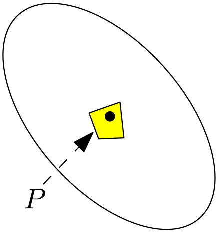

Outline

s-t Cut Polytope

Def. Given a digraph $G = (V, E)$, C is a s-t cut in G, if s and t are disconnected in $(V, E \setminus C)$.

- $\mathcal{P}_{\text{min-cut}} := \text{conv}\big(\{\chi^C : C \text{ is a } s\text{-}t \text{ cut in } G\}\big)$

Theorem $\mathcal{P}_{\text{min-cut}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in P} x_e \geq 1 \qquad \forall \text{ simple } s\text{-}t \text{ path } P \qquad (*)$$

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A: Use shortest path algorithm with weights $(x_e)_{e \in E}$.

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Proof of Lemma.

- Given $x \in [0,1]^E$ satisfying (*)
- $d_x(v), v \in V$: length of shortest path from s to v, with x being the weights; so $d_x(s) = 0$ and $d_x(t) \geq 1$

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- $d_x(v), v \in V$: length of shortest path from s to v, with x being the weights; so $d_x(s) = 0$ and $d_x(t) \geq 1$
- randomly choose a real $\theta \in (0,1)$
- $S := \{v \in V : d_x(v) \leq \theta\}, T := V \setminus S = \{v \in V : d_x(v) > \theta\}$
- $C := E(S,T)$

Claim For an edge $(u,v) \in E$, we have

$$\Pr[(u,v) \in C] \le \max\{d_x(v) - d_x(u), 0\}.$$

Proof.

- $(u,v) \in C$ happens only if $d_x(u) < \theta \le d_x(v)$.
- This happens with probability at most $\max\{d_x(v) - d_x(u), 0\} \le x_{(u,v)}$. □

Proof of Lemma, Continued

- $\mathbb{E}_\theta[\chi^C] \le x$
- We can define a random set C' so that $C' \supseteq C$ happens with probability 1, and $\mathbb{E}_\theta[\chi^{C'}] = x$.
- So $x \in \text{conv}(\{\chi^{C'} : C' \text{ is a } s\text{-}t \text{ cut in } G\})$ □

Outline

Spanning Tree Polytope

- Given a connected graph $G = (V, E)$
- $\mathcal{P}_{\mathrm{ST}} :=$ conv $\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\mathrm{ST}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\begin{aligned} \sum_{e \in E} x_e &= n - 1 & & \\ \sum_{e \in E[S]} x_e &\leq |S| - 1 & & \forall S \subseteq V, 2 \leq |S| \leq n-1 \quad (*) \\ x_e &\geq 0 & & \forall e \in E \end{aligned}$$

- Spanning trees correspond to bases of graphic matroid for G
- Later we prove a more general theorem on matroid polytopes

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A: $\xrightarrow{\text{reduce}}$ densest sub-graph $\xrightarrow{\text{reduce}}$ maximum flow

Checking if $\sum_{e \in E[S]} x_e \leq |S| - 1, \forall S \subseteq V$

- We need to check if $\exists S \subseteq V, \frac{\sum_{e \in E[S]} x_e}{|S|-1} > 1$:

- Guess a vertex $v \in S$; set $w_v = 0$ and $w_u = 1$ for every $u \in V \setminus \{v\}$
- The problem becomes to check if $\exists S \subseteq V, \frac{\sum_{e \in E[S]} x_e}{\sum_{u \in S} w_u} > 1$

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- This is a (weighted) densest subgraph problem
- Exercise: It can be solved using maximum flow

Outline

General Graph Perfect Matching Polytope

General Perfect Matching Polytope

- Given a graph $G = (V, E)$, where $|V|$ is even
- $\mathcal{P}_{\text{GPM}} :=$ conv $\left(\left\{\chi^M : M \subseteq E \text{ is a perfect matching in } G\right\}\right)$

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Theorem (General Perfect Matching Polytope Theorem)
$\mathcal{P}_{\text{GPM}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in \delta(v)} x_e = 1 \qquad \forall v \in V$$

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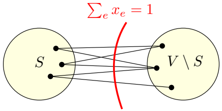
Proof of General Perfect Matching Polytope Theorem

- Clearly, every $x \in \mathcal{P}_{\text{GPM}}$ satisfies all the LP constraints
- We prove the LP polytope is integral; this implies lemma
- We choose the counter-example G with the smallest $|V| + |E|$, and focus on a non-integral vertex x of the LP polytope

Proof of General Perfect Matching Polytope Theorem

- $x_e = 0$ for some $e \in E$: e could be removed.
- $x_e = 1$ for some $e \in E$: e and its 2 end vertices could be removed.
- So $x_e \in (0, 1)$ for every $e \in E$.
- Every $v \in V$ has degree at least 2.
- Every $v \in V$ has degree exactly 2: G is union of disjoint cycles, x would not be a vertex of LP polytope.
- Assume some $v \in V$ has degree at least 3; $|E| \geq |V| + 1$.
- x is the unique solution to a system ot n linear equations from the LP.
- So, some linear equation is

$$\sum_{e \in E(S, V \setminus S)} x_e = 1 \text{ for some } S \subseteq V \text{ with } |S| \geq 3, |V \setminus S| \geq 3$$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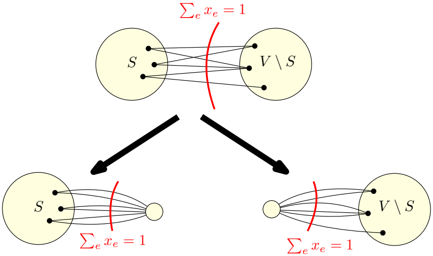

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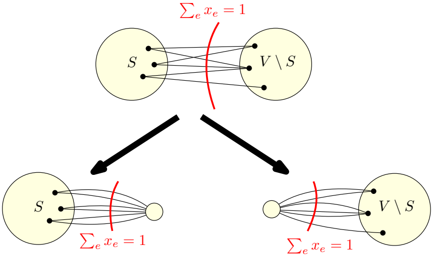
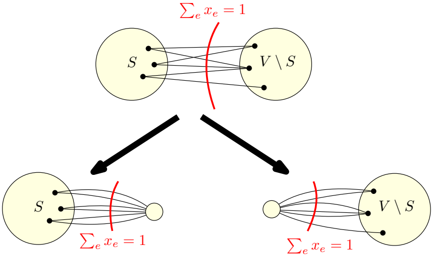


Proof of General Perfect Matching Polytope Theorem

- Consider two instances: $(G/V, x'), (G/(V \setminus S), x'')$
- Both x' and x'' satisfy the LP constraints for their respective graphs.



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$$x' \in \text{conv}(\{\chi^M : M \text{ is a perfect matching in } G/S\})$$
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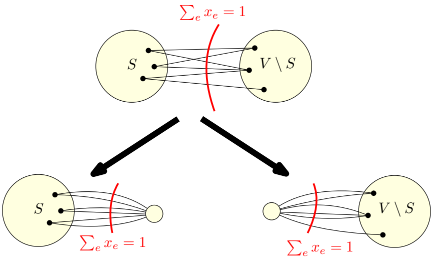

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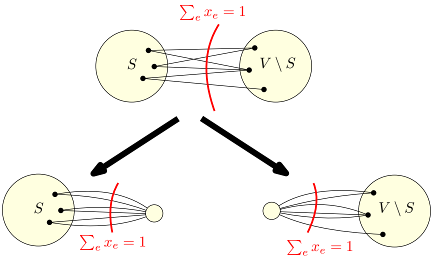

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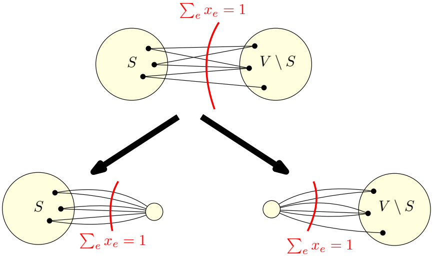

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- Decompose x' and x'' into a convex combinations of matchings
- Each $e \in E(S, V \setminus S)$ has the same fraction in combinations
- "Concatenate" two convex combinations into one convex combinations of matching in G. So x can not be a vertex. □

Theorem (General Perfect Matching Polytope Theorem)
$\mathcal{P}_{\text{GPM}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

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- inequality in (*) can be replaced by $\sum_{e \in E[S]} x_e \leq \frac{|S|-1}{2}$
- more convenient to obtain general matching polytope

General Matching Polytope

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Remark

- For all the polytopes, we identified a set of linear inequalities that are sufficient to define the polytope.
- However, not all the constraints are facet-defining.
- Only facet-defining constraints are necessarily; other constraints could be removed. (We keep all the constraints.for convenience of description.)

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- However, not all the constraints are facet-defining.
- Only facet-defining constraints are necessarily; other constraints could be removed. (We keep all the constraints.for convenience of description.)

- Example: in spanning tree polytope, $\sum_{e \in E[S]} x_e \leq |S| - 1$ is not needed if $(S, E[S])$ is disconnected, or contains a bridge. In this case, the constraint does not define a facet.

Outline

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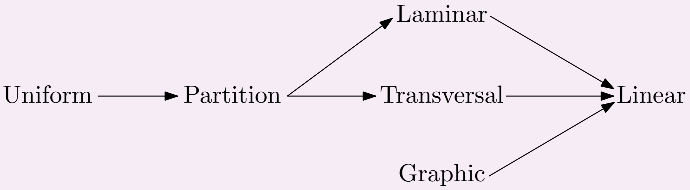
Recall Definition and Examples of Matroid

Def. A (finite) matroid $\mathcal{M}$ is a pair $(E, \mathcal{I})$, where E is a finite set (called the ground set) and $\mathcal{I}$ is a family of subsets of E (called independent sets) with the following properties:

1. $\emptyset \in \mathcal{I}$.
2. (downward-closed property) If $B \subsetneq A \in \mathcal{I}$, then $B \in \mathcal{I}$.
3. (augmentation/exchange property) If $A, B \in \mathcal{I}$ and $|B| < |A|$, then there exists $e \in A \setminus B$ such that $B \cup \{e\} \in \mathcal{I}$.

Relationship between matroids



Other Terminologies Related To a Matroid $\mathcal{M} = (E, \mathcal{I})$

- A subset of E that is not independent is dependent.
- A maximal independent set is called a basis (plural: bases)
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- Graphic matroid for a connected graph $G = (V, E)$:
  basis $\iff$ spanning tree   circuit $\iff$ cycle

Lemma All bases of a matroid have the same size.

Proof.

- Assume two A and A' are both bases of $\mathcal{M}$ and $|A| > |A'|$
- By exchange property: $\exists i \in A \setminus A', A' \cup \{i\} \in \mathcal{I}$
- contradiction with that A' is a basis $\square$

- Recall: Matroid Rank Function:

Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the rank of any $A \subseteq E$ is defined as

$$r_{\mathcal{M}}(A) = \max\left\{|A'| : A' \subseteq A, A' \in \mathcal{I}\right\}.$$

The function $r_{\mathcal{M}} : 2^E \to \mathbb{Z}_{\geq 0}$ is called the rank function of $\mathcal{M}$.

- $r_{\mathcal{M}}(A)$ is size of maximum independent subset of A

Trivial properties of $r_{\mathcal{M}}$

- $r_{\mathcal{M}}(\emptyset) = 0$
- $r_{\mathcal{M}}(A \cup \{i\}) - r_{\mathcal{M}}(A) \in \{0, 1\}$ for every $A \subseteq E, i \in E \setminus A$

Theorem The rank function $r_{\mathcal{M}}$ of a matroid $\mathcal{M} = (E, \mathcal{I})$ is submodular.

Greedy algorithm finds max ind. subset of any given $X \subseteq E$:

1: $S \leftarrow \emptyset$
2: **while** $\exists e \in X \setminus S$ s.t. $S \cup \{e\} \in \mathcal{I}$ **do**
3: let e be an arbitrary element satisfying the condition
4: $S \leftarrow S \cup \{e\}$
5: **return** S

Proof of Submodularity of $r_{\mathcal{M}}$.

Greedy algorithm finds max ind. subset of any given $X \subseteq E$:

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S ← ∅
while ∃e ∈ X \ S s.t. S ∪ {e} ∈ I do
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- Take $A \subsetneq E, i, j \in E \setminus A, i \neq j$, need to prove: $r_{\mathcal{M}}(A \cup \{i, j\}) - r_{\mathcal{M}}(A \cup \{i\}) \leq r_{\mathcal{M}}(A \cup \{j\}) - r_{\mathcal{M}}(A)$

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- if not, then LHS $= 1$, RHS $= 0$
- S: max ind. subset of A, S': max ind. subset of $A \cup \{i\}$
- $|S| = r_{\mathcal{M}}(A), |S'| = r_{\mathcal{M}}(A \cup \{i\})$, $S' = S$ or $S' = S \cup \{i\}$

Greedy algorithm finds max ind. subset of any given $X \subseteq E$:

```
S ← ∅
while ∃e ∈ X \ S s.t. S ∪ {e} ∈ I do
    let e be an arbitrary element satisfying the condition
    S ← S ∪ {e}
return S
```

Proof of Submodularity of $r_{\mathcal{M}}$.

- Take $A \subsetneq E, i, j \in E \setminus A, i \neq j$, need to prove: $r_{\mathcal{M}}(A \cup \{i,j\}) - r_{\mathcal{M}}(A \cup \{i\}) \leq r_{\mathcal{M}}(A \cup \{j\}) - r_{\mathcal{M}}(A)$
- if not, then LHS $= 1$, RHS $= 0$
- S: max ind. subset of A, S': max ind. subset of $A \cup \{i\}$
- $|S| = r_{\mathcal{M}}(A), |S'| = r_{\mathcal{M}}(A \cup \{i\})$, $S' = S$ or $S' = S \cup \{i\}$
- RHS $= 0 \implies S \cup \{j\} \notin \mathcal{I}$, LHS $= 1 \implies S' \cup \{j\} \in \mathcal{I}$
- contradiction □

Lemma A function $r : 2^E \to \mathbb{R}$ is the rank function of a matroid if and only if

1. $r(\emptyset) = 0$
2. $r(A \cup \{i\}) - r(A) \in \{0, 1\}$ for all $A \subseteq E, i \notin E \setminus A$
3. r is submodular.

Proof.

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- Claim: $(E, \mathcal{I})$ is a matroid and r is its rank function.

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- $A, A' : r(A) = |A|, r(A') = |A'|, |A| < |A'|$
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- $U := A \cup A' : r(U) \geq r(A') > r(A), \qquad A \subsetneq U$
- ③ $\implies \exists i \in U \setminus A = A' \setminus A : r(A \cup \{i\}) > r(A)$
- $i \in A' \setminus A$ and $r(A \cup \{i\}) = r(A) + 1 = |A \cup \{i\}|$
- so, $A \cup \{i\} \in \mathcal{I} \implies$ exchange property □

Derivatives of Matroids

Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and an element $e \in E$, the matroid obtained from $\mathcal{M}$ by removing e, denoted as $\mathcal{M} \setminus e$, is defined as follows:

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Def. For a matroid $\mathcal{M} = (E, \mathcal{I})$, the dual matroid $\mathcal{M}^* = (E, \mathcal{I}^*)$ is defined so that the bases in $\mathcal{M}^*$ are exactly the complements of the bases in $\mathcal{I}$.

Theorem $\mathcal{M}^*$ is a matroid.

Outline

Matroid Polytope

- Given a matroid $\mathcal{M} = (E, \mathcal{I})$
- The matroid polytope for $\mathcal{M}$ is defined as

$$\mathcal{P}_{\mathcal{M}} := \text{conv}(\{\chi^A : A \in \mathcal{I}\}).$$

- Recall: $\chi^A \in \{0,1\}^E, \quad \chi_i^A = \begin{cases} 1 & i \in A \\ 0 & i \notin A \end{cases}$

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Theorem (Matroid Polytope Theorem) For a matroid $\mathcal{M} = (E, \mathcal{I})$, we have

$$\mathcal{P}_{\mathcal{M}} = \Big\{x \in [0,1]^E : x(S) \le r_{\mathcal{M}}(S), \forall S \subseteq E\Big\},$$

where $x(S) := \sum_{i \in S} x_i$ for every $S \subseteq E$.

Proof of Matroid Polytope Theorem

- $\mathcal{Q} := \left\{x \in [0,1]^E : \sum_{i \in A} x_i \leq r_{\mathcal{M}}(A), \forall A \subseteq E\right\}$
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- Focus on the counter example with the smallest $|E|$
- assume some vertex x of $\mathcal{Q}$ is non-integral
- If $x_e = 0$ for some $e \in E$, removing e gives a smaller counterexample
- If $x_e = 1$ for some $e \in E$, contracting e gives a smaller counterexample
- So, $x_e \in (0,1)$ for every $e \in E$.

Proof of Matroid Polytope Theorem

Def. We say a set $A \subseteq E$ is tight if $x(A) = r_{\mathcal{M}}(A)$. Let $\mathcal{T}$ be the family of all tight subsets of E.

Lemma If $A, B \in \mathcal{T}$, then both $A \cup B$ and $A \cap B$ are in $\mathcal{T}$.

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Lemma If $A, B \in \mathcal{T}$, then both $A \cup B$ and $A \cap B$ are in $\mathcal{T}$.

Proof.

$$\begin{aligned} x(A) + x(B) &= r_{\mathcal{M}}(A) + r_{\mathcal{M}}(B) \\ &\geq r_{\mathcal{M}}(A \cup B) + r_{\mathcal{M}}(A \cap B) \geq x(A \cup B) + x(A \cap B). \end{aligned}$$

- equality: A and B are tight
- first inequality: $r_{\mathcal{M}}$ is submodular
- second inequality: $x(S) \leq r_{\mathcal{M}}(S)$ for every $S \subseteq E$

But $x(A) + x(B) = x(A \cup B) + x(A \cap B)$. So, both inequalities hold with equality. □

Proof of Matroid Polytope Theorem

Def. A chain is a sequence of subsets $S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_t$ of E.

- We use span$(\mathcal{S})$ for span$(\{\chi^S : S \in \mathcal{S}\})$, for any $\mathcal{S} \subseteq \mathcal{T}$.

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Lemma (Key Lemma) Let $\mathcal{C}$ be a longest chain of tight subsets of E (i.e., subsets in $\mathcal{T}$). Then, we have span($\mathcal{C}$) $=$ span($\mathcal{T}$).

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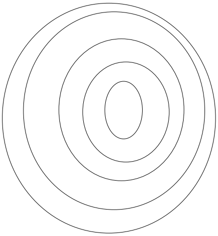
Lemma (Key Lemma) Let $\mathcal{C}$ be a longest chain of tight subsets of E (i.e., subsets in $\mathcal{T}$). Then, we have span$(\mathcal{C})$ = span$(\mathcal{T})$.

Proof of Key Lemma

- We say two sets B and T conflict with each other, if $B \not\subseteq T$ and $T \not\subseteq B$.
- Define $\tau(B) := \{T \in \mathcal{C} : B \text{ conflicts with } T\}, \forall B$
- Assume span$(\mathcal{C}) \subsetneq$ span$(\mathcal{T})$
- Let $B = \arg\min_{B \in \mathcal{T}, \chi^B \notin \text{span}(\mathcal{C})} |\tau(B)|$

Proof of Matroid Polytope Theorem



Proof of Key Lemma

- Let $T \in \mathcal{C}$ be a set contradicting with B;
- We prove $\tau(B \cup T), \tau(B \cap T) \subsetneq \tau(B)$.

Proof of Matroid Polytope Theorem

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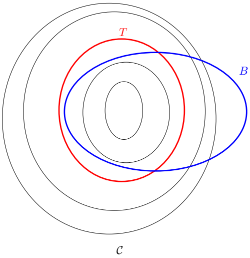

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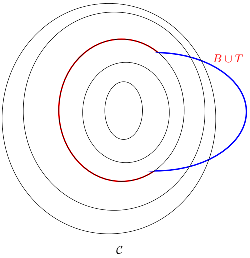

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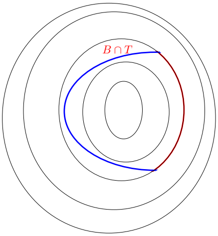

$\mathcal{C}$

Proof of Matroid Polytope Theorem

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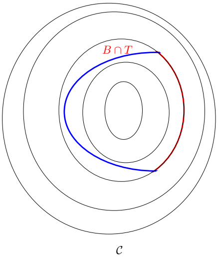

- For $\tau(B \cup T) \subseteq \tau(B)$:
  - $S \subsetneq T$: S does not conflict with $B \cup T$, and may conflict with B.
  - $S \supsetneq T$: S not conflict with $B \implies S$ not conflict with $B \cup T$.
- For $\tau(B \cap T) \subseteq \tau(B)$:
  - $S \subsetneq T$: S not conflict with $B \implies S$ not conflict with $B \cap T$.
  - $S \supsetneq T$: S does not conflict with $B \cap T$, and may conflict with B.
- "$\neq$" : B conflicts with T, but $B \cup T$ and $B \cap T$ do not.

Proof of Matroid Polytope Theorem

Proof of Key Lemma

- By our choice of B, we have $\chi^{B \cup T}, \chi^{B \cap T} \in \text{span}(\mathcal{C})$.
- However, as $\chi^B = \chi^{B \cup T} + \chi^{B \cap T} - \chi^T$ and all the three vectors are in $\text{span}(\mathcal{T})$, contradiction with $\chi^B \notin \text{span}(\mathcal{C})$. $\square$

Recall the key lemma:

Lemma (Key Lemma) Let $\mathcal{C}$ be a longest chain of tight subsets of E (i.e., subsets in $\mathcal{T}$). Then, we have $\text{span}(\mathcal{C}) = \text{span}(\mathcal{T})$.

- Therefore, $x \in [0,1]^E$ is defined by the system of linear equations correspondent to $\mathcal{C}$.
- $|\mathcal{C}| = |E|$, the chain $\mathcal{C}$ is of full length.
- The system gives an integer solution x. Contradiction. $\square$

What we proved:

Matroid Polytope

- Given a matroid $\mathcal{M} = (E, \mathcal{I})$
- The matroid polytope for $\mathcal{M}$ is defined as

$$\mathcal{P}_{\mathcal{M}} := \text{conv}(\{\chi^A : A \in \mathcal{I}\}).$$

Theorem (Matroid Polytope Theorem) For a matroid $\mathcal{M} = (E, \mathcal{I})$, we have

$$\mathcal{P}_{\mathcal{M}} = \left\{x \in [0,1]^E : x(S) \leq r_{\mathcal{M}}(S), \forall S \subseteq E\right\},$$

where $x(S) := \sum_{i \in S} x_i$ for every $S \subseteq E$.

Outline

Matroid Basis Polytope

- Given a matroid $\mathcal{M} = (E, \mathcal{I})$
- The matroid basis polytope for $\mathcal{M}$ is defined as

$$\mathcal{P}_{\mathcal{M}}^{\text{basis}} := \text{conv}(\{\chi^A : A \in \mathcal{I}, \text{rank}_{\mathcal{M}}(A) = \text{rank}_{\mathcal{M}}(E)\}).$$

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where $x(S) := \sum_{i \in S} x_i$ for every $S \subseteq E$.

Proof.

- $\mathcal{P}_{\mathcal{M}}^{\text{basis}}$ is a face (not necessarily a facet) of $\mathcal{P}_{\mathcal{M}}$.
- $\mathcal{P}_{\mathcal{M}}$ is integral $\implies$ $\mathcal{P}_{\mathcal{M}}^{\text{basis}}$ is integral □

Recall: Spanning Tree Polytope

Spanning Tree Polytope

- Given a connected graph $G = (V, E)$
- $\mathcal{P}_{\mathrm{ST}} :=$ conv $\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\mathrm{ST}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in E[S]} x_e \leq |S| - 1 \qquad \forall S \subseteq V, 2 \leq |S| \leq n-1 \qquad (*)$$

$$x_e \geq 0 \qquad \forall e \in E$$

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$$x(E') \leq n - \mathsf{CC}(E'), \forall E' \subseteq E; \quad x(E) = n - 1,$$

where $\mathsf{CC}(E')$ is the number of connected components in (V, E').

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  where $\mathsf{CC}(E')$ is the number of connected components in (V, E').
- It suffices to consider the case where $E' = E[S]$ for some connected set $S \subseteq V$, in which case $n - \mathrm{CC}(E') = |S| - 1$.
- $\implies$ Spanning Tree Polytope Theorem.

Theorem (Matroid Intersection Polytope Theorem) Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be two matroids with the common ground set E. Then

$$\text{conv}\big(\{\chi^A : A \in \mathcal{I}_1 \cap \mathcal{I}_2\}\big) = \mathcal{P}_{\mathcal{M}_1} \cap \mathcal{P}_{\mathcal{M}_2}$$
$$= \Big\{x \in [0,1]^E : x(S) \leq r_{\mathcal{M}_1}(S), x(S) \leq r_{\mathcal{M}_2}(S), \forall S \subseteq E\Big\}.$$

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- We will not prove the theorem.
- A similar theorem works if we require A to be a basis for the matroid $\mathcal{M}_1$ or $\mathcal{M}_2$:

$$\text{conv}\big(\{\chi^A : A \in \mathcal{I}_1 \cap \mathcal{I}_2, \text{rank}_{\mathcal{M}_1}(A) = \text{rank}_{\mathcal{M}_1}(E)\}\big)$$
$$= \mathcal{P}^{\text{basis}}_{\mathcal{M}_1} \cap \mathcal{P}_{\mathcal{M}_2}$$

Applications

Bipartite Matching Polytope

- Given bipartite graph $G = (L \cup R, E)$
- $\mathcal{P}_{\text{BM}} := \text{conv}\big(\{\chi^M : M \text{ is a matching in } G\}\big)$

Theorem $\mathcal{P}_{\text{BM}}$ is the set of $x \in \mathbb{R}^E$ satisfying the following constraints:

$$\sum_{e \in \delta(v)} x_e \leq 1, \forall v \in L \cup R; \qquad x_e \geq 0, \forall e \in E.$$

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- A matching is an independent set of two partition matroids, one for each side of the bipartite graph.
- Matching polytope is intersection of two partition matroid polytopes.

Applications

Arborescence Polytope

- Given a directed graph $G = (V, E)$, a root $r \in V$
- $\mathcal{P}_{\mathrm{Arbo}} :=$ conv$(\{\chi^{E'} : E'$ is an arborescence of G rooted at $r\})$

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- We define two matroids:
  - Graphic Matroid: we ignore the directions of G, and require E' to be a spanning forest
  - Partition Matroid: we require every vertex other than r has in-degree at most 1

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- E' is an arborescence if it is a basis of both polytopes.

Summary

- linear programming, simplex method, interior point method, ellipsoid method
- Polytopes with totally-unimodular coefficient matrix:
  - integral LP polytopes: bipartite matching polytope, s-t flow polytope, weighted interval scheduling polytope

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