Advanced Algorithms

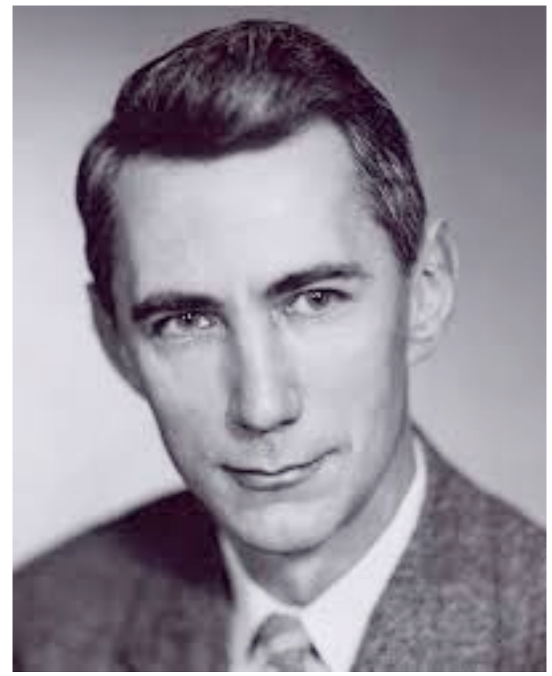
Introduction

刘明谋 Nanjing University, Suzhou, 2025

Early works about TCS

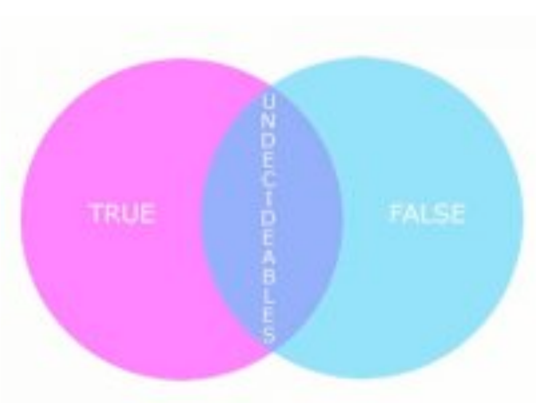
The founders & the origins

Kurt Gödel

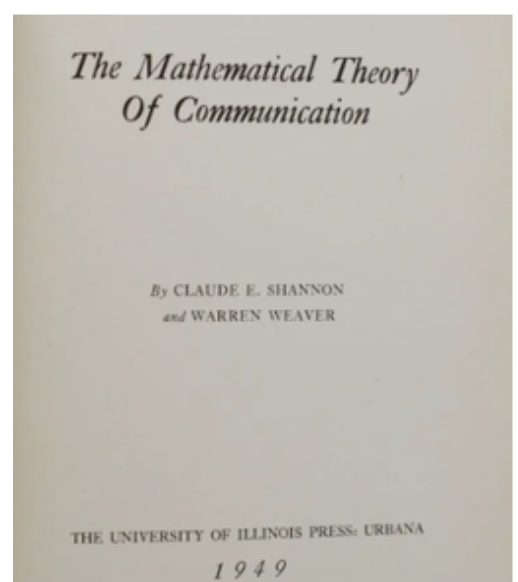


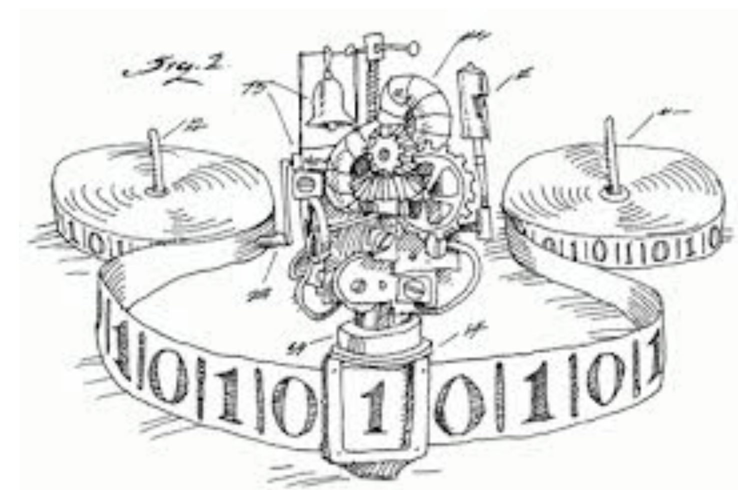
Claude Shannon

Alan Turing



Incompleteness Theorems

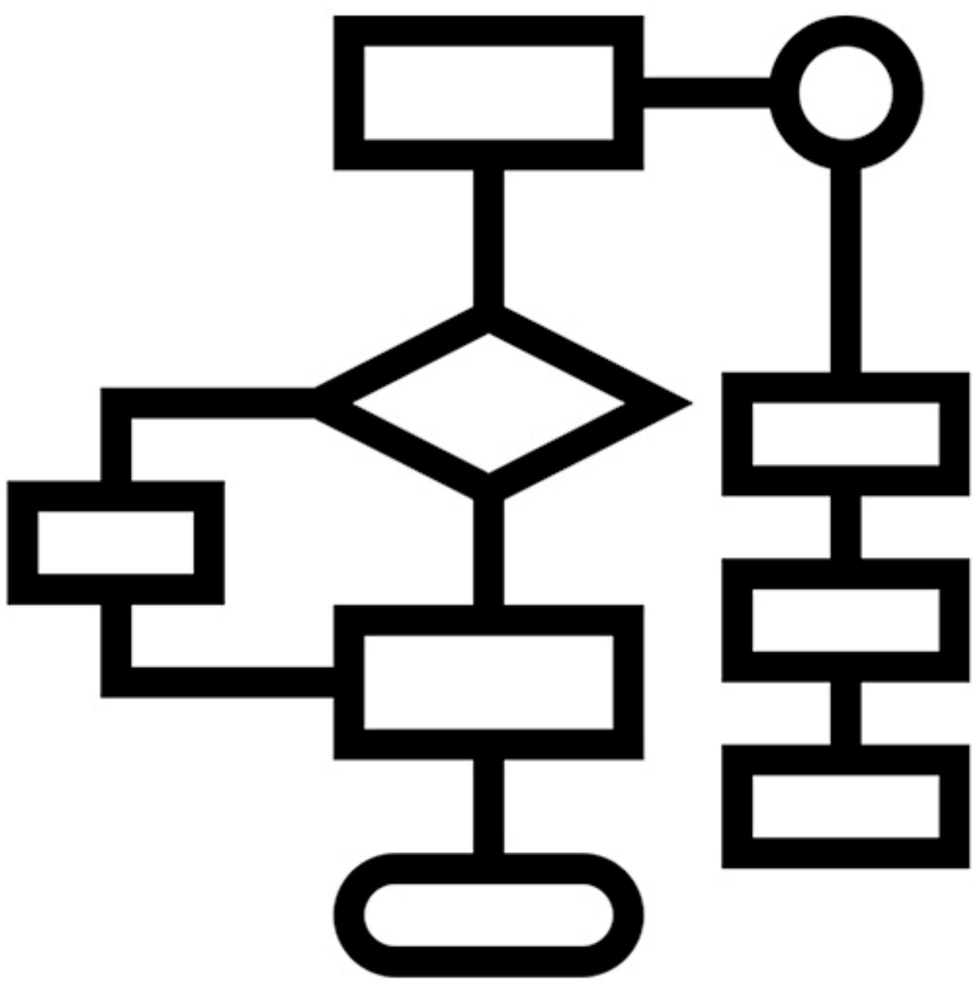


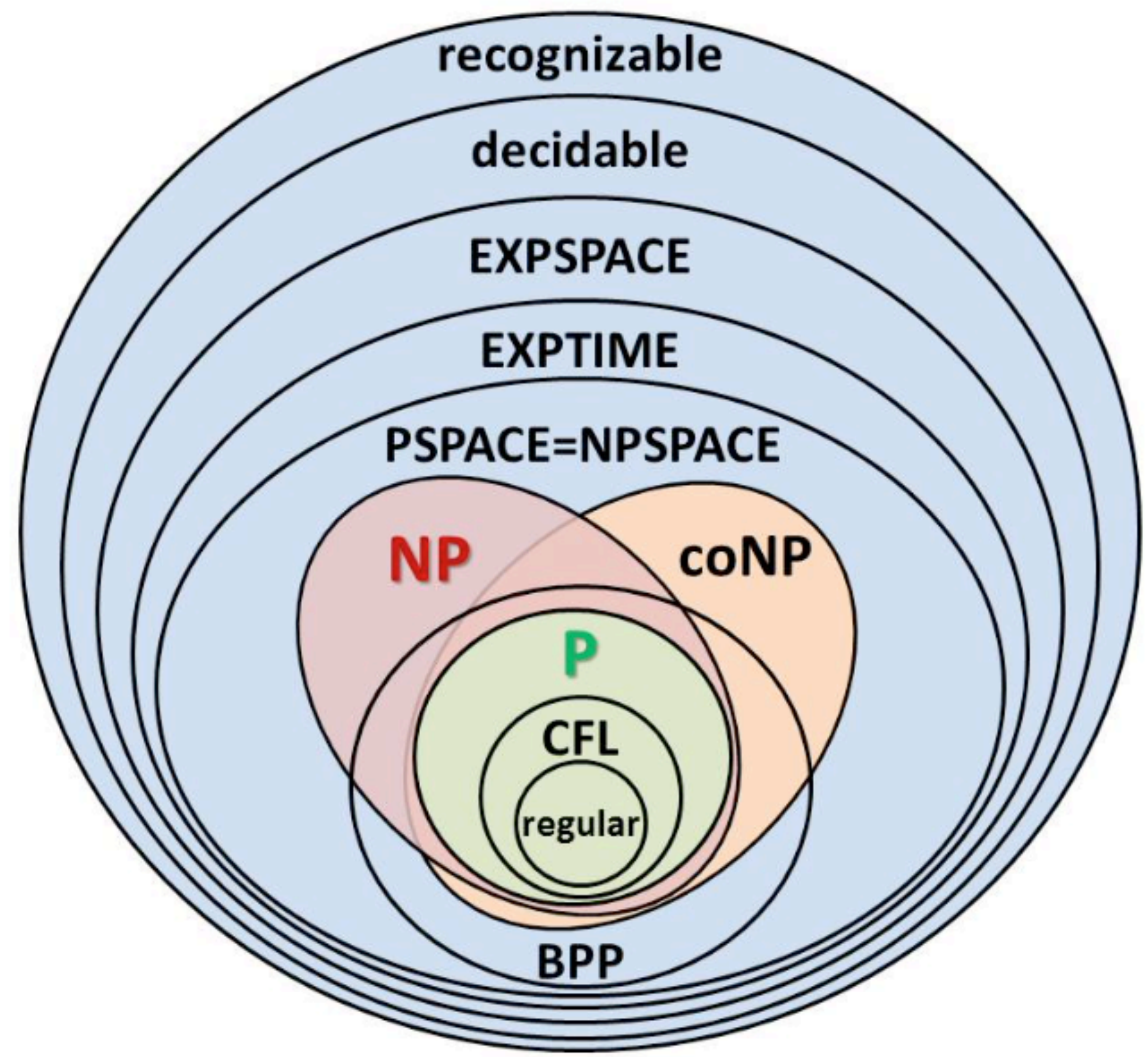


Turing machine

Morden works

Algorithms & Complexity





Course Info

- Instructors:
  - 刘明谋
  - lmm@nju.edu.cn
- Office hour:
  - 南雍东207， 周二 7-10pm
- QQ group: 1029885231
- Course webpage: http://tcs.nju.edu.cn/wiki/

高级算法（苏州校...

群号: 1029885231

Lecture Schedule

- 基于哈希的大数据算法
- 哈希表与面向大数据的现代计算场景
- 测度的集中与处理高维数据
- 网络流与线性/整数规划
- 其他重要话题

Lecture Schedule

- 基于哈希的大数据算法
  - Fingerprints
  - 布隆过滤器
  - 数据流算法
  - 实用哈希函数

- 哈希表与面向大数据的现代计算场景
- 测度的集中与处理高维数据
- 网络流与线性/整数规划
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- 基于哈希的大数据算法
- 哈希表与面向大数据的现代计算场景
  - 负载均衡
  - 哈希表、简洁数据结构、可扩展数据结构
  - 外存算法、并行算法、在线算法

- 测度的集中与处理高维数据

- 网络流与线性/整数规划

- 其他重要话题

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- 哈希表与面向大数据的现代计算场景
- 测度的集中与处理高维数据
  - Johnson-Lindenstrauss 变换
    - 应用：近似最近邻搜索和 LSH 位置敏感哈希
  - 子空间嵌入及其应用：线性回归
  - 快速的降维方法：稀疏子空间嵌入、快速JL变换

- 网络流与线性/整数规划
- 其他重要话题

Lecture Schedule

- 基于哈希的大数据算法
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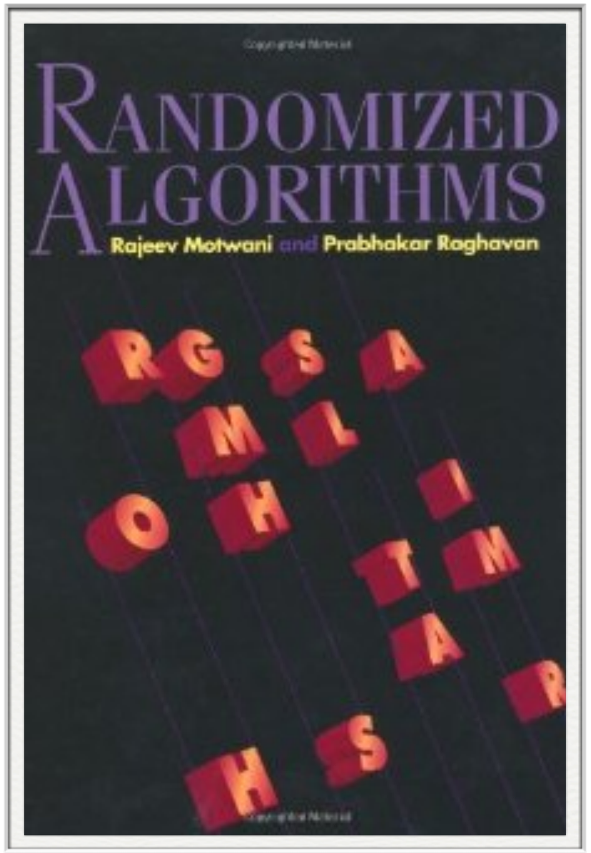
- 测度的集中与处理高维数据
- 网络流与线性/整数规划
    - 网络流
        - 最大流&最小割、最大流&费用流算法及其应用
    - 线性规划：原始-对偶、求解方法
    - 整数规划

- 其他重要话题

Lecture Schedule

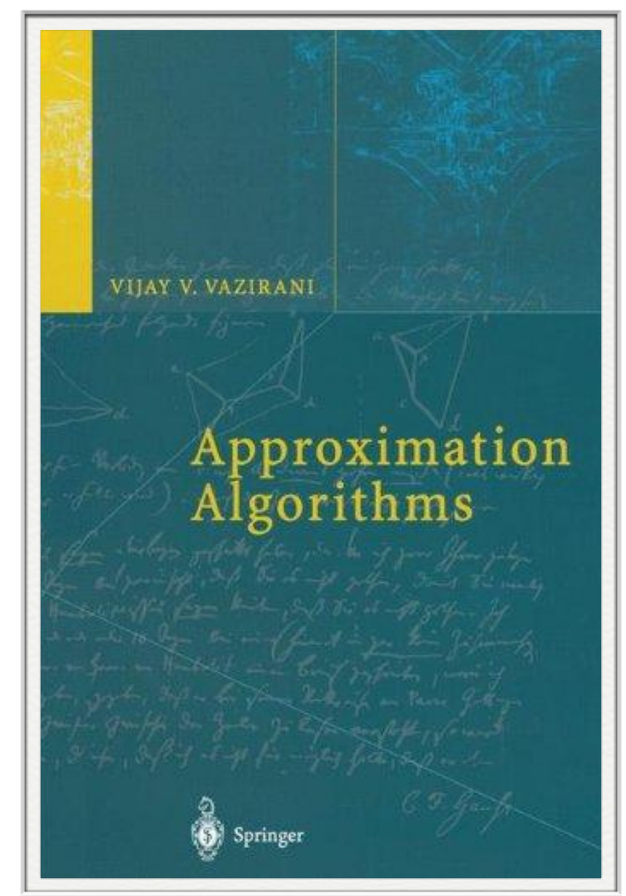
- 基于哈希的大数据算法
- 哈希表与面向大数据的现代计算场景
- 测度的集中与处理高维数据
- 网络流与线性/整数规划
- 其他重要话题
  - 计算复杂度?
  - 近似算法？
  - 计算几何?
  - 马尔科夫链蒙特卡洛算法?

Textbooks

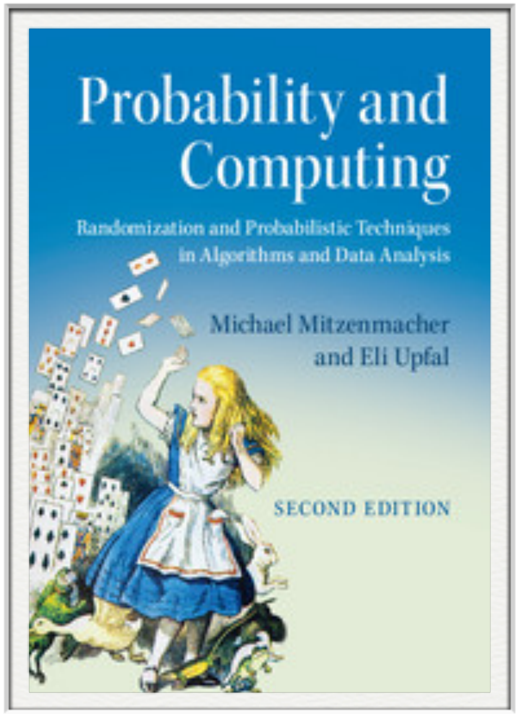


Rajeev Motwani and Prabhakar Raghavan.
Randomized Algorithms.
Cambridge University Press, 1995.

Vijay Vazirani
Approximation Algorithms.
Spinger-Verlag, 2001.

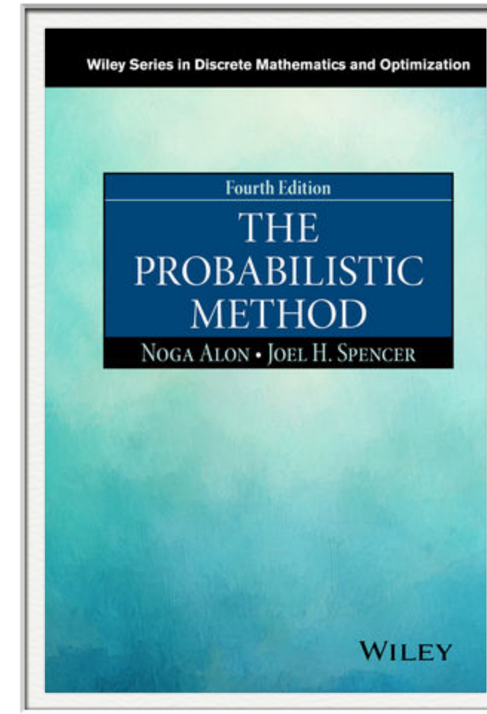


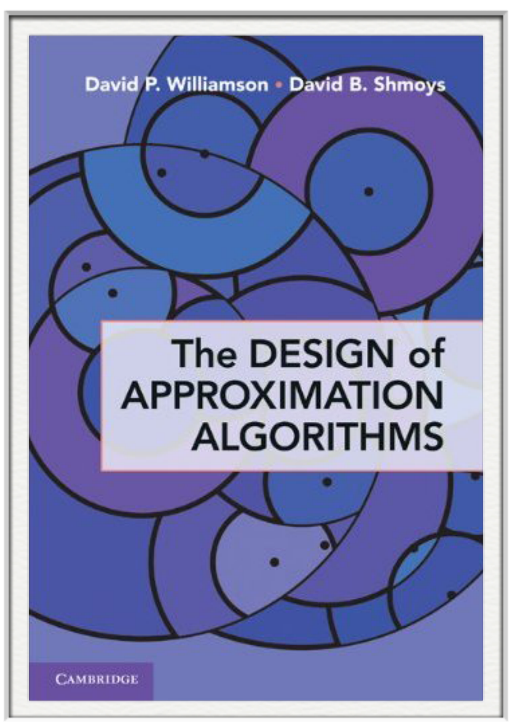
References



Mitzenmacher and Upfal.
Probability and Computing,
2nd Ed.

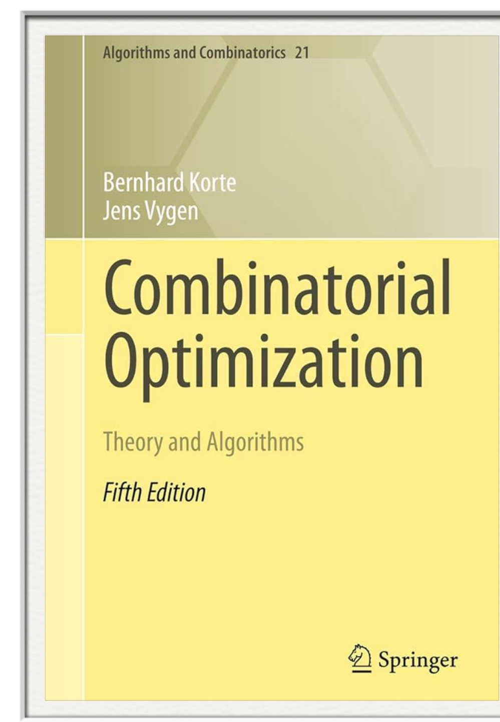
Alon and Spencer
The Probabilistic Method,
4th Ed.





Williamson and Shmoys
The Design of
Approximation Algorithms

Korte and Vygen
Combinatorial Optimization:
Theory and Algorithms



Problem sets & the final
课后作业和期末

- 课后作业较少，一节课大约一道题
- 成绩=期末*30%+课后作业*70%
- 如果人数较少：一对一口试
- 如果人数不是很多：essay写证明（用自己的语言重新描述某个前沿算法的证明，细化到你的同学能看懂的程度）
- 如果人数较多：笔试

Fingerprint

Checksum

SHA-1

MD5

CRC-32

Polynomial Identity Testing (PIT)

Input: two polynomials $f, g \in \mathbb{F}[x]$ of degree d.
Output: $f \equiv g$?

$\mathbb{F}[x]$: **polynomial ring** in x over **field** $\mathbb{F}$

$f \in \mathbb{F}[x]$ of degree d: $\quad f(x) = \sum_{i=0}^{d} a_i x^i$ where $a_i \in \mathbb{F}$

field

Input: a polynomial $f \in \mathbb{F}[x]$ of degree d.
Output: $f \equiv 0$?

f is given as **black-box**

Input: a polynomial $f \in \mathbb{F}[x]$ of degree d.

Output: $f \equiv 0$?

- Deterministic algorithm (polynomial interpolation):

pick arbitrary *distinct* $x_0, x_1, \ldots, x_d \in \mathbb{F}$;

check if $f(x_i) = 0$ for all $0 \leq i \leq d$;

Fundamental Theorem of Algebra.

Any non-zero d-degree polynomial $f \in \mathbb{F}[x]$ has at most d roots.

- Randomized algorithm (fingerprinting):

pick a uniform random $r \in S$;

check if $f(r) = 0$;

let $S \subseteq \mathbb{F}$ be arbitrary

(whose size to be fixed later)

Input: a polynomial $f \in \mathbb{F}[x]$ of degree d.

Output: $f \equiv 0$?

pick a uniform random $r \in S$;

check if $f(r) = 0$;

let $S \subseteq \mathbb{F}$ be arbitrary

~~(whose size to be fixed later)~~

$$|S| = 2d$$

if $f \equiv 0$: always correct

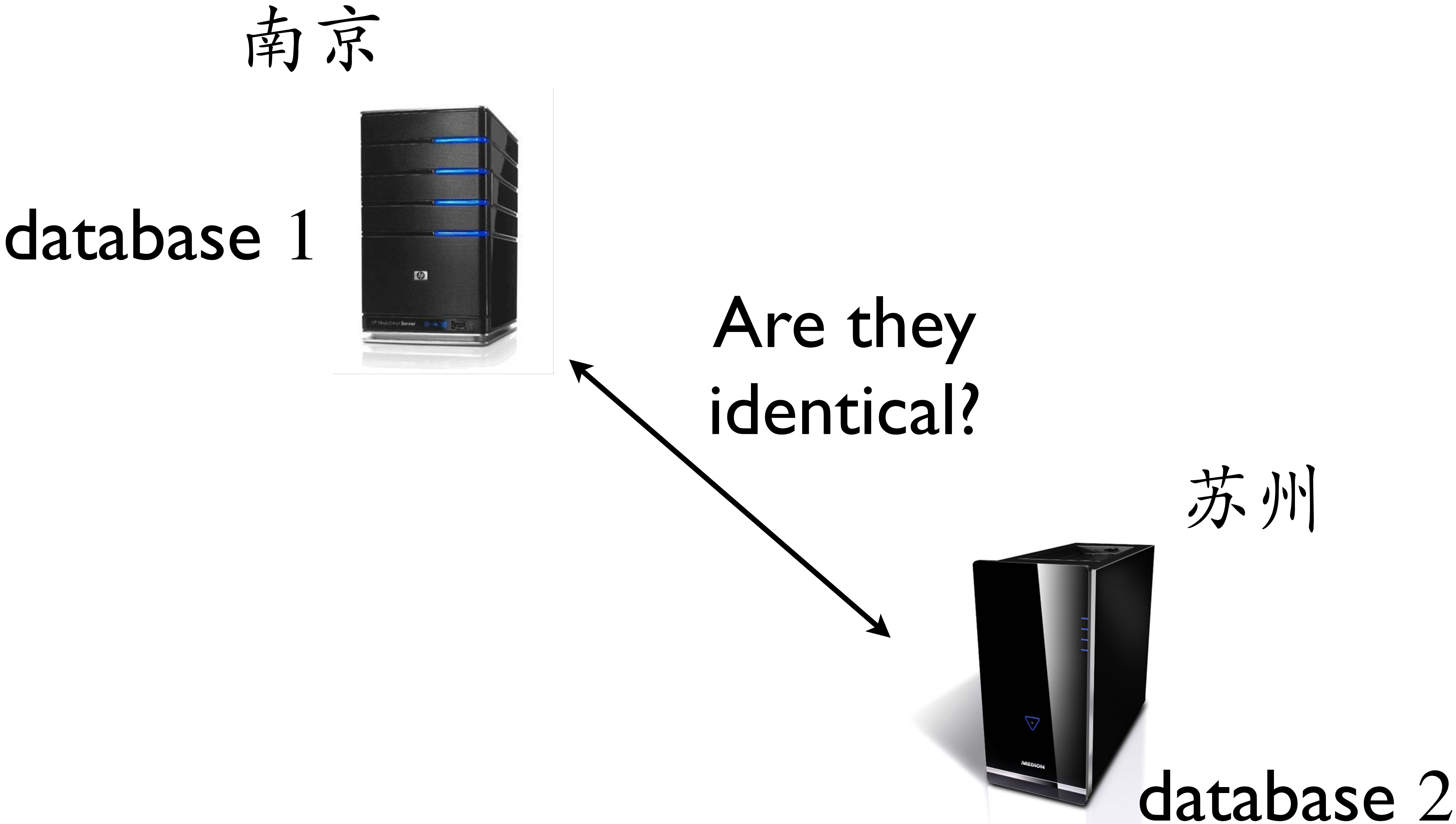
if $f \not\equiv 0$:

$$\Pr[f(r) = 0] \le \frac{d}{|S|} = \frac{1}{2}$$

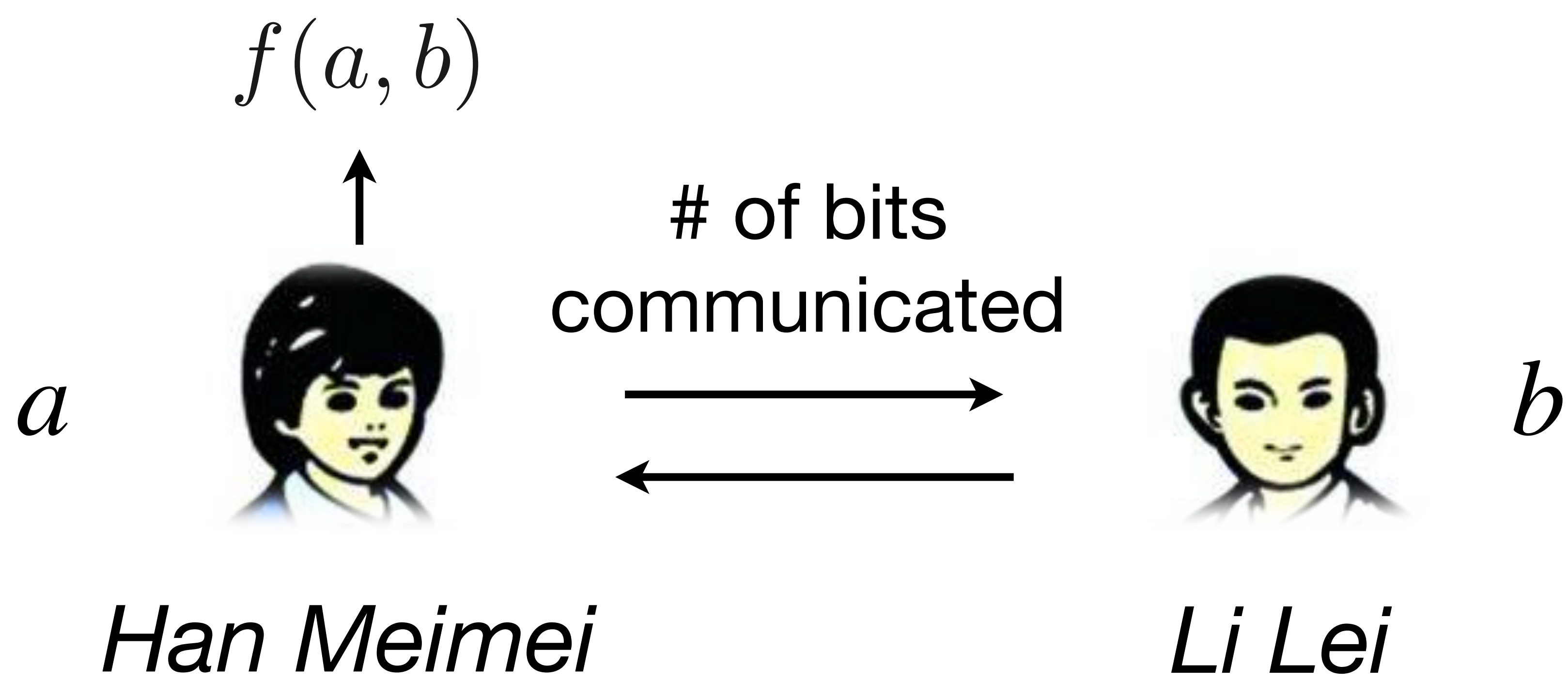
Fundamental Theorem of Algebra.

Any non-zero d-degree polynomial $f \in \mathbb{F}[x]$ has at most d roots.

Checking Identity



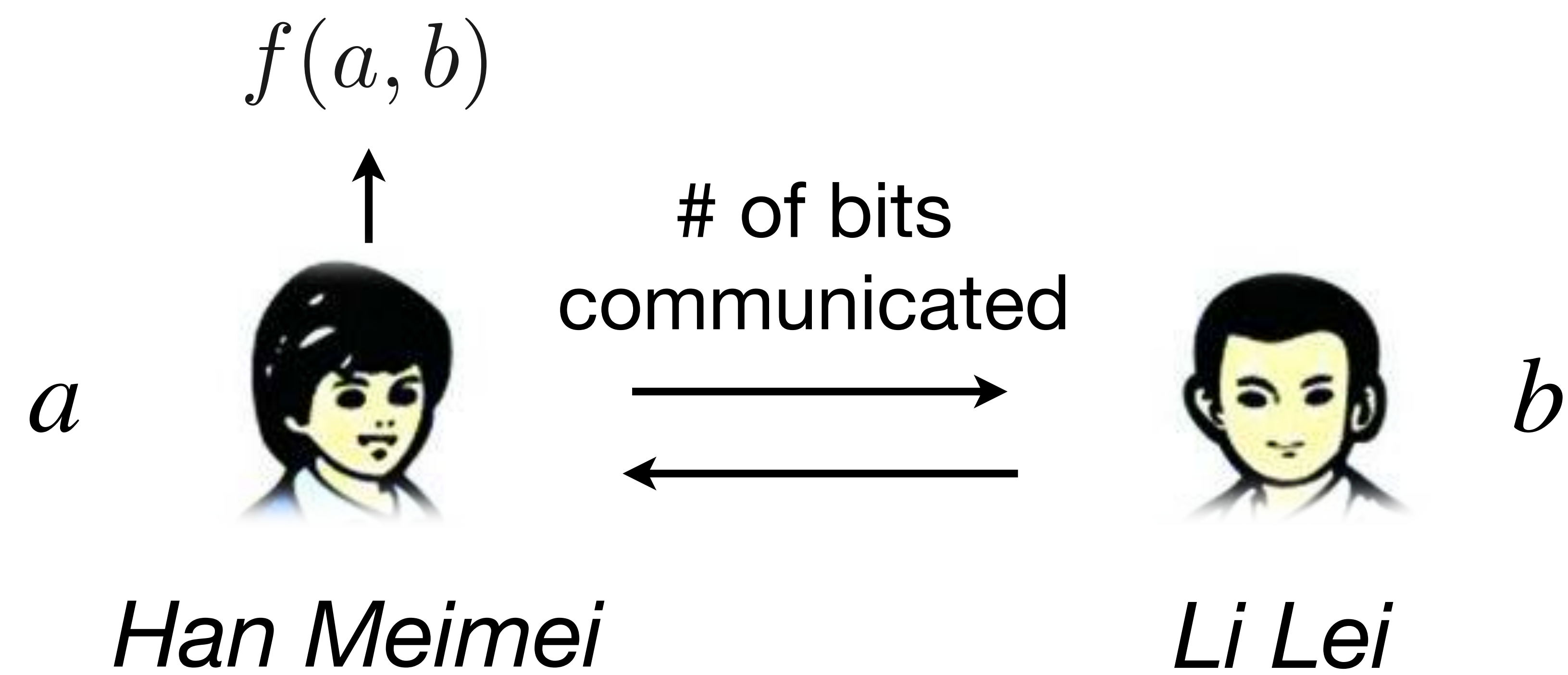
Communication Complexity



$$\mathrm{EQ}:\{0,1\}^n\times\{0,1\}^n\to\{0,1\}$$

$$\mathrm{EQ}(a,b)=\begin{cases}1 & a=b\\ 0 & a\neq b\end{cases}$$

Communication Complexity



$$\mathrm{EQ} : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$$

Theorem (Yao 1979).

Every deterministic communication protocol solving EQ communicates n bits in the worst-case.

Communication Complexity

$$f = \sum_{i=0}^{n-1} a_i x^i$$

$f(r)=g(r)$?

$$g = \sum_{i=0}^{n-1} b_i x^i$$

$a \in \{0,1\}^n$

$r, g(r)$

$b \in \{0,1\}^n$

pick uniform
random $r \in [2n]$

by **PIT**:

one-sided error $\leq \dfrac{1}{2}$

# of bit communicated: **too large!**

Communication Complexity

$$f = \sum_{i=0}^{n-1} a_i x^i$$

$f(r)=g(r)$?

$$g = \sum_{i=0}^{n-1} b_i x^i$$

$a \in \{0,1\}^n$

$r, g(r)$

O(log n) bits

$b \in \{0,1\}^n$

pick uniform
random $r \in [p]$

- choose a **prime** $p \in [n^2, 2n^2]$
- let $f, g \in \mathbb{Z}_p[x]$
- by **PIT**: one-sided error is $\dfrac{n}{p} = O\left(\dfrac{1}{n}\right)$

(correct w.h.p.)

Public coin

$$f = \sum_{i=0}^{n-1} a_i x_i \qquad\qquad g = \sum_{i=0}^{n-1} b_i x_i$$

$f(r)=g(r)$?

$a \in \{0,1\}^n$

$\xleftarrow{g(r)}$

1 bit

$b \in \{0,1\}^n$

pick uniform random $r \in \{0,1\}^n$

- choose a prime $p = 2$
- let $f, g \in \mathbb{Z}_p[x]$
- one-sided error is $\frac{1}{2}$

Polynomial Identity Testing (PIT)

Input: a polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d.
Output: $f \equiv 0$?

$\mathbb{F}[x_1, \ldots, x_n]$: ring of n-variate polynomials in $x_1, \ldots, x_n$ over field $\mathbb{F}$

$f \in \mathbb{F}[x_1, \ldots, x_n]$:

$$f(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

Degree of f: maximum $i_1 + i_2 + \cdots + i_n$ with $a_{i_1, i_2, \ldots, i_n} \neq 0$

Polynomial Identity Testing (PIT)

Input: a polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d.
Output: $f \equiv 0$?

$$f(x_1, \ldots, x_n) = \sum_{\substack{i_1, \ldots, i_n \geq 0 \\ i_1 + \cdots + i_n \leq d}} a_{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

f is given as **black-box**: given any $\vec{x} \in \mathbb{F}^n$, return $f(\vec{x})$

or as product form: e.g. Vandermonde determinant

$$M = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{bmatrix} \qquad f(\vec{x}) = \det(M) = \prod_{j<i} (x_i - x_j)$$

Polynomial Identity Testing (PIT)

Input: a polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d.
Output: $f \equiv 0$?

f is given as product form

if ∃ a *poly-time deterministic* algorithm for **PIT**:

either: $\mathbf{NEXP} \neq \mathbf{P/poly}$

or: $\mathbf{\#P} \neq \mathbf{FP}$

Input: a polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d.

Output: $f \equiv 0$?

Fix an arbitrary $S \subseteq \mathbb{F}$:

pick $r_1, \ldots, r_n \in S$ uniformly and independently at random;

check if $f(r_1, \ldots, r_n) = 0$;

$$f \equiv 0 \implies f(r_1, \ldots, r_n) = 0$$

Schwartz-Zippel Theorem.

$$f \not\equiv 0 \implies \Pr\left[f(r_1, \ldots, r_n) = 0\right] \le \frac{d}{|S|}$$

# of roots for any $f \not\equiv 0$ in any cube S^n is $\le d \cdot |S|^{n-1}$

Schwartz-Zippel Theorem.

$$f \not\equiv 0 \implies \Pr\left[f(r_1, \ldots, r_n) = 0\right] \le \frac{d}{|S|}$$

$$f(x_1, x_2, \ldots, x_n) = \sum_{\substack{i_1, i_2, \ldots, i_n \ge 0 \\ i_1 + i_2 + \cdots + i_n \le d}} a_{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

f can be treated as a single-variate polynomial of x_n:

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=0}^{d} x_n^i f_i(x_1, x_2, \ldots, x_{n-1})$$

$$= g_{x_1, x_2, \ldots, x_{n-1}}(x_n)$$

$$\Pr[f(r_1, r_2, \ldots, r_n) = 0] = \Pr[g_{r_1, r_2, \ldots, r_{n-1}}(r_n) = 0]$$

$$g_{r_1, r_2, \ldots, r_{n-1}} \not\equiv 0?$$

done?

Schwartz-Zippel Theorem.

$$f \not\equiv 0 \implies \Pr\left[f(r_1,\ldots,r_n)=0\right] \le \frac{d}{|S|}$$

induction on n :

basis: $n{=}1$ single-variate case, proved by the *fundamental Theorem of algebra*

I.H.: Schwartz-Zippel Thm is true for all smaller n

Schwartz-Zippel Theorem.

$$f \not\equiv 0 \implies \Pr\left[f(r_1, \ldots, r_n) = 0\right] \le \frac{d}{|S|}$$

induction step:

k: highest power of x_n in f $\Rightarrow$ $\begin{cases} f_k \not\equiv 0 \\ \text{degree of } f_k \le d - k \end{cases}$

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=0}^{k} x_n^i f_i(x_1, x_2, \ldots, x_{n-1})$$

$$= x_n^k f_k(x_1, x_2, \ldots, x_{n-1}) + \bar{f}(x_1, x_2, \ldots, x_n)$$

where $\bar{f}(x_1, x_2, \ldots, x_n) = \sum_{i=0}^{k-1} x_n^i f_i(x_1, x_2, \ldots, x_{n-1})$

highest power of x_n in $\bar{f}$ $< k$

Schwartz-Zippel Theorem.

$$f \not\equiv 0 \implies \Pr\left[f(r_1, \ldots, r_n) = 0\right] \le \frac{d}{|S|}$$

$$f(x_1, x_2, \ldots, x_n) = x_n^k f_k(x_1, x_2, \ldots, x_{n-1}) + \bar{f}(x_1, x_2, \ldots, x_n)$$

$$\begin{cases} f_k \not\equiv 0 \\ \text{degree of } f_k \le d - k \end{cases}$$

highest power of x_n in $\bar{f} < k$

law of total probability:

$$\Pr[f(r_1, r_2, \ldots, r_n) = 0]$$

I.H. $\Rightarrow$ $\le \frac{d-k}{|S|}$

$$= \Pr[f(\vec{r}) = 0 \mid f_k(r_1, \ldots, r_{n-1}) = 0] \cdot \Pr[f_k(r_1, \ldots, r_{n-1}) = 0]$$

$$+ \Pr[f(\vec{r}) = 0 \mid f_k(r_1, \ldots, r_{n-1}) \neq 0] \cdot \Pr[f_k(r_1, \ldots, r_{n-1}) \neq 0]$$

$$= \Pr[g_{r_1, \ldots, r_{n-1}}(r_n) = 0 \mid f_k(r_1, \ldots, r_{n-1}) \neq 0] \le \frac{k}{|S|}$$

where $g_{x_1, \ldots, x_{n-1}}(x_n) = f(x_1, \ldots, x_n)$

Schwartz-Zippel Theorem.

$$f \not\equiv 0 \implies \Pr\left[f(r_1, \ldots, r_n) = 0\right] \leq \frac{d}{|S|}$$

$$\Pr[f(r_1, r_2, \ldots, r_n) = 0] \leq \frac{d-k}{|S|} + \frac{k}{|S|} = \frac{d}{|S|}$$

Input: a polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d.
Output: $f \equiv 0$?

Fix an arbitrary $S \subseteq \mathbb{F}$:

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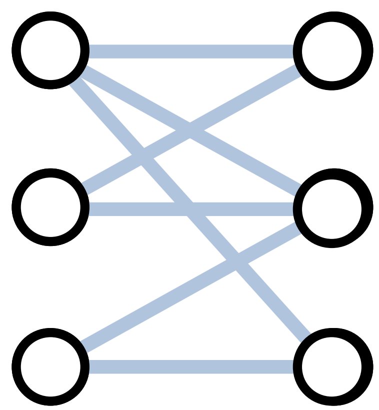
# of roots for any $f \not\equiv 0$ in any cube S^n is $\leq d \cdot |S|^{n-1}$

Applications of *Schwartz-Zippel*

- test whether a graph has perfect matching;
- test isomorphism of rooted trees;
- distance property of Reed-Muller codes;
- proof of hardness vs randomness tradeoff;
- algebraic construction of ***probabilistically checkable proofs*** (PCP);
- ....

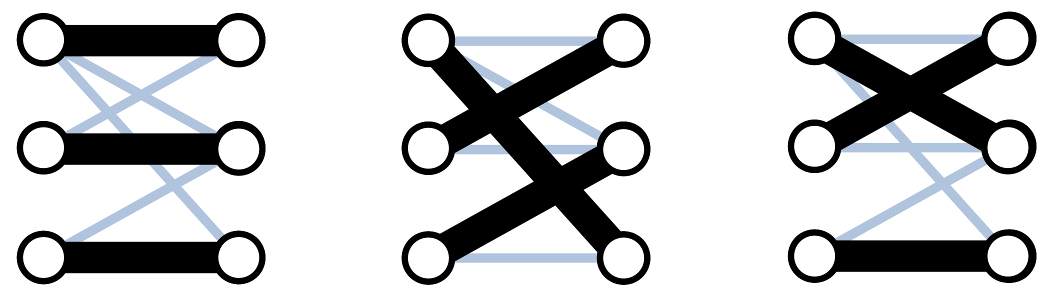
Bipartite Perfect Matching

bipartite graph

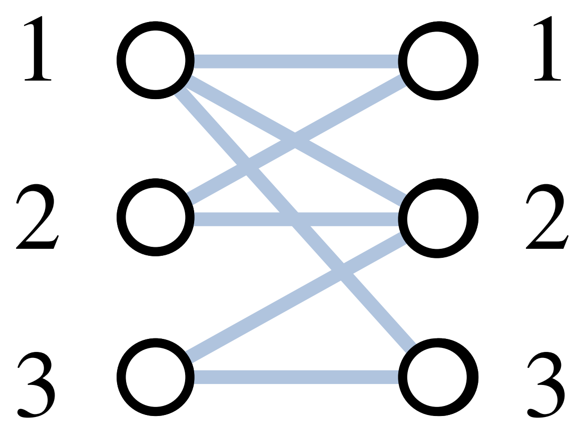


$G([n],[n],E)$

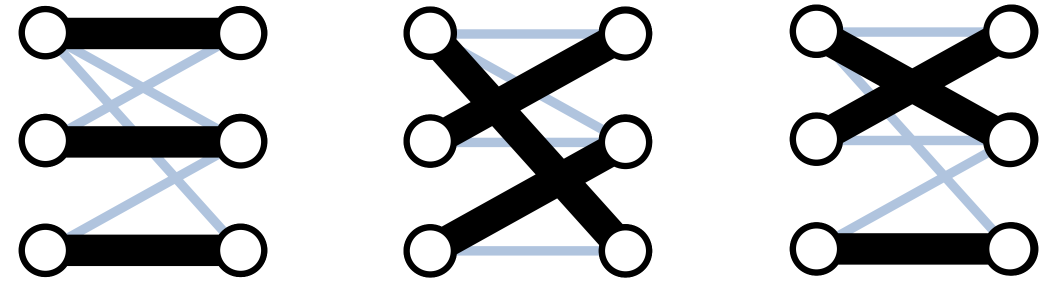
perfect matchings



- determine whether G has a perfect matching:
  - **Hall's theorem:** enumerates all subset of $[n]$
  - **Hungarian method:** $O(n^3)$
  - **Hopcroft-Karp algorithm:** $O(m\sqrt{n})$



$G([n],[n],E)$



$$A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & 0 \\ 0 & x_{32} & x_{33} \end{bmatrix}$$

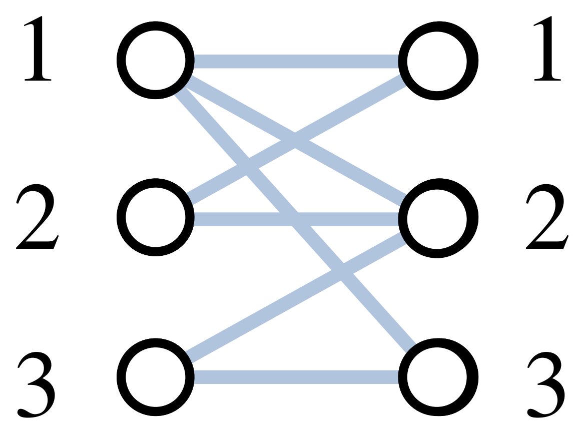
$$\begin{aligned}\det(A) = &\; x_{11}x_{22}x_{33} \\ &+ x_{13}x_{21}x_{32} \\ &- x_{12}x_{21}x_{33}\end{aligned}$$

Edmonds matrix: an $n \times n$ matrix A defined as

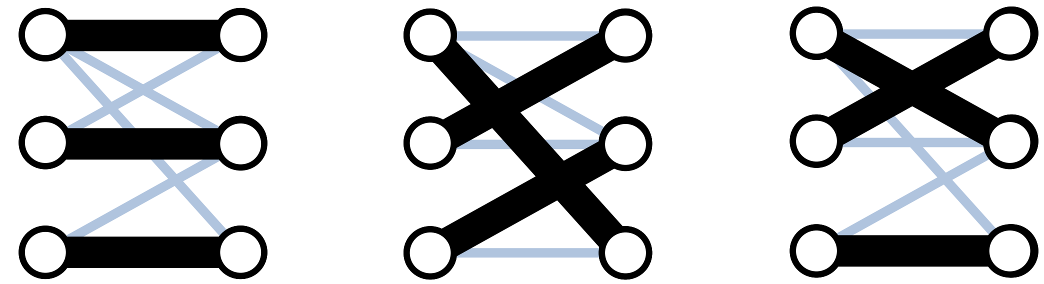
$$\forall i, j \in [n], \quad A(i,j) = \begin{cases} x_{i,j} & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases}$$

Theorem: $\det(A) \not\equiv 0 \iff \exists$ a perfect matching in G

$$\det(A) = \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \prod_{i \in [n]} A(i, \pi(i)) = \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \begin{cases} \prod_{i \in [n]} x_{i,\pi(i)} & \pi \text{ is a P.M.} \\ 0 & \text{otherwise} \end{cases}$$



$G([n],[n],E)$



$$A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & 0 \\ 0 & x_{32} & x_{33} \end{bmatrix}$$

$$\det(A) = x_{11}x_{22}x_{33} \\ +x_{13}x_{21}x_{32} \\ -x_{12}x_{21}x_{33}$$

Edmonds matrix: an $n \times n$ matrix A defined as

$$\forall i,j \in [n], \quad A(i,j) = \begin{cases} x_{i,j} & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases}$$

Theorem: $\det(A) \not\equiv 0 \iff \exists$ a perfect matching in G

- $\det(A)$ is an m-variate degree-n polynomial:
  - Use *Schwartz-Zippel* to check whether $\det(A) \not\equiv 0$
  - Computing determinants is generic and can be done in parallel (Chistov's algorithm)

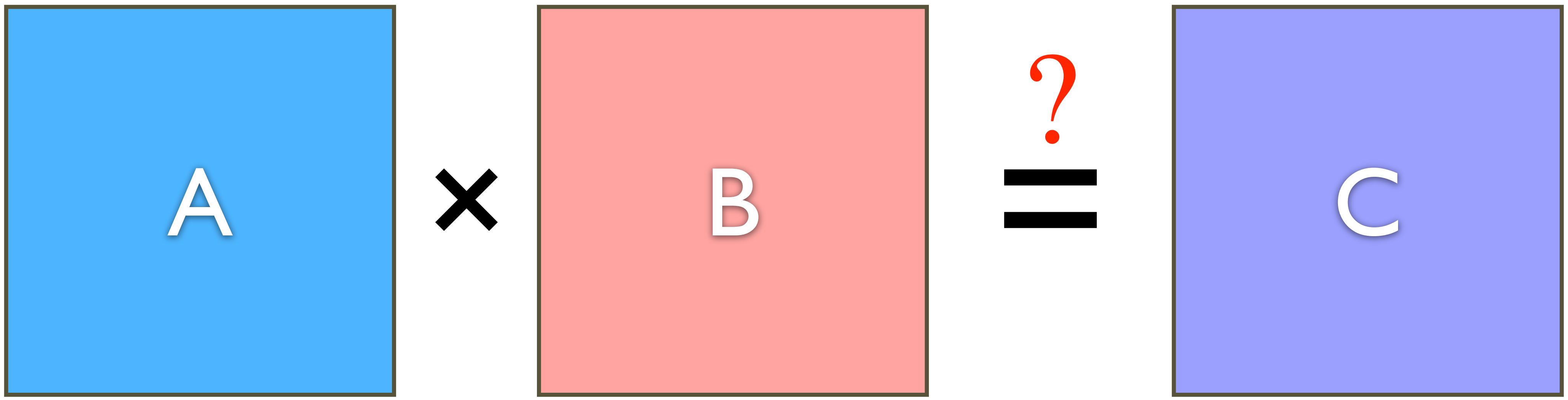
Fingerprinting (checksum)

$$\begin{array}{ccc} X & = & Y \quad ? \\ \downarrow & & \downarrow \\ \text{FING}(X) & = & \text{FING}(Y) \quad ? \end{array}$$

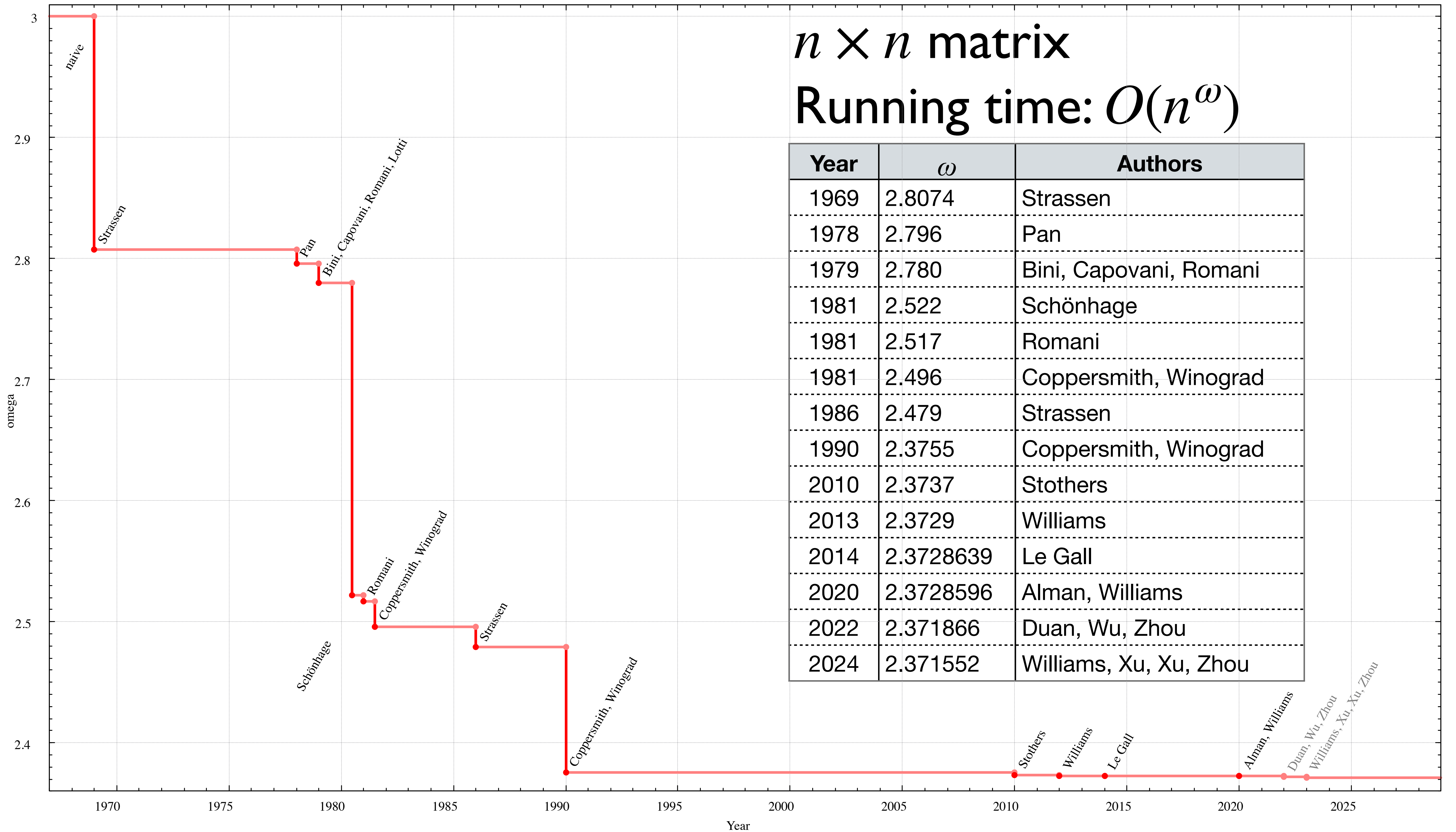
- FING( ) is a function: $X = Y \implies \text{FING}(X) = \text{FING}(Y)$
- if $X \neq Y$, $\Pr[\text{ FING}(X) = \text{FING}(Y)\ ]$ is small.
- Fingerprints are easy to compute and compare.

Checking Matrix Multiplication

- three $n \times n$ matrices A, B, C:



Matrix Multiplication Algorithms



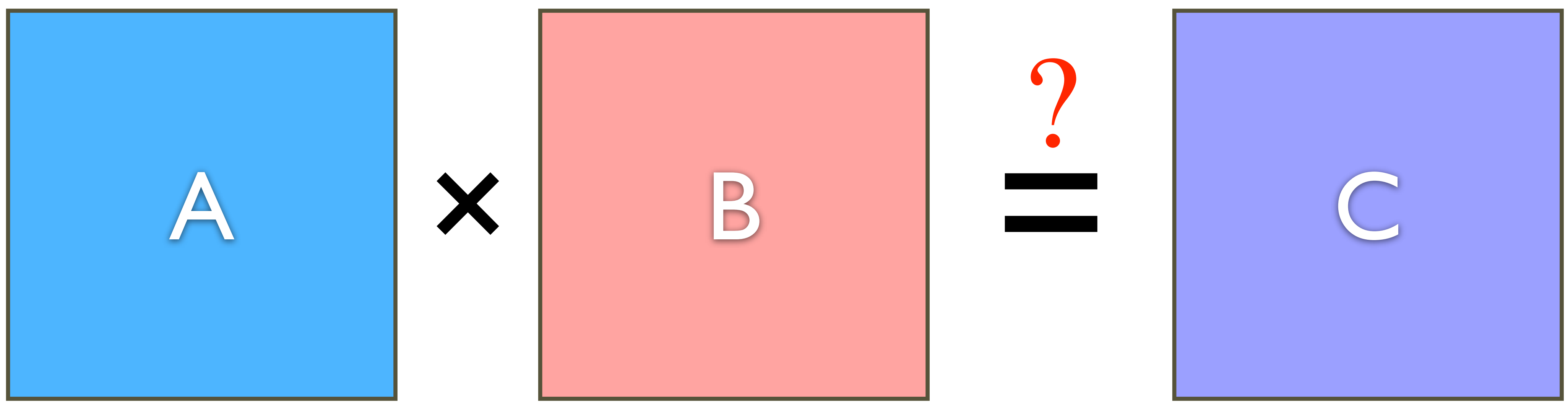
$n \times n$ matrix

Running time: $O(n^{\omega})$

Year	ω	Authors
1969	2.8074	Strassen
1978	2.796	Pan
1979	2.780	Bini, Capovani, Romani
1981	2.522	Schönhage
1981	2.517	Romani
1981	2.496	Coppersmith, Winograd
1986	2.479	Strassen
1990	2.3755	Coppersmith, Winograd
2010	2.3737	Stothers
2013	2.3729	Williams
2014	2.3728639	Le Gall
2020	2.3728596	Alman, Williams
2022	2.371866	Duan, Wu, Zhou
2024	2.371552	Williams, Xu, Xu, Zhou

Checking Matrix Multiplication

- three $n \times n$ matrices A, B, C:



Freivald's Algorithm:

pick a uniform random $r \in \{0,1\}^n$;

check whether $A(Br) = Cr$;

time: $O(n^2)$

if $AB = C$: always correct

if $AB \neq C$:

Freivald's Algorithm:

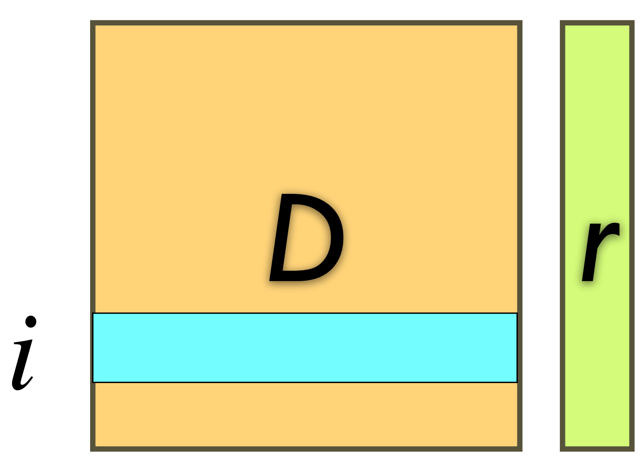
pick a uniform random $r \in \{0,1\}^n$;

check whether $A(Br) = Cr$;

if $AB \neq C$: let $D = AB - C \neq \mathbf{0}_{n \times n}$

suppose $D_{ij} \neq 0$

$$\Pr[ABr = Cr] = \Pr[Dr = \mathbf{0}] \leq \frac{2^{n-1}}{2^n} = \frac{1}{2}$$



$$(Dr)_i = \sum_{k=1}^{n} D_{ik} r_k = 0$$

$$r_j = -\frac{1}{D_{ij}} \sum_{k \neq j} D_{ik} r_k$$

Freivald's Algorithm:

pick a uniform random $r \in \{0,1\}^n$;

check whether $A(Br) = Cr$;

if $AB = C$: always correct

Theorem (Feivald 1979).

For $n \times n$ matrices A, B, C, if $AB \neq C$, for uniform random $r \in \{0,1\}^n$,

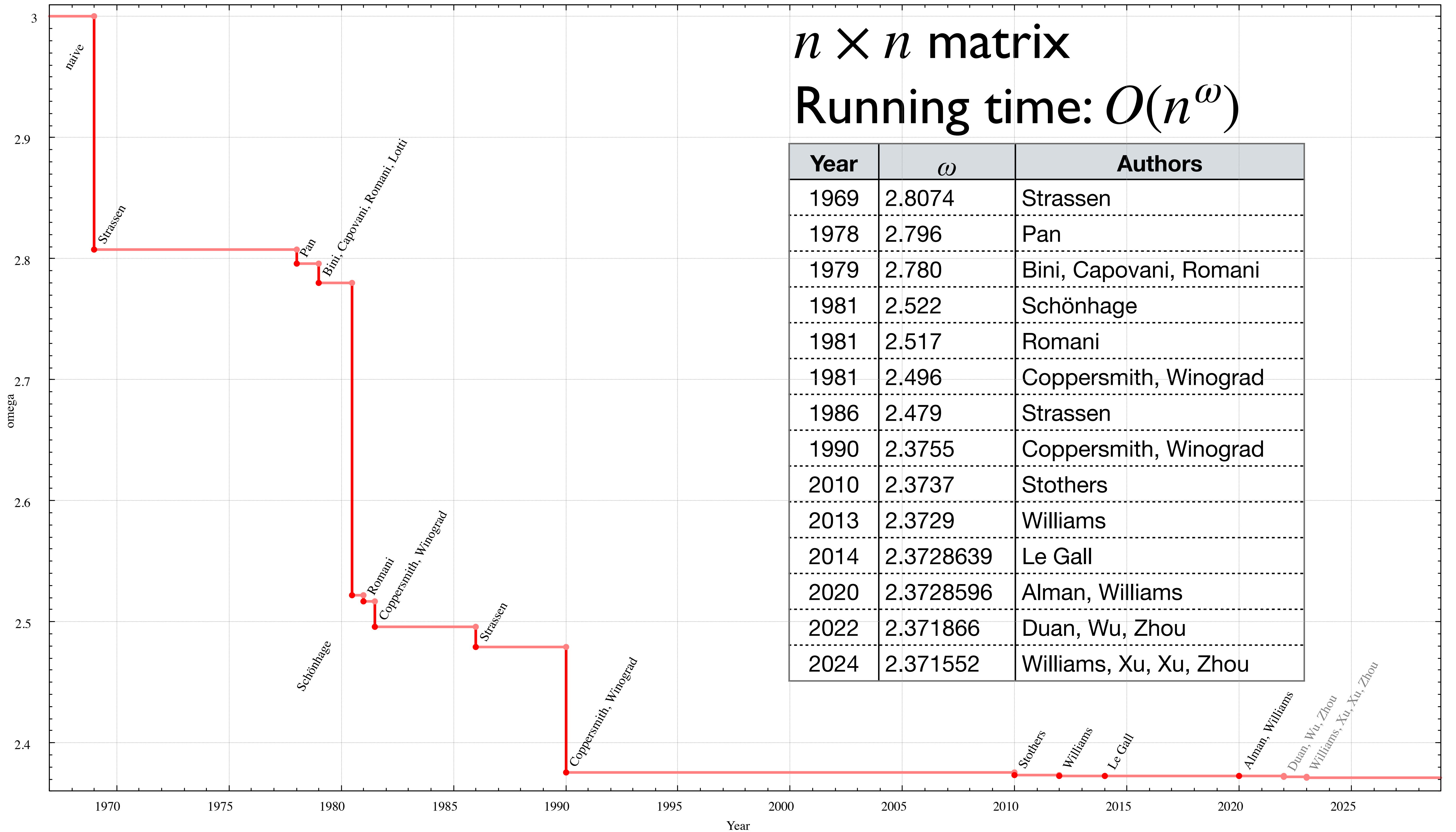
$$\Pr[ABr = Cr] \leq \frac{1}{2}$$

repeat independently for $O(\log n)$ times

Total running time: $O(n^2 \log n)$

Correct *with high probability* (w.h.p.).

Matrix Multiplication Algorithms



$n \times n$ matrix

Running time: $O(n^{\omega})$

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2022	2.371866	Duan, Wu, Zhou
2024	2.371552	Williams, Xu, Xu, Zhou

Checking Matrix Multiplication

- three $n \times n$ matrices A, B, C:

$$A \times B \stackrel{?}{=} C$$

Freivald's Algorithm:

pick a uniform random $r \in \{0,1\}^n$;

check whether $A(Br) = Cr$;

For an $n \times n$ matrix M:

$$\text{FING}(M) = Mr \text{ for uniform random } r \in \{0,1\}^n$$

Polynomial Identity Testing (PIT)

Input: a polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d.
Output: $f \equiv 0$?

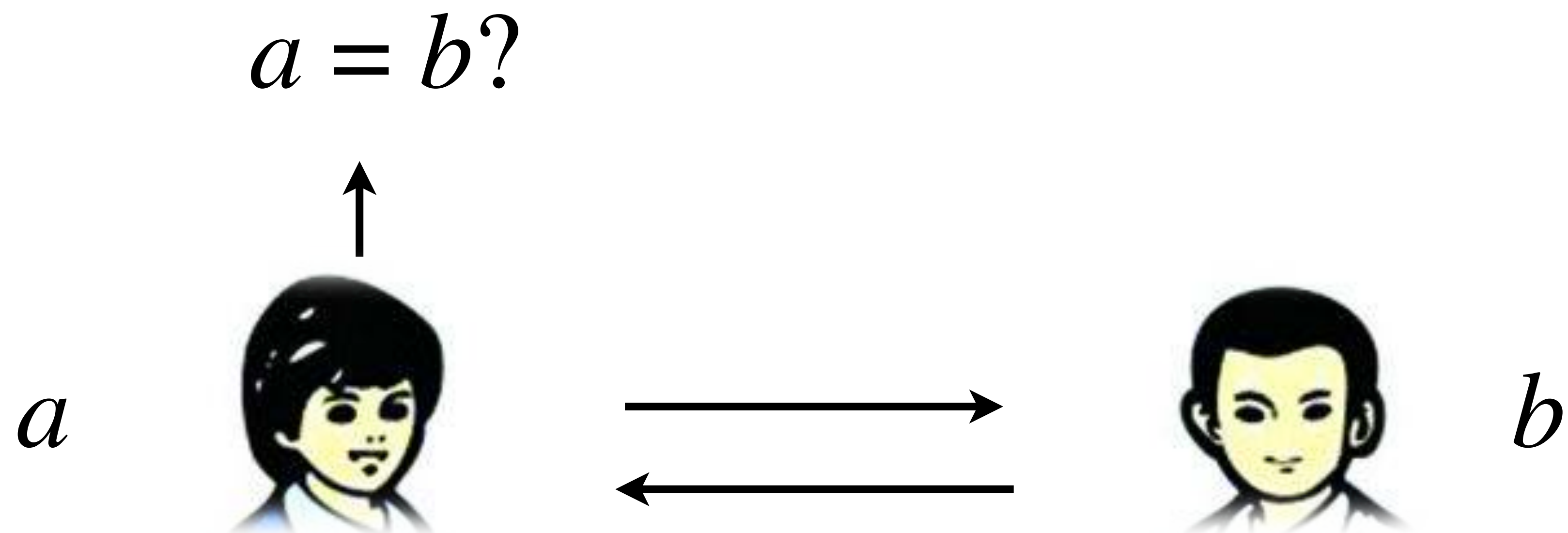
Fix an arbitrary $S \subseteq \mathbb{F}$:

pick $r_1, \ldots, r_n \in S$ uniformly and independently at random;
check if $f(r_1, \ldots, r_n) = 0$;

For a polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$:

FING$(f) = f(r_1, \ldots, r_n)$ for uniform independent $r_1, \ldots, r_n \in S$

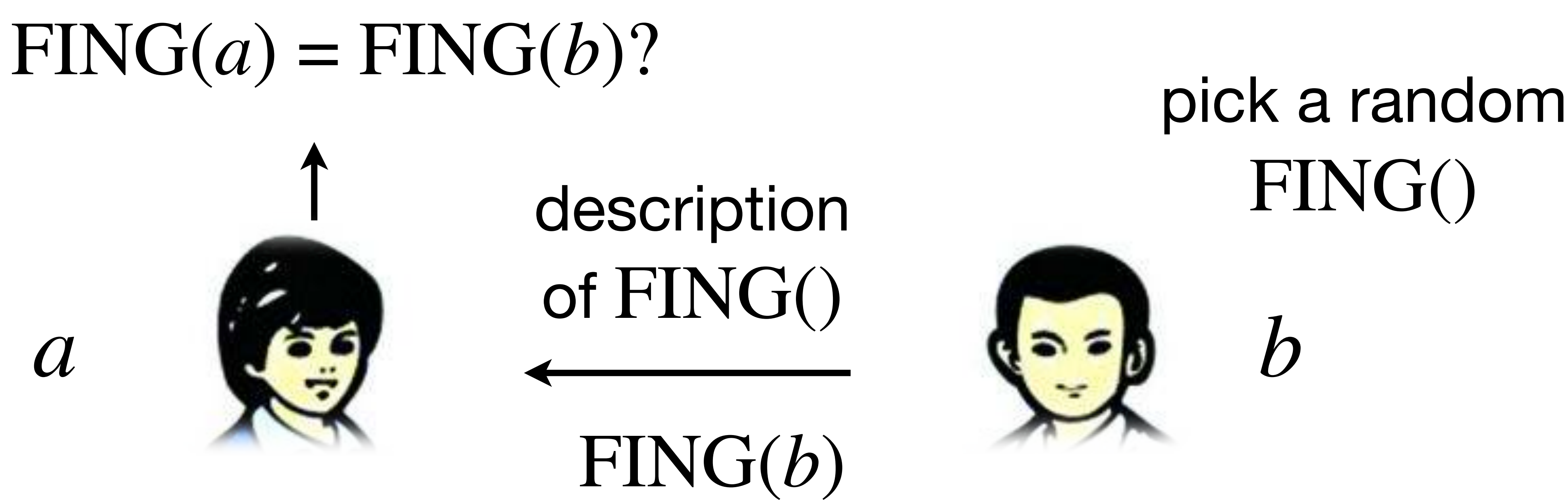
Communication Complexity



$$\mathrm{EQ} : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$$

$$\mathrm{EQ}(a,b) = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

Fingerprinting



- FING( ) is a function: $a = b \implies$ FING(a) = FING(b)
- if $a \neq b$, Pr[FING(a) = FING(b)] is small.
- Fingerprints are short.

$$f = \sum_{i=0}^{n-1} a_i x^i$$

$f(r)$=$g(r)$?

$$g = \sum_{i=0}^{n-1} b_i x^i$$

$a \in \{0,1\}^n$

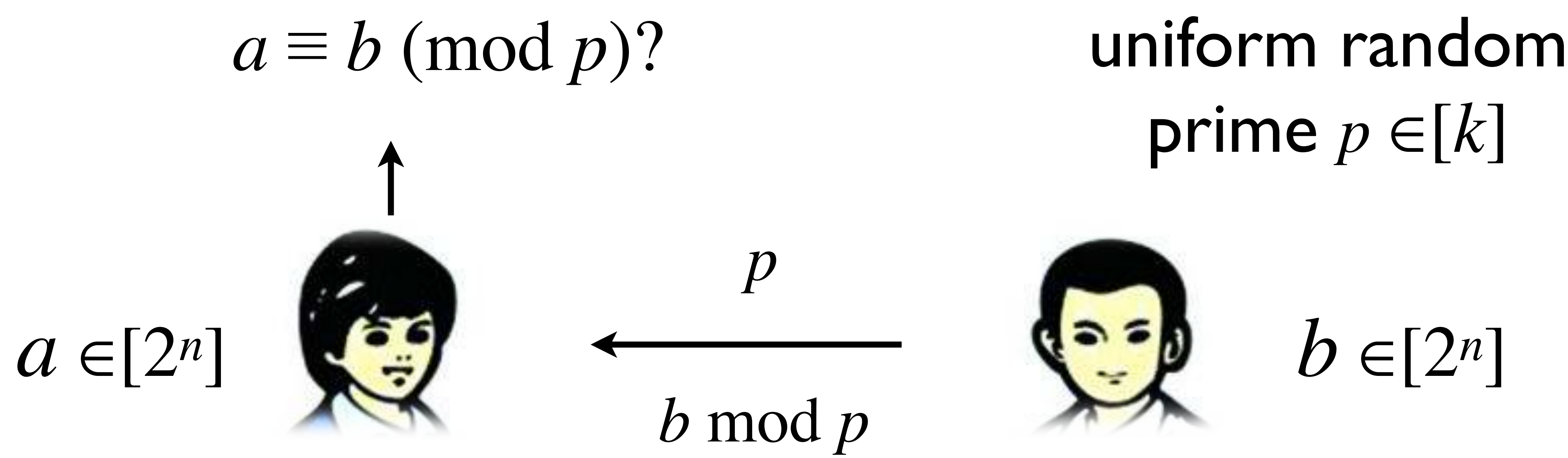
$r, g(r)$

$b \in \{0,1\}^n$

pick uniform random $r \in [p]$

$f, g \in \mathbb{Z}_p[x]$ for a prime $p \in [n^2, 2n^2]$

$$\mathrm{FING}(b) = \sum_{i=0}^{n-1} b_i r^i \text{ for random } r$$



$\mathrm{FING}(x) = x \bmod p$ for uniform random prime $p \in [k]$

communication complexity: $O(\log k)$

if $a = b$ $\Rightarrow$ $a \equiv b \pmod{p}$

if $a \neq b$: $\Pr[a \equiv b \pmod{p}] \leq ?$

for a $z = | a - b | \neq 0$: $\Pr[z \bmod p = 0] \leq ?$

uniform random prime $p \in [k]$

for a $z = |a - b| \neq 0$: $\Pr[z \bmod p = 0] \leq ?$

$$\left.\begin{array}{l} z \in [2^n] \\ \text{each prime divisor} \geq 2 \end{array}\right\} \Rightarrow \#\text{ of prime divisors of } z \leq n$$

$$\Pr[z \bmod p = 0] = \frac{\#\text{ of prime divisors of } z \quad \leq n}{\#\text{ of primes in } [k] \quad = \pi(k)}$$

$\pi(N)$: # of primes in $[N]$

Prime Number Theorem (PNT):

$$\pi(N) \sim \frac{N}{\ln N} \text{ as } N \to \infty$$

$a \equiv b \pmod{p}$?

uniform random prime $p \in [k]$

$a \in [2^n]$

$\xleftarrow[b \bmod p]{p}$

$b \in [2^n]$

for a $z = |a - b| \neq 0$: $\Pr[z \bmod p = 0] \leq$?

$$\Pr[z \bmod p = 0] = \frac{\text{\# of prime divisors of } z \quad \leq n}{\text{\# of primes in } [k] \quad = \pi(k)}$$

choose $k = n^3$ $\quad \leq \dfrac{n \ln k}{k} = \dfrac{3 \ln n}{n^2} = O\left(\dfrac{1}{n}\right)$

$a \equiv b \pmod{p}$?

uniform random prime $p \in [k]$

$a \in [2^n]$  $\xleftarrow[b \bmod p]{p}$  $b \in [2^n]$

$\mathrm{FING}(b) = b \bmod p$ for uniform random prime $p \in [n^3]$

communication complexity: $O(\log n)$

if $a = b$ $\Rightarrow$ $a \equiv b \pmod{p}$

if $a \neq b$ $\Rightarrow$ $\Pr[\, a \equiv b \pmod{p} \,] = O\left(\frac{1}{n}\right)$

Public coin

$f(r)=g(r)$?

$f = \langle \mathbf{a}, \mathbf{x} \rangle$

$g = \langle \mathbf{b}, \mathbf{x} \rangle$

$a \in \{0,1\}^n$

$\xleftarrow{g(r)}$

1 bits

$b \in \{0,1\}^n$

pick uniform
random $r \in \{0,1\}^n$

FING$(b) = \langle \mathbf{b}, \mathbf{r} \rangle$ for random $\mathbf{r}$

Pattern Matching

Input: string $x \in \{0,1\}^n$, pattern $y \in \{0,1\}^m$

Check whether y is a substring of x.

- naive algorithm: $O(mn)$ time
- **Knuth-Morris-Prat** (KMP) algorithm: $O(m + n)$ time
    - finite state automaton
        - Aho–Corasick algorithm

Pattern Matching via Fingerprinting

$x[i, i+m-1] = y$?

y

$x[i, i+m-1]$

y : | y_1 | y_2 | $\cdots$ | y_m | $\in \{0,1\}^m$

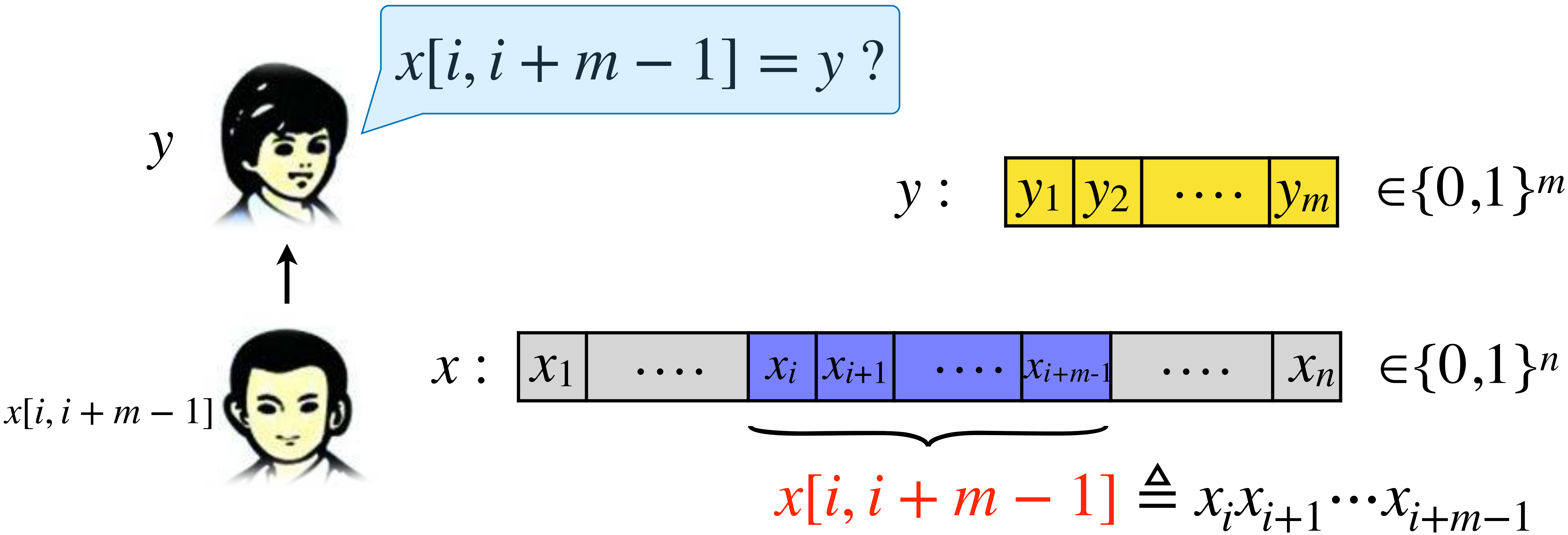
x : | x_1 | $\cdots$ | x_i | x_{i+1} | $\cdots$ | x_{i+m-1} | $\cdots$ | x_n | $\in \{0,1\}^n$

$$\underbrace{x_i x_{i+1} \cdots x_{i+m-1}}$$

$x[i, i+m-1] \triangleq x_i x_{i+1} \cdots x_{i+m-1}$

> pick a random FING();
> for $i = 1,2,\ldots,n-m+1$ do:
>   if FING($x[i, i+m-1]$) $=$ FING(y) then return i;
> return "*no match*";

Karp-Rabin Algorithm



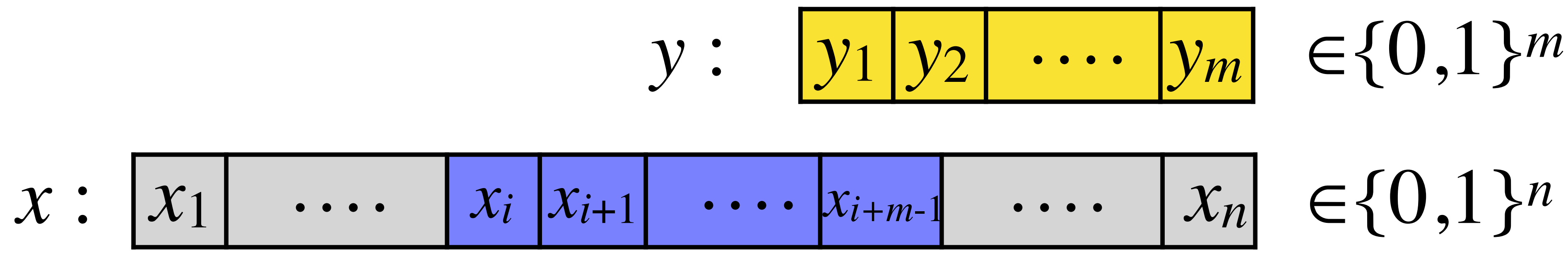
Karp-Rabin Algorithm: FING$(a) = a \bmod p$

pick a uniform random prime $p \in [mn^3]$;

for $i = 1, 2, \ldots, n - m + 1$ **do**:

  if $x[i, i+m-1] \equiv y \pmod{p}$ **then return** i;

return "*no match*";



Karp-Rabin Algorithm: FING$(a) = a \bmod p$

pick a uniform random prime $p \in [mn^3]$;

for $i = 1,2,\ldots,n-m+1$ **do:**

  if $x[i, i+m-1] \equiv y \pmod{p}$ **then return** i;

return "*no match*";

For each i, if $x[i, i+m-1] \neq y$:

$$\Pr\left[x[i, i+m-1] \equiv y \pmod{p}\right] \leq m\ln(mn^3)/mn^3 \ = o(1/n^2)$$

By **union bound:** when y is not a substring of x

$$\Pr[\text{ the algorithm ever makes a mistake }]$$

$$\leq \Pr\left[\exists i,\ x[i, i+m-1] \equiv y \pmod{p}\right] \ = o(1/n)$$

$y: \; \boxed{y_1 \,|\, y_2 \,|\, \cdots\cdots \,|\, y_m} \in \{0,1\}^m$

$x: \; \boxed{x_1 \,|\, \cdots\cdots \,|\, \underbrace{x_i \,|\, x_{i+1} \,|\, \cdots\cdots \,|\, x_{i+m-1}} \,|\, \cdots\cdots \,|\, x_n} \in \{0,1\}^n$

$$x[i, i+m-1] \triangleq x_i x_{i+1} \cdots x_{i+m-1}$$

Karp-Rabin Algorithm: $\mathsf{FING}(a) = a \bmod p$

pick a uniform random prime $p \in [mn^3]$;

for $i = 1,2,\ldots,n-m+1$ do:

  if $\boxed{x[i,i+m-1] \equiv y \pmod{p}}$ then return i;

return "*no match*"; Testable in O(1) time

Observe: $x[i+1, i+m] = x_{i+m} + 2\left(x[i, i+m-1] - 2^{m-1}x_i\right)$

$$\mathsf{FING}(x[i+1,i+m]) = \left(x_{i+m} + 2\left(\mathsf{FING}(x[i,i+m-1]) - 2^{m-1}x_i\right)\right) \bmod p$$

Checking Distinctness

Input: n numbers $x_1, x_2, \ldots, x_n \in \{1,2,\ldots,n\}$

Determine whether every number appears exactly once.

$$A = \{x_1, x_2, \ldots, x_n\}$$
$$B = \{1, 2, \ldots, n\}$$

Input: two multisets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$
where $a_1, \ldots, a_n, b_1, \ldots, b_n \in \{1,\ldots,n\}$

Output: $A = B$ (as multisets)?

$A = B \iff \forall x$: # of times x appearing in A
= # of times x appearing in B

Input: two multisets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ where $a_1, \ldots, a_n, b_1, \ldots, b_n \in \{1, \ldots, n\}$

Output: $A = B$ (as multisets)?

- naive algorithm: use $\mathrm{O}(n)$ time and $\mathrm{O}(n)$ space
- fingerprinting: random fingerprint function FING( )
  - check $\mathrm{FING}(A) = \mathrm{FING}(B)$?
  - time cost: time to compute and check fingerprints $\mathrm{O}(n)$
  - space cost: space to store fingerprints $\mathrm{O}(\log p)$

multisets $A=\{a_1, a_2, \ldots, a_n\}$ $\Rightarrow$ $f_A(x) = \prod_{i=1}^{n}(x - a_i)$

$f_A \in \mathbb{Z}_p[x]$ for prime p (to be specified)

$\mathrm{FING}(A) = f_A(r)$ for uniform random $r \in \mathbb{Z}_p$

multisets $A=\{a_1, a_2, \ldots, a_n\}$

$B=\{b_1, b_2, \ldots, b_n\}$

where $a_i, b_i \in \{1, 2, \ldots, n\}$

$$\begin{cases} f_A(x) = \prod_{i=1}^{n}(x-a_i) \\ f_B(x) = \prod_{i=1}^{n}(x-b_i) \end{cases}$$

$f_A, f_B \in \mathbb{Z}_p[x]$ for prime p (to be specified)

$$\left.\begin{array}{l} \mathrm{FING}(A) = f_A(r) \\ \mathrm{FING}(B) = f_B(r) \end{array}\right\} \text{ for uniform random } r \in \mathbb{Z}_p$$

$$\boxed{A \neq B \implies f_A \not\equiv f_B \text{ on reals } \mathbb{R}}$$

(but possibly $f_A \equiv f_B$ on finite field $\mathbb{Z}_p$)

if $A = B$: $\mathrm{FING}(A) = \mathrm{FING}(B)$

if $A \neq B$: $\mathrm{FING}(A) = \mathrm{FING}(B)$

- $f_A \equiv f_B$ on finite field $\mathbb{Z}_p$

in $f_A - f_B$ on $\mathbb{R}$:

$\exists$ coefficient $c \neq 0$

$c \bmod p = 0$

- $f_A \not\equiv f_B$ on $\mathbb{Z}_p$ but $f_A(r) = f_B(r)$

Schwartz-Zippel

with probability $\leq n/p$

multisets $A=\{a_1, a_2, \ldots, a_n\}$

$B=\{b_1, b_2, \ldots, b_n\}$

where $a_i, b_i \in \{1, 2, \ldots, n\}$

$$\Rightarrow \begin{cases} f_A(x) = \prod_{i=1}^{n}(x-a_i) \\ f_B(x) = \prod_{i=1}^{n}(x-b_i) \end{cases}$$

$f_A, f_B \in \mathbb{Z}_p[x]$ for uniform random prime $p \in [L, U]$

(L, U to be specified)

$$\left.\begin{aligned} \mathrm{FING}(A) &= f_A(r) \\ \mathrm{FING}(B) &= f_B(r) \end{aligned}\right\} \text{ for uniform random } r \in \mathbb{Z}_p$$

Prime Number Theorem (PNT):

$$\pi(N) \sim \frac{N}{\ln N} \text{ as } N \to \infty$$

in $f_A - f_B$ on $\mathbb{R}$:

$\exists$ coefficient $c \neq 0$

$c \bmod p = 0$

$$\Pr[\ c \bmod p = 0\ ] \leq \frac{\#\text{ of prime factors of } c}{\#\text{ of primes in } [L, U]}$$

$\boxed{|c| \leq n^n}$ $\Rightarrow$ $\leq \dfrac{n \log_2 n}{\pi(U) - \pi(L)} \sim \dfrac{n \log_2 n}{U/\ln U - L/\ln L}$

- $f_A \not\equiv f_B$ on $\mathbb{Z}_p$ but $f_A(r) = f_B(r)$ $\xrightarrow{\text{Schwartz-Zippel}}$ with probability $\leq n/p$ $\leq n/L$

multisets $A=\{a_1, a_2, \ldots, a_n\}$

$B=\{b_1, b_2, \ldots, b_n\}$

where $a_i, b_i \in \{1, 2, \ldots, n\}$

$$\begin{cases} f_A(x) = \prod_{i=1}^{n}(x-a_i) \\ f_B(x) = \prod_{i=1}^{n}(x-b_i) \end{cases}$$

$f_A, f_B \in \mathbb{Z}_p[x]$ for uniform random prime $p \in [L, U]$

with $U = 2L = (n \log n)^2$

$$\left.\begin{aligned} \mathrm{FING}(A) &= f_A(r) \\ \mathrm{FING}(B) &= f_B(r) \end{aligned}\right\}$$ for uniform random $r \in \mathbb{Z}_p$

if $A \neq B$: $\mathrm{FING}(A) = \mathrm{FING}(B)$

- $f_A \equiv f_B$ on finite field $\mathbb{Z}_p$ with probability $\leq \dfrac{n \log_2 n}{U/\ln U - L/\ln L} = O(1/n)$
- $f_A \not\equiv f_B$ on $\mathbb{Z}_p$ but $f_A(r) = f_B(r)$ $\xrightarrow{\text{Schwartz-Zippel}}$ with probability $\leq n/p \leq n/L = O(1/n)$

Input: two multisets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ where $a_1, \ldots, a_n, b_1, \ldots, b_n \in \{1, \ldots, n\}$

Output: $A = B$ (as multisets)?

Lipton's Algorithm (1989):

$$\left.\begin{aligned} \mathrm{FING}(A) &= \prod_{i=1}^{n}(r - a_i) \bmod p \\ \mathrm{FING}(B) &= \prod_{i=1}^{n}(r - b_i) \bmod p \end{aligned}\right\}$$

for uniform random prime $p \in [(n \log n)^2/2, (n \log n)^2]$ and uniform random $r \in \mathbb{Z}_p$

if $A \neq B$ as multisets:

$$f_A(x) = \prod_{i=1}^{n}(x - a_i) \bmod p \qquad f_B(x) = \prod_{i=1}^{n}(x - b_i) \bmod p$$

$\Pr[\ \mathrm{FING}(A) = \mathrm{FING}(B)\ ]$

$\leq \Pr[\ f_A \equiv f_B\ ] + \Pr[\ f_A(r) = f_B(r) \mid f_A \not\equiv f_B\ ] = O(1/n)$

Input: n numbers $x_1, x_2, \ldots, x_n \in \{1,2,\ldots,n\}$

Determine whether every number appears exactly once.

Lipton's Algorithm (1989):

$$\left.\begin{array}{l} \mathrm{FING}(A) = \prod_{i=1}^{n}(r - a_i) \bmod p \\ \text{check if:} \\ \mathrm{FING}(A) = \prod_{i=1}^{n}(r - i) \bmod p? \end{array}\right\}$$

for uniform random prime $p \in [(n \log n)^2/2, (n \log n)^2]$ and uniform random $r \in \mathbb{Z}_p$

- time cost: $\mathrm{O}(n)$
- space cost: $\mathrm{O}(\log n)$
- error probability (false positive): $\mathrm{O}(1/n)$
- data stream: input comes one at a time