Foundations of Data Science

Limit Theorems

尹一通、刘明谋 Nanjing University, 2024 Fall

Limit Theorems

Let $X_1, X_2, \ldots$ be *i.i.d.* random variables with $\mu = \mathbb{E}[X_1]$ and $\mathbf{Var}[X_1] = \sigma^2$.

And let $\overline{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ be the <u>sample mean</u>.

- <u>Law of large numbers</u> (**LLN**): sample mean → expectation

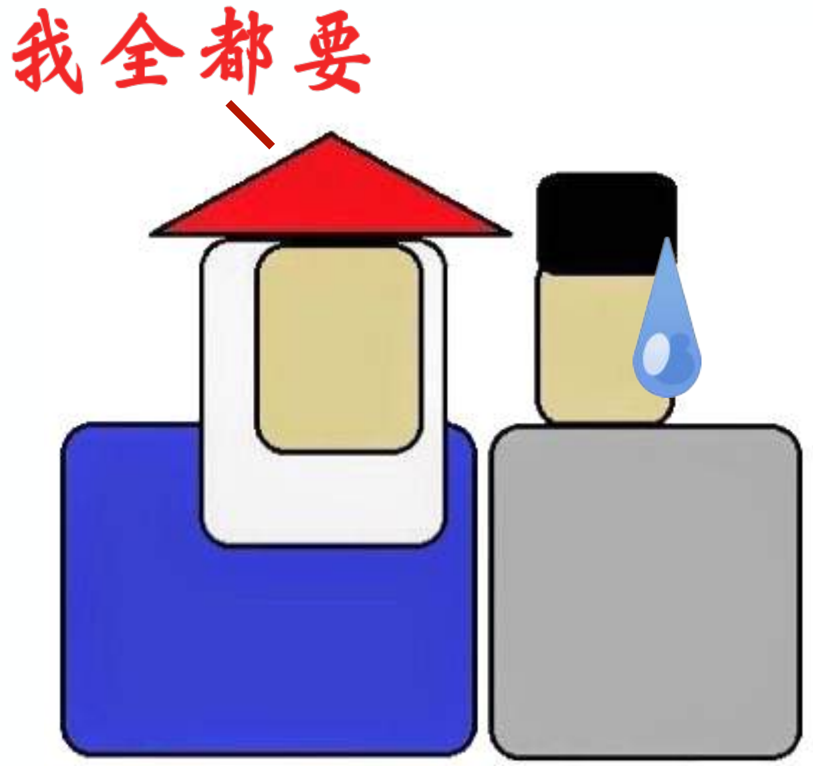
$$\overline{X}_n \longrightarrow \mu \quad \text{as } n \to \infty$$

- <u>Central limit theorem</u> (**CLT**): standardized sample mean → standard normal

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \longrightarrow N(0,1) \quad \text{as } n \to \infty$$

Convergence

- A real sequence $\{a_n\}$ converges to $a \in \mathbb{R}$, denoted $\lim_{n\to\infty} a_n = a$ or $a_n \to a$, if for all $\epsilon > 0$, there is N such that $|a_n - a| < \epsilon$ for all $n > N$
- A sequence $f_1, f_2, \ldots : \Omega \to \mathbb{R}$ is said to <u>**converge pointwise**</u> to $f : \Omega \to \mathbb{R}$, if and only if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \Omega$
- For random variables $X_1, X_2, \ldots$ and X on probability space $(\Omega, \Sigma, \Pr)$:
  - random variables $X_1, X_2, \ldots : \Omega \to \mathbb{R}$ and $X : \Omega \to \mathbb{R}$ are functions
  - CDFs $F_{X_1}, F_{X_2}, \ldots : \mathbb{R} \to [0,1]$ and $F_X : \mathbb{R} \to [0,1]$ are functions
- Should $X_n \to X$ be: $X_n \to X$ pointwise or $F_{X_n} \to F_X$ pointwise?



Convergence of Random Variables

$0.$  $\cdots \to U_{[0,1]}$

Modes of Convergence

- Let $X, X_1, X_2, \ldots : \Omega \to \mathbb{R}$ be random variables on prob. space $(\Omega, \Sigma, \Pr)$.

- $\{X_n\}$ converges in distribution (依分布收敛) to X, denoted $X_n \xrightarrow{D} X$, if

$$F_{X_n}(x) = \Pr(X_n \le x) \to F_X(x) = \Pr(X \le x) \quad \text{as} \quad n \to \infty$$

  for all $x \in \mathbb{R}$ at which $F_X(x)$ is continuous

- $\{X_n\}$ converges in probability (依概率收敛) to X, denoted $X_n \xrightarrow{P} X$, if

$$\Pr(|X_n - X| > \epsilon) = 0 \quad \text{as} \quad n \to \infty \quad \text{for all } \epsilon > 0$$

- $\{X_n\}$ converges almost surely to X, denoted $X_n \xrightarrow{a.s.} X$, if $\exists A \in \Sigma$ such that

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \text{for all } \omega \in A, \quad \text{and} \quad \Pr(A) = 1$$

Modes of Convergence

- Let $X_1, X_2, \dots$ and X be random variables on probability space $(\Omega, \Sigma, \Pr)$.

- $X_n \xrightarrow{D} X$ (convergence in distribution / in law / weak convergence of measure) if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

$F_{X_n} \to F_X$ pointwise on continuous set

for all $x \in \mathbb{R}$ at which $F_X(x)$ is continuous

- $X_n \xrightarrow{P} X$ (convergence in probability / in measure) if

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0 \quad \text{for all } \epsilon > 0$$

$X_n \to X$ in measure

- $X_n \xrightarrow{a.s.} X$ (convergence almost surely / almost everywhere / w.p. 1) if

$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1$$

$X_n \to X$ pointwise on a set of measure 1

Convergence in Distribution

- Let $X_1, X_2, \ldots$ and X be random variables on probability space $(\Omega, \Sigma, \Pr)$.
- $X_n \xrightarrow{D} X$ (convergence in distribution / in law / weak convergence of measure) if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

$F_{X_n} \to F_X$ pointwise on continuous set

  for all $x \in \mathbb{R}$ at which $F_X(x)$ is continuous

- The restriction on continuity set is necessary, consider:

  uniform X_n on $(0,1/n)$, which satisfies $X_n \xrightarrow{D} X$, where $\Pr(X = 0) = 1$

- $X_n \xrightarrow{D} X$ and $F_X = F_Y \implies X_n \xrightarrow{D} Y$ (convergence in distribution depends only on distribution)
- $X_n \xrightarrow{D} X$ is a weak convergence of measures

Convergence in Probability

- Let $X_1, X_2, \ldots$ and X be random variables on probability space $(\Omega, \Sigma, \Pr)$.
- $X_n \xrightarrow{P} X$ (convergence <u>in probability</u>) if

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0 \quad \text{for all } \epsilon > 0$$

$X_n \to X$ in measure

- Functions $X_n : \Omega \to \mathbb{R}$ converges to $X : \Omega \to \mathbb{R}$ <u>in measure</u> $\Pr$
- $X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X$
  - **Counterexample for converse**: X is uniform on $[0,1]$ and $X_n = 1 - X$
- If $X_n \xrightarrow{D} c$, where $c \in \mathbb{R}$ is a constant, then $X_n \xrightarrow{P} c$
  - **Proof**: $\Pr(|X_n - c| > \epsilon) = \Pr(X_n < c - \epsilon) + \Pr(X_n > \epsilon + c) \to 0$ if $X_n \xrightarrow{D} c$

Almost Sure Convergence

- Let $X_1, X_2, \ldots$ and X be random variables on probability space $(\Omega, \Sigma, \Pr)$.
- $X_n \xrightarrow{a.s.} X$ (convergence <u>almost surely</u> / <u>almost everywhere</u> / **w.p. 1**) if

$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1$$

$X_n \to X$ pointwise on a set of measure 1

- $X_n : \Omega \to \mathbb{R}$ converges to $X : \Omega \to \mathbb{R}$ **almost everywhere** except a null set
- $X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X$
  - **Counterexample for converse**: $\{X_n\}$ are independent $\mathrm{Bernoulli}(1/n)$. We have $X_n \xrightarrow{P} 0$, but we do not have $X_n = 0$ almost everywhere as $n \to \infty$.

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Strength of Convergence

- $(X_n \xrightarrow{a.s.} X) \implies (X_n \xrightarrow{P} X) \implies (X_n \xrightarrow{D} X)$

Proof* $(X_n \xrightarrow{a.s.} c \implies X_n \xrightarrow{P} c)$: Let event $C \triangleq \{X_n \to c\}$, then $\Pr(C) = 1$.

For any $\epsilon > 0$, let event $A_k \triangleq \{\forall n \geq k, |X_n - c| < \epsilon\}$.

Assume $X_n \xrightarrow{a.s.} X$, then $\exists k$, such that $\forall n \geq k, C \subseteq A_n$. Therefore $C \subseteq \bigcup_{k=1}^{\infty} A_k$.

Since $A_1 \subseteq A_2 \subseteq \ldots$, and $A_k \subseteq \{|X_n - c| < \epsilon\}$,

$$\lim_{n\to\infty} \Pr(|X_n - c| < \epsilon) \geq \lim_{n\to\infty} \Pr(A_k) = \Pr(\cup_{k=1}^{\infty} A_k) \geq \Pr(C) = 1.$$

Therefore,

$$\lim_{n\to\infty} \Pr(|X_n - c| \geq \epsilon) = 0, \text{ i.e. } X_n \xrightarrow{P} X.$$

Other Convergence Modes*

- $X_n \xrightarrow{1} X$ (convergence in mean) if

$$\lim_{n \to \infty} \mathbb{E}\left[|X_n - X|\right] = 0$$

- $X_n \xrightarrow{r} X$ (convergence in rth mean / in the L^r-norm) if

$$\lim_{n \to \infty} \mathbb{E}\left[|X_n - X|^r\right] = 0$$

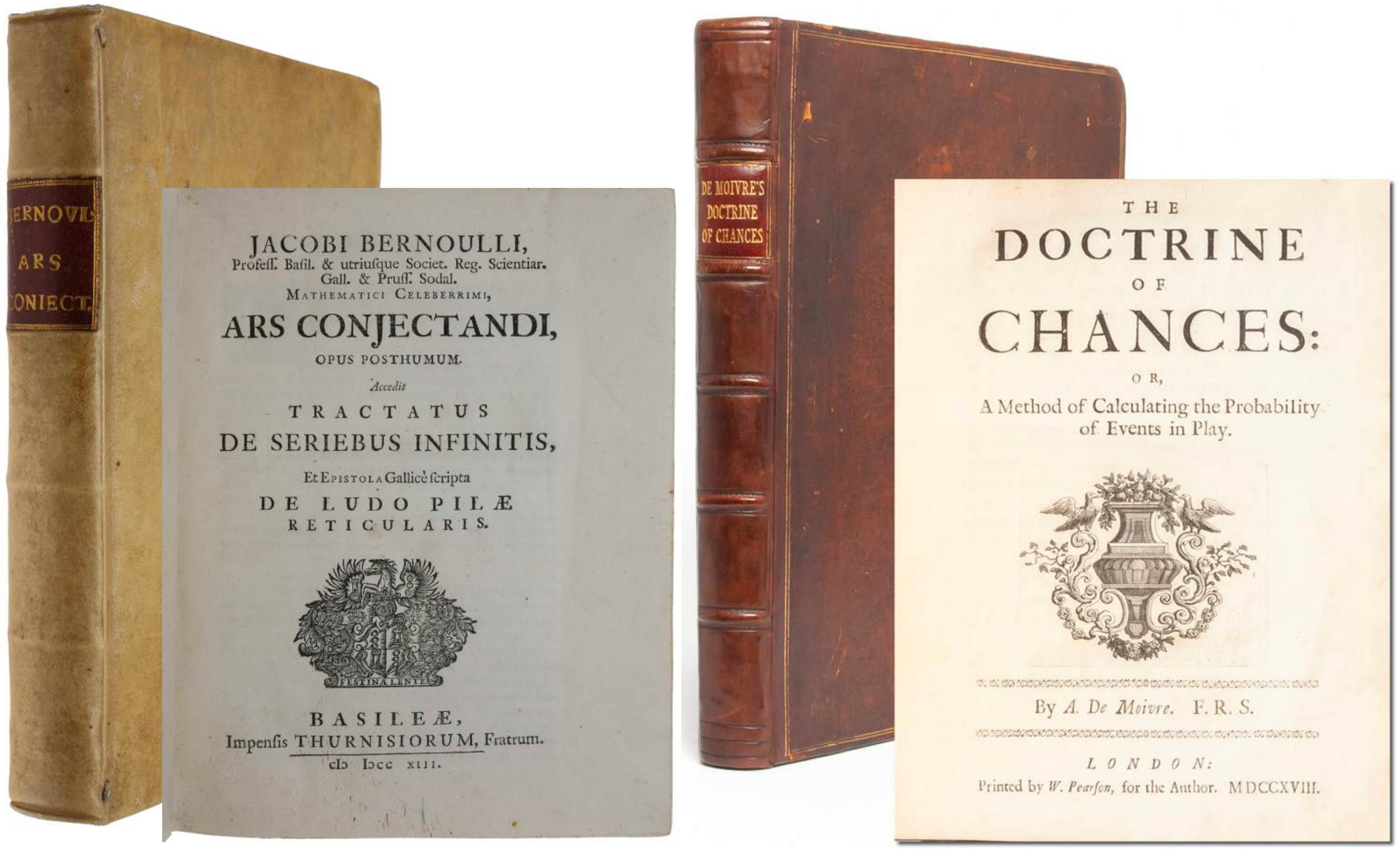
$$(X_n \xrightarrow{a.s.} X) \Longrightarrow (X_n \xrightarrow{P} X) \Longrightarrow (X_n \xrightarrow{D} X)$$

$$\Uparrow$$

$$(X_n \xrightarrow{s} X) \Longrightarrow (X_n \xrightarrow{r} X) \Longrightarrow (X_n \xrightarrow{1} X)$$

(for $s \geq r \geq 1$)

LLN and CLT


JACOBI BERNOULLI,
Profess. Basil. & utriusque Societ. Reg. Scientiar. Gall. & Pruss. Sodal.
Mathematici Celeberrimi,
ARS CONJECTANDI,
OPUS POSTHUMUM.
Accedit
TRACTATUS
DE SERIEBUS INFINITIS,
Et Epistola Gallicè scripta
DE LUDO PILÆ
RETICULARIS.
BASILEÆ,
Impensis THURNISIORUM, Fratrum.
ARS
DE MOIVRE'S
DOCTRINE
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DOCTRINE
OF
CHANCES:
OR,
A Method of Calculating the Probability of Events in Play.
By A. De Moivre. F. R. S.
LONDON:
Printed by W. Pearson, for the Author. MDCCXVIII.

Bernoulli’s Law of Large Number

In *Ars Conjectandi* (1713)

- Let $X_1, X_2, \ldots$ be *i.i.d.* Bernoulli trials with $\mathbb{E}[X_1] = p \in [0,1]$. Then

$$\Pr\left(\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - p\right| > \epsilon\right) \to 0 \quad \text{as } n \to \infty \quad \text{for all } \epsilon > 0$$

i.e. $\overline{X}_n \xrightarrow{P} p$, where $\overline{X}_n$ is the sample mean $\overline{X}_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$

Proof: By Chebyshev’s inequality, $\Pr(|\overline{X}_n - p| > \epsilon) \leq \dfrac{p(1-p)}{n\epsilon^2} \to 0$ as $n \to \infty$

(This is of course not the original proof of Bernoulli.)

Law of Large Numbers (LLN)

Let $X_1, X_2, \ldots$ be *i.i.d.* random variables with finite mean $\mathbb{E}[X_1] = \mu$.

And let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the **sample mean.**

- **<u>Weak law</u>** (Khinchin's law) **<u>of large number</u>**:

$$\overline{X}_n \xrightarrow{P} \mu \quad \text{as} \quad n \to \infty$$

- **<u>Strong law</u>** (Kolmogorov's law) **<u>of large number</u>**:

$$\overline{X}_n \xrightarrow{a.s.} \mu \quad \text{as} \quad n \to \infty$$

(*The deviation* $|\overline{X}_n - \mu|$ *is always small for all sufficiently large* n)

Weak LLN Assuming Bounded Variance

- Let $X_1, X_2, \ldots$ be independent random variables with finite mean $\mathbb{E}[X_i] = \mu$ and finitely bounded variance $\mathbf{Var}[X_i] \leq \sigma^2$.

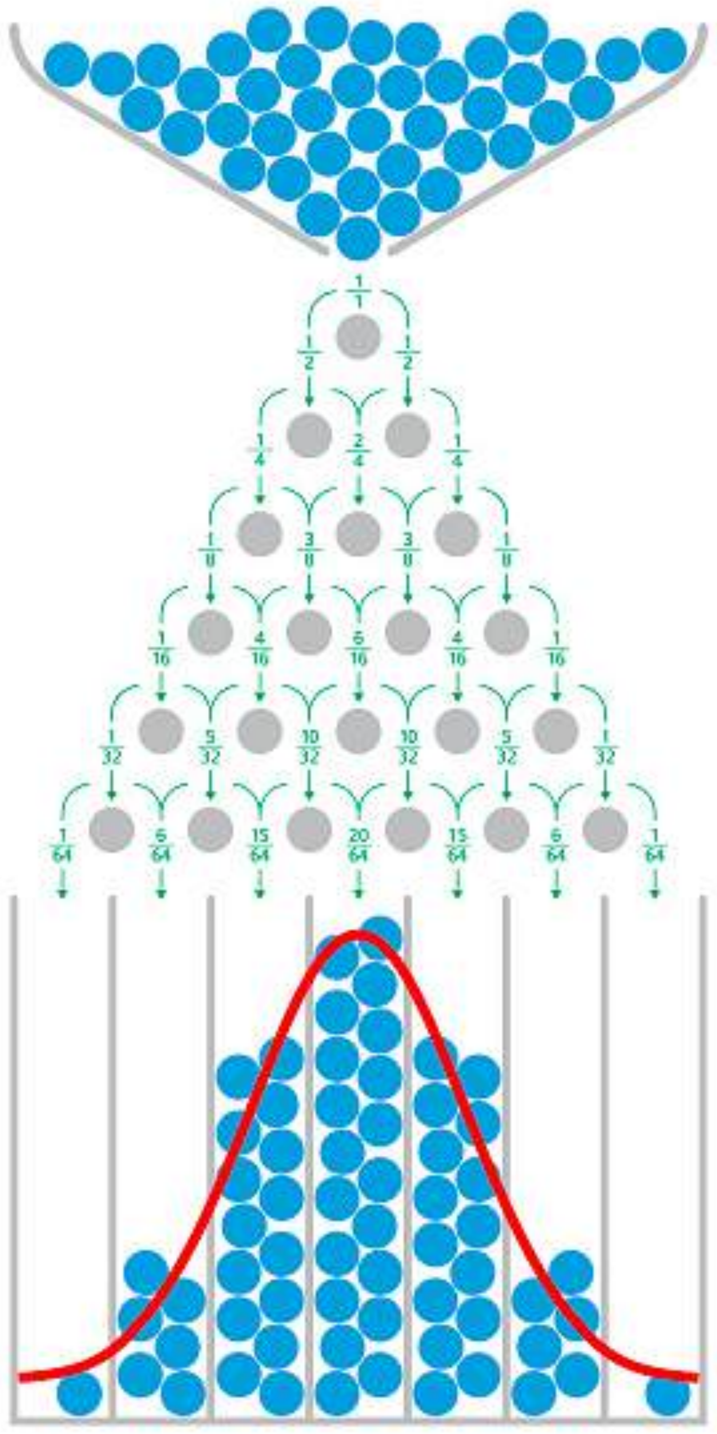
  Then the sample mean $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ has

$$\overline{X}_n \xrightarrow{P} \mu \text{ as } n \to \infty$$

Proof: By **Chebysev's inequality**, $\Pr(|\overline{X}_n - \mu| > \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \to 0$ as $n \to \infty$

De Moivre—Laplace Theorem

(棣莫弗－拉普拉斯定理)



- Let $p \in (0,1)$ and $X_n \sim B(n,p)$. Then its standardization

$$\frac{X_n - np}{\sqrt{np(1-p)}} \xrightarrow{D} N(0,1) \quad \text{as } \; n \to \infty$$

- For any $p \in (0,1)$ and any $\epsilon > 0$, there is an n_0 such that for all $n > n_0$ and all k,

$$\binom{n}{k} p^k (1-p)^{n-k} \in (1 \pm \epsilon) \frac{1}{\sqrt{2\pi np(1-p)}} \mathrm{e}^{-\frac{(k-np)^2}{2np(1-p)}}$$

By Stirling's formula $n! \simeq n^n e^{-n}\sqrt{2\pi n}$ and Maclaurin series $\ln(1+x) \simeq x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

Central Limit Theorem (CLT)

- Let $X_1, X_2, \ldots$ be *i.i.d.* random variables with $\mathbb{E}[X_1] = \mu$ and $\mathbf{Var}[X_1] = \sigma^2$.

  And let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the **sample mean.**

- <u>**Classical**</u> (Lindeberg–Lévy) <u>**central limit theorem**</u>:

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1) \quad \text{as } n \to \infty$$

CLT for Non-Identically Distributed RVs*

- Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}[X_i] = \mu_i$ and $S_n^2 = \sum_{i=1}^{n} \mathbf{Var}[X_i]$.

  Assume:

  - $\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} \mathrm{E}\left[\left|X_i - \mu_i\right|^{2+\delta}\right] = 0$ for some $\delta > 0$ (Lyapunov's condition)

  -or, $\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^{n} \mathrm{E}\left[(X_i - \mu_i)^2 \cdot \mathbf{1}_{\left\{\left|X_i - \mu_i\right| > \varepsilon s_n\right\}}\right] = 0$ for every $\epsilon > 0$ (Lindeberg's condition)

  Then

$$\frac{\sum_{i=1}^{n} (X_i - \mu_i)}{S_n} \xrightarrow{D} N(0,1)$$

Convergence Rate of CLT

(Berry–Esseen theorem)

- Berry–Esseen theorem: Let $X_1, X_2, \dots$ be *i.i.d.* random variables with $\mathbb{E}[X_1] = \mu$, $\mathbf{Var}[X_1] = \sigma^2$, and $\rho = \mathbb{E}[|X_1 - \mu|^3]$. And let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

  There is an absolute constant C, such that for any z

$$\left| \Pr\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \le z \right) - \Phi(z) \right| \le \frac{C\rho}{\sigma^3\sqrt{n}}$$

where Φ stands for the CDF for standard normal distribution $N(0,1)$