Advanced Algorithms (Fall 2025)

Primal-Dual Algorithms

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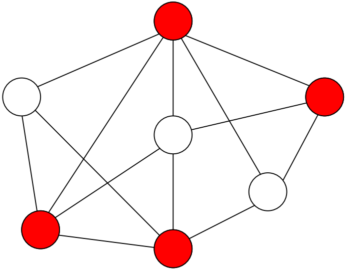
Outline



Weighted Vertex Cover Problem

Input: graph $G = (V, E)$, vertex weights $w \in \mathbb{Z}^V_{>0}$

Output: vertex cover S of G, to minimize $\sum_{v \in S} w_v$

LP Relaxation

$$\min \quad \sum_{v \in V} w_v x_v$$

$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$

$$x_v \geq 0 \qquad \forall v \in V$$

Dual LP

$$\max \quad \sum_{e \in E} y_e$$

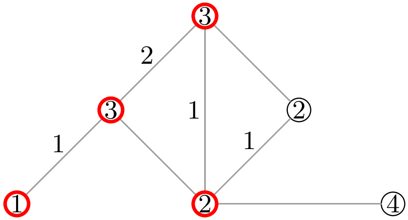
$$\sum_{e \in \delta(v)} y_e \leq w_v \qquad \forall v \in V$$

$$y_e \geq 0 \qquad \forall e \in E$$

- Algorithm constructs integral primal solution x and dual solution y simultaneously.

Primal-Dual Algorithm for Weighted Vertex Cover Problem

$x \leftarrow 0, y \leftarrow 0$, all edges said to be uncovered
while there exists at least one uncovered edge **do**
    take such an edge e arbitrarily
    increasing y_e until the dual constraint for one end-vertex v of e becomes tight
    $x_v \leftarrow 1$, claim all edges incident to v are covered
return x



Lemma

1. x satisfies all primal constraints
2. y satisfies all dual constraints
3. $P \leq 2D \leq 2D^* \leq 2 \cdot \text{opt}$

   $P := \sum_{v \in V} x_v$: value of x

   $D := \sum_{e \in E} y_e$: value of y

   D^* : dual LP value

Proof of $P \leq 2D$.

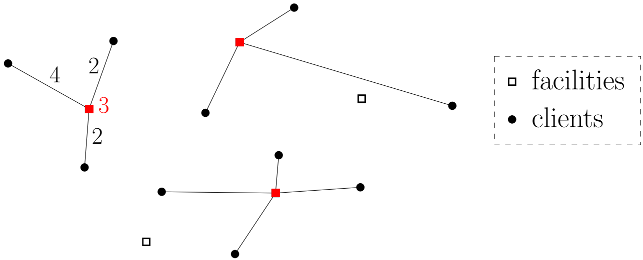
$$P = \sum_{v \in V} w_v x_v \leq \sum_{v \in V} x_v \sum_{e \in \delta(v)} y_e = \sum_{(u,v) \in E} y_{(u,v)} (x_u + x_v)$$

$$\leq 2 \sum_{e \in E} y_e = 2D. \qquad \square$$

- a more general framework: construct an arbitrary maximal dual solution y; choose the vertices whose dual constraints are tight
- y is maximal: increasing any coordinate y_e makes y infeasible

- primal-dual algorithms do not need to solve LPs
- LPs are used in analysis only
- faster than LP-rounding algorithm in general

Outline



Uncapacitated Facility Location Problem

Input: F: pontential facilities C: clients

d: (symmetric) metric over $F \cup C$ $(f_i)_{i \in F}$: facility opening costs

Output: $S \subseteq F$, so as to minimize $\sum_{i \in S} f_i + \sum_{j \in C} d(j, S)$

- 1.488-approximation [Li, 2011]
- 1.463-hardness of approximation, 1.463 $\approx$ root of $x = 1 + 2e^{-x}$

- y_i: open facility i?
- $x_{i,j}$: connect client j to facility i?

Basic LP Relaxation

$$\min \quad \sum_{i \in F} f_i y_i + \sum_{i \in F, j \in C} d(i,j) x_{i,j}$$

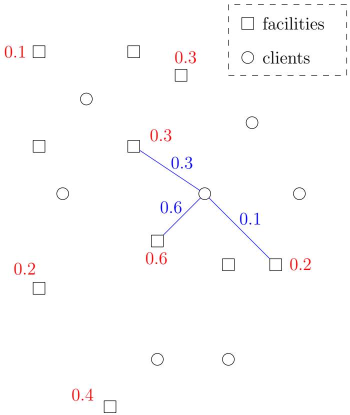
$$\sum_{i \in F} x_{i,j} \geq 1 \qquad \forall j \in C$$

$$x_{i,j} \leq y_i \qquad \forall i \in F, j \in C$$

$$x_{i,j} \geq 0 \qquad \forall i \in F, j \in C$$

$$y_i \geq 0 \qquad \forall i \in F$$

Obs. When $(y_i)_{i \in F}$ is determined, $(x_{i,j})_{i \in F, j \in C}$ can be determined automatically.



Basic LP Relaxation

$$\min \sum_{i\in F} f_i y_i + \sum_{i\in F, j\in C} d(i,j) x_{i,j}$$

$$\sum_{i\in F} x_{i,j} \geq 1 \qquad \forall j \in C$$

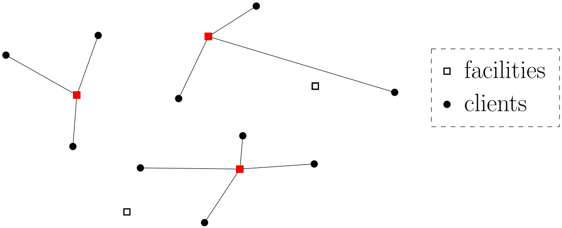
$$x_{i,j} \leq y_i \qquad \forall i \in F, j \in C$$

$$x_{i,j} \geq 0 \qquad \forall i \in F, j \in C$$

$$y_i \geq 0 \qquad \forall i \in F$$

- LP is not of covering type
- harder to understand the dual

- consider an equivalent covering LP
- idea: treat a solution as a set of stars



- $(i, J), i \in F, J \subseteq C$: star with center i and leaves J
- cost$(i, J) := f_i + \sum_{j \in J} d(i, j)$: cost of star (i, J)
- $x_{i,J} \in \{0, 1\}$: if star (i, J) is chosen

Equivalent LP

$$\min \quad \sum_{(i,J)} \mathrm{cost}(i, J) \cdot x_{i,J}$$

$$\sum_{(i,J):j \in J} x_{i,J} \geq 1 \qquad \forall j \in C$$

$$x_{i,J} \geq 0 \qquad \forall (i, J)$$

Dual LP

$$\max \quad \sum_{j \in C} \alpha_j$$

$$\sum_{j \in J} \alpha_j \leq \mathrm{cost}(j, J) \qquad \forall (i, J)$$

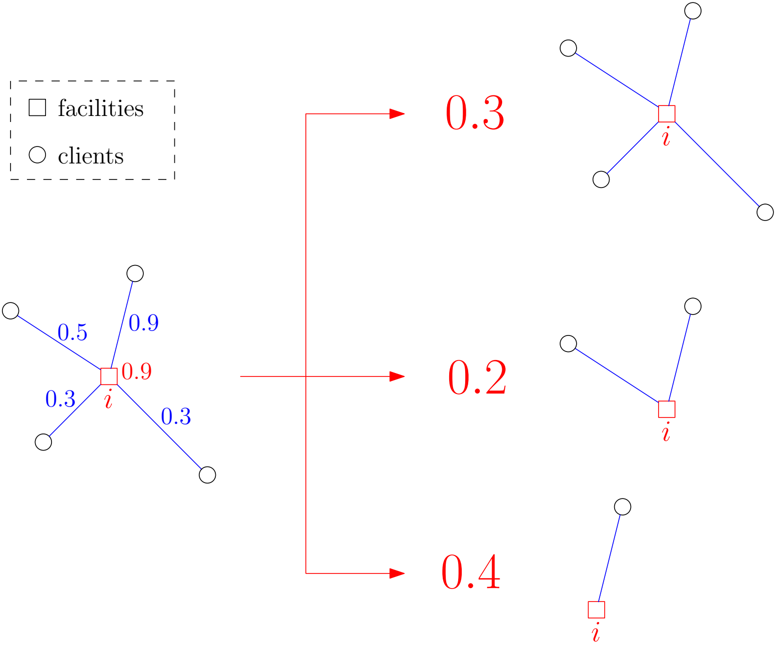
$$\alpha_j \geq 0 \qquad \forall j \in C$$

- both LPs have exponential size, but the final algorithm can run in polynomial time

$$\begin{aligned} \min \quad & \sum_{(i,J)} \text{cost}(i,J) \cdot x_{i,J} \\ \sum_{(i,J): j \in J} & x_{i,J} \geq 1 \qquad \forall j \in C \\ & x_{i,J} \geq 0 \qquad \forall (i,J) \end{aligned}$$

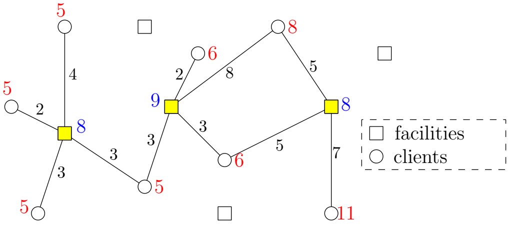
$$\begin{aligned} \max \quad & \sum_{j \in C} \alpha_j \\ \sum_{j \in J} & \alpha_j \leq \text{cost}(j,J) \qquad \forall (i,J) \\ & \alpha_j \geq 0 \qquad \forall j \in C \end{aligned}$$

- α_j: budget of j
- dual constraints: total budget in any star is $\leq$ its cost
- $\implies$ opt $\geq$ total budget $=$ dual value


facilities
clients
0.5
0.9
0.9
0.3
0.3
i
0.3
0.2
0.4
i
i
i

Construction of Dual Solution α

- α_j's can only increase
- α is always feasible
- if a dual constraint becomes tight, freeze all clients in star
- unfrozen clients are called active clients



Construction of Dual Solution α

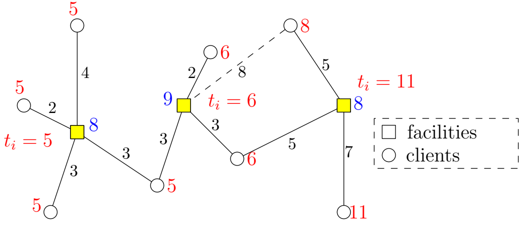
1: $\alpha_j \leftarrow 0, \forall j \in C$

2: **while** exists at least one active client **do**

3: increase the budgets α_j for all active clients j at uniform rate, until (at least) one new client is frozen

Construction of Dual Solution α

- ■: tight facilities; they are temporarily open
- □: pemanently closed
- t_i: time when facility i becomes tight
- construct a bipartite graph: (i, j) exists $\iff \alpha_j > d(i, j)$,



$\alpha_j > d(i, j)$: j contributes to i, (solid lines)

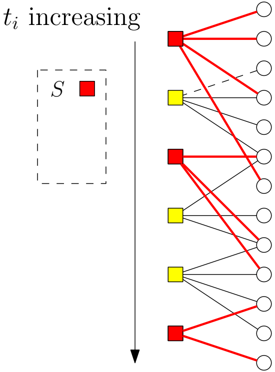
$\alpha_j = d(i, j)$: j does not contribute to i, but its budget is just enough for it to connect to i (dashed lines)

$\alpha_j < d(i, j)$: budget of j is not enough to connect to i

Construction of Integral Primal Solution

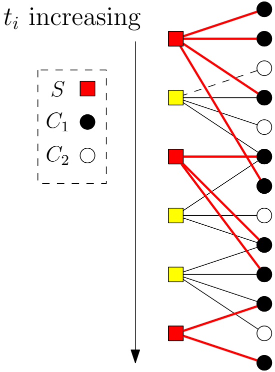
Construction of Integral Primal Solution

1: $S \leftarrow \emptyset$, all clients are unowned
2: **for** every temporarily open facility i, in increasing order of t_i **do**
3: **if** all (solid-line) neighbors of i are unowned **then**
4: $S \leftarrow S \cup \{i\}$, open facility i
5: connect to all its neighbors to i
6: let i own them
7: connect unconnected clients to their nearest facilities in S



- S: set of open facilities
- C_1: clients that make contributions
- C_2: clients that do not make contributions

- f: total facillity cost
- c_j: connection cost of client j
- $c = \sum_{j \in C} c_j$: total connection cost
- $D = \sum_{j \in C} \alpha_j$: value of α



Lemma

- $f + \sum_{j \in C_1} c_j \le \sum_{j \in C_1} \alpha_j$
- for any client $j \in C_2$, we have $c_j \le 3\alpha_j$

Lemma

- $f + \sum_{j \in C_1} c_j \le \sum_{j \in C_1} \alpha_j$
- for any client $j \in C_2$, we have $c_j \le 3\alpha_j$

- So, $f + c = f + \sum_{j \in C} c_j \le 3 \sum_{j \in C} \alpha_j = 3D \le 3 \cdot \mathrm{opt}.$

- stronger statement:

$$3f + c = 3f + \sum_{j \in C} c_j \le 3 \sum_{j \in C} \alpha_j = 3D \le 3 \cdot \mathrm{opt}.$$

Proof of $\forall j \in C_2, c_j \leq 3\alpha_j$

- at time α_j, j is frozen.
- let i be the temporarily open facility it connects to
- $i \in S$: then $c_j \leq \alpha_j$. assume $i \notin S$.
- there exists a client j', which made contribution to i, and owned by another facility $i' \in S$
- $d(j,i) \leq \alpha_j$
- $d(j',i) < \alpha_{j'}, d(j',i') < \alpha_{j'}$
- $\alpha_{j'} = t'_i \leq t_i \leq \alpha_j$
- $d(j,i') \leq d(j,i)+d(i,j')+d(j',i') \leq \alpha_j + \alpha_j + \alpha_j = 3\alpha_j$