# Advanced Algorithms

Spectral methods and algorithms

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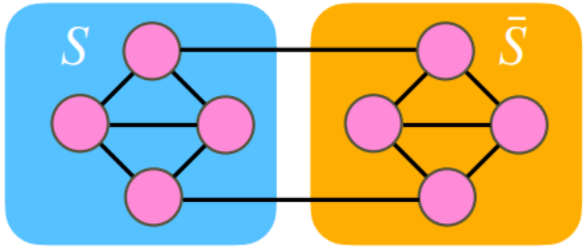
# Recap




$$\varphi(S) := \frac{|E(S,\bar{S})|}{d|S|}$$

We saw a spectral partitioning algorithm on day 1:
To find a sparse cut with small **conductance** in a $d$-regular graph, we

1. Compute the second largest eigenvector $\boldsymbol{x} \in \mathbb{R}^n$ of the adjacency matrix.
2. Sort the vertices so that $x_1 \geq x_2 \geq \cdots \geq x_n$.
3. Let $S_i := \begin{cases} \{1, \ldots, i\} & \text{if } i \leq \frac{n}{2} \\ \{i+1, \ldots, n\} & \text{otherwise} \end{cases}$, and output $S_i = \underset{1 \leq i \leq n}{\operatorname{argmin}} \varphi(S_i)$.

**Theorem**: $\exists i, \varphi(S_i) \leq 2\sqrt{\varphi(G)}$

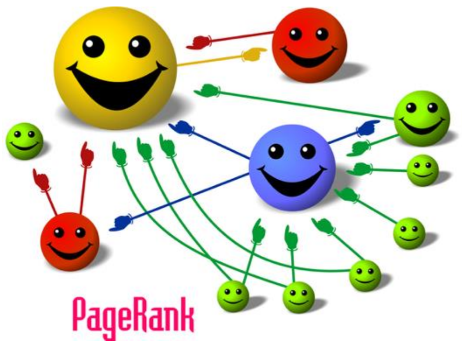
**Why eigenvectors?**

# Overview

**Analysis of the spectral partitioning algorithm**

- Introduction to spectral graph theory
  - Connectedness
  - Bipartiteness (2-coloring)
- Cheeger's inequality on d-regular graphs
  - Easy direction: a sparse cut implies $\lambda_2$ is small
  - Hard direction: a small $\lambda_2$ means we can find a sparse cut from $v_2$
  - Improvements of Cheeger's
  - Generalizations of Cheeger's

# Spectral graph theory




**Spectral theory**

eigenvalues + eigenvectors + related linear algebra

**Graph structures**

- Connectedness
- Coloring / Clustering
- Mixing of random walks
- Expander graphs

In Theoretical CS

- Pagerank
- Sparsification
- Solving linear systems
- Counting / Sampling
- Expander codes
- Hardness of approximation
- Derandomization
- Max flow and more

And Beyond

- Image segmentation
- Electrical networks
- Reliable / Efficient networks
- Epidemic modelling
- Economic networks

# Graphs as matrices

Eigenvalues and eigenvectors

$$Av = \lambda v$$

- $\lambda$ : **eigenvalue**
- $v$ : **eigenvector**
- **characteristic polynomial** of $A$: $\det(A - xI)$
- $\det(A - xI) = 0$ gives all the eigenvalues
- **multiplicity** of $\lambda$:
  - Geometric: **dimension** of the eigenspace corresponding to $\lambda$
  - Algebraic: how many times $\lambda$ appears as a **root**
  - For ***diagonalizable matrices***, they are the same

Undirected graph $G = (V, E)$ has **adjacency matrix**

$$A_{u,v} = 1 \text{ iff } uv \in E$$

$A$ is an $n \times n$ real symmetric matrix:

- It has **real eigenvalues** $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$
- there is an **orthonormal basis** of **eigenvectors** $v_1, v_2, \dots, v_n$ such that

$$Av_i = \alpha_i v_i, \forall i$$

$$v_i^{\mathsf{T}} v_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases}$$

**Adjacency matrix is NOT only a data structure**
**Its algebraic properties as a matrix are useful too:**
**rank, determinant, eigenspaces, …**

# Complexity of Linear algebra

All the following can be solved in $\tilde{O}(n^{\omega})$ arithmetic operations:

- Matrix multiplication
- Matrix inverse
- Determinant
- Characteristic polynomial
- Solving linear equations $Ax = b$
- Singular value decomposition
- Eigen-decomposition of symmetric matrices

In fact, almost linear time (in theory) for matrices that we will care about..

# Spectrum of the adjacency matrix

Let $G = (V, E)$ be an undirected graph, $\alpha_1$ be the largest eigenvalue of the adjacency matrix $A(G)$

**Claim**: $d_{\text{avg}} \leq \alpha_1 \leq d_{\max}$

Proof of the upperbound:

Let $v$ be the eigenvector corresponding to $\alpha_1$, so that $Av = \alpha_1 v$

Without loss of generality we can assume that $\max_i v_i > 0$

Choose an index $j$ so that $v_j = \max_i v_i$

Then $Av = \alpha_1 v$ in the $j$-th row means that

$$\alpha_1 v_j = \sum_i A_{ji}\, v_i \leq d_{\max} \cdot v_j$$

Here the inequality follows from $v_j = \max_i v_i$, and there are at most $d_{\max}$ neighbors of $j$

Since $v_j > 0$, $\alpha_1 v_j \leq d_{\max} \cdot v_j \Rightarrow \alpha_1 \leq d_{\max}$

**Remark**. This argument can be adapted to prove: for a connected $G$, $\alpha_1 = d_{\max}$ iff $G$ is regular

# Spectrum of Bipartite graphs

Spectrum also tells us something about graph coloring

We start with 2-colorability (Bipartiteness)

Let $G = (V, E)$ be an undirected graph, and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ be its eigenvalues

**Claim**: The spectrum of $A(G)$ is symmetric about 0 (i.e., $\alpha_i = -\alpha_{n-i+1}$) iff $G$ is bipartite

# Spectrum of Bipartite graphs

$$A(G) \quad = \quad \begin{array}{cc} & \begin{array}{cc} U & V \end{array} \\ \begin{array}{c} U \\ V \end{array} & \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix} \end{array}$$

**Lemma**: Let $G$ be bipartite, and $\alpha$ be an eigenvalue of $A(G)$ with multiplicity $k$, then $-\alpha$ is also an eigenvalue of $A(G)$ with multiplicity $k$

Proof: If $\alpha = 0$, the lemma is vacuously true. So we assume $\alpha \neq 0$.

Let $\binom{x}{y}$ be an eigenvector of $A$ corresponding to $\alpha$: $\binom{By}{B^\top x} = \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix} \binom{x}{y} = \alpha \binom{x}{y}$

So $B^\top x = \alpha y, By = \alpha x$. On the other hand, $A \binom{x}{-y} = \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix} \binom{x}{-y} = \binom{-By}{B^\top x} = \binom{-\alpha x}{\alpha y} = -\alpha \binom{x}{-y}$

This means $-\alpha$ is also an eigenvalue of $A$

Finally, notice that the multiplicity of $\alpha$ being $k \Leftrightarrow$ there exists $k$ linearly independent eigenvectors corresponding to $\alpha$

Apply the above argument to every one of those, we get that the multiplicity of $-\alpha$ is also $k$

# Spectrum of Bipartite graphs

$$A(G) = \begin{array}{c c} & \begin{array}{cc} U & V \end{array} \\ \begin{array}{c} U \\ V \end{array} & \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix} \end{array}$$

**Lemma**: If the spectrum of $A(G)$ is symmetric about 0 (i.e., $\alpha_i = -\alpha_{n-i+1}$), then $G$ is bipartite

Proof: Note that for every odd integer $k$, $\sum_i \alpha_i^k = 0$

Since the eigenvalues of $A^k$ is $\alpha_1^k, \alpha_2^k, \dots, \alpha_n^k$, thus for every odd integer $k$,

$$\mathrm{trace}(A^k) = \sum_i \alpha_i^k = 0$$

On the other hand, $\mathrm{trace}(A^k)$ has a combinatorial meaning:

$$(A^k)_{i,j} = \text{the number of } k\text{−walks going from } i \text{ to } j$$

Since $\mathrm{trace}(A^k) = \sum_i (A^k)_{i,i} = 0$, and $(A^k)_{i,i} \geq 0$, so we must have $(A^k)_{i,i} = 0$

This means: for every odd integer $k$, there is no cycle of length $k$. Thus, all cycles are of even length.

# Side note: Graph Coloring

For $k$-coloring, we do not expect a spectral characterization (why?)

For an approximation, the chromatic number $\chi(G)$ satisfies

$$\left\lceil \frac{\alpha_1}{-\alpha_n} \right\rceil + 1 \leq \chi(G) \leq \lfloor \alpha_1 \rfloor + 1$$

The upperbound is known as Wilf's Theorem, and the lowerbound as Hoffman's bound

See Dan Spielman's [Spectral Graph Theory book](#) for a proof

# Many matrices associated with a graph

- **Adjacency matrix** $A(G)$

$$A_{u,v} = 1 \text{ iff } uv \in E$$

- **Laplacian matrix**: let $D(G)$ be the diagonal degree matrix

$$L(G) \coloneqq D(G) - A(G)$$

Later in class:

- **Normalized Laplacian matrix**: $\mathcal{L}(G) \coloneqq D^{-\frac{1}{2}} L(G)\, D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$
- Random walk matrix
  - Consider $\overrightarrow{p_{t+1}} = \overrightarrow{p_t}(D^{-1}A)$
  - The **transition matrix** $P \coloneqq D^{-1}A$

# Laplacian matrix $L(G) \coloneqq D(G) - A(G)$

For regular graphs, $L(G) = dI - A(G)$, eigenspace is roughly the same as $A(G)$

This is not true for irregular graphs, and the difference is important

$$L_{ij} = \begin{cases} \deg(i), & \text{if } i = j \\ -1, & \text{if } ij \in E \\ 0, & \text{otherwise} \end{cases}$$

Consider the Laplacian on a single edge $e = (u, v)$, $L_e = b_e b_e^\top$

$$L_e = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{cc} & \begin{array}{ccccc} \phantom{\cdots} & u & \phantom{\cdots} & v & \phantom{\cdots} \end{array} \\ & \left( \begin{array}{ccccc} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & -1 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \end{array} \right) \begin{array}{c} \\ u \\ \\ v \\ \\ \end{array} \end{array}$$

# Decomposition of Laplacian

$$L_e = \begin{array}{cc} \begin{array}{ccccc} & u & & v & \end{array} & \\ \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & -1 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} & \begin{array}{c} \\ u \\ \\ v \\ \\ \end{array} \end{array}$$

$$L(G) \coloneqq D(G) - A(G) = \sum_{e \in E(G)} L_e = \sum_{e \in E(G)} b_e b_e^\top$$

**Theorem**: $\vec{1}$ is an eigenvector of $L$ with eigenvalue 0

Proof: Notice that each row of $L$ sum up to 0, so $L\vec{1} = 0$

**Theorem**: The smallest eigenvalue of $L$ is 0

Proof: Note that for every $x$,

$$x^\top L\, x = \sum_e x^\top b_e b_e^\top\, x = \sum_e (x_u - x_v)^2 \geq 0$$

Thus $L$ is a positive semi-definite (PSD) matrix, with all eigenvalues non-negative.
We also saw that 0 is an eigenvalue, this concludes the proof.

PSD often simply written as $L \succcurlyeq 0$

# $\lambda_2$ of the Laplacian

$$L(G) \coloneqq D(G) - A(G) = \sum_{e \in E(G)} b_e b_e^{\top}$$

**Theorem**: The second smallest eigenvalue of $L(G)$ is 0 iff $G$ is disconnected

Proof: Suppose that is $G$ disconnected, with components $G = G_1 \uplus G_2$

$$L(G) = \begin{array}{c c} & \begin{array}{cc} V_1 & V_2 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \end{array} & \begin{bmatrix} L(G_1) & 0 \\ 0 & L(G_2) \end{bmatrix} \end{array}$$

$\vec{1}_{G_1}, \vec{1}_{G_2}$ are eigenvectors with eigenvalue 0, and are linearly independent

Conversely, if $G$ is connected, and let $x \neq 0$ be any vector such that $L\, x = 0$

$$x^{\top} L\, x = \sum_{e} (x_u - x_v)^2 = 0 \;\Rightarrow\; \forall uv \in E, x_u = x_v$$

Since $G$ is connected, $\forall uv \in E, x_u = x_v \Rightarrow \;\forall u \in V, v \in V, x_u = x_v \Rightarrow\; x = c\vec{1}$

This argument can be adapted to prove:
$\lambda_k(L) = 0$ iff $G$ has k connected components

# Spectrum of the Laplacian

$$L(G) := D(G) - A(G) = \sum_{e \in E(G)} b_e b_e^\top$$

Denote eigenvalues of the Laplacian by $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$

**Corollary**: $\lambda_2(L) > 0$ iff $G$ is connected

Robust generalizations:

$\lambda_2(L)$ is small $\Leftrightarrow$ $G$ is "almost disconnected"

$\lambda_k(L)$ is small $\Leftrightarrow$ $G$ is "close to having k disconnected components"

$\alpha_1 \approx -\alpha_n$ $\Leftrightarrow$ $G$ has an "almost bipartite component"

Intuition behind spectral algorithms for finding
- sparse cuts
- $k$-way cuts
- Maximum cuts

# Recap: Graph conductance

We first define what it means to be "almost disconnected"

The **conductance** of a set $S \subseteq V$ is defined as $\varphi(S) \coloneqq \frac{|E(S,\bar{S})|}{\mathrm{vol}(S)}$, where $\mathrm{vol}(S) \coloneqq \sum_{v \in S} \deg(v)$

When the graph is $d$-regular, $\varphi(S) \coloneqq \frac{|E(S,\bar{S})|}{d|S|}$

Note: the **expansion** of a set $S$ is defined as $\frac{|E(S,\bar{S})|}{|S|}$

For $d$-regular graphs, they're basically the same.

The conductance of a graph $G$ is defined as $\varphi(G) \coloneqq \min_{S:\mathrm{vol}(S) \leq m} \varphi(S)$

Note that $0 \leq \varphi(G) \leq 1$

# Recap: Expander graphs and sparse cuts

A graph $G$ with constant $\varphi(G)$ (e.g. $\varphi(G) = 0.1$) is called an **expander graph**

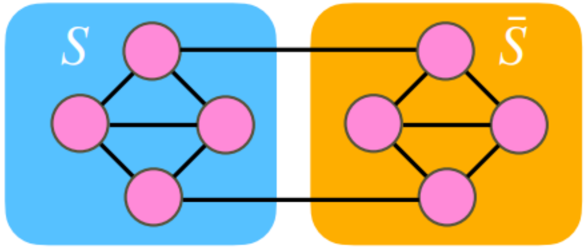
A set $S$ with small $\varphi(S)$ is called a **sparse cut**

Both concepts are very useful

Finding a sparse cut is useful in designing divide-and-conquer algorithms, and have applications in

- image segmentation
- data clustering
- community detection
- VLSI-design
- ….

# Recap: Spectral partitioning



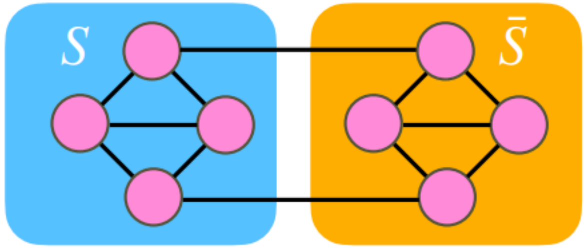

$$\varphi(S) := \frac{|E(S,\bar{S})|}{d|S|}$$

To find a sparse cut with small **conductance** in a **general graph**, we

1. Compute the second smallest eigenvector $\boldsymbol{x} \in \mathbb{R}^n$ of the **normalized Laplacian** $\mathcal{L}(G) := D^{-\frac{1}{2}} L(G)\, D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$
2. Sort the vertices so that $x_1 \geq x_2 \geq \cdots \geq x_n$.
3. Let $S_i := \begin{cases} \{1, \dots, i\} & \text{if } i \leq \frac{n}{2} \\ \{i+1, \dots, n\} & \text{otherwise} \end{cases}$, and output $S_i = \operatorname{argmin}_{1 \leq i \leq n} \varphi(S_i)$.

Intuition: $\varphi(G) \approx \lambda_2(\mathcal{L})$

# Recap: Spectral partitioning




$\varphi(S) := \frac{|E(S,\bar{S})|}{d|S|}$

To find a sparse cut with small **conductance** in a $d$-**regular graph**, we

1. Compute the second smallest eigenvector $\boldsymbol{x} \in \mathbb{R}^n$ of the **normalized Laplacian** $\frac{L}{d}$
2. Sort the vertices so that $x_1 \geq x_2 \geq \cdots \geq x_n$.
3. Let $S_i := \begin{cases} \{1, \dots, i\} & \text{if } i \leq \frac{n}{2} \\ \{i+1, \dots, n\} & \text{otherwise} \end{cases}$, and output $S_i = \underset{1 \leq i \leq n}{\operatorname{argmin}} \, \varphi(S_i)$.

Intuition: $\varphi(G) \approx \lambda_2(\mathcal{L}) = \lambda_2(L/d)$

# Courant-Fischer Theorem

**Theorem**: For a real symmetric matrix $A$, the maximum eigenvalue

$$\lambda_n(A) = \max_{x \neq 0} \frac{x^\top A\, x}{x^\top x} \longrightarrow$$

**Rayleigh quotient**

$$R_A(x) = \frac{x^\top A\, x}{x^\top x}$$

**Proof:** Since equality can be attained. It suffices to show $\frac{x^\top A\, x}{x^\top x} \leq \lambda_n(A)$

Let $v_1, v_2, .., v_n$ be an orthonormal basis of $A$

$$\begin{aligned} x^\top A x &= (a_1 v_1 + \cdots + a_n v_n)^\top A(a_1 v_1 + \cdots + a_n v_n) \\ &= \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 \leq \lambda_n(a_1^2 + \cdots + a_n^2) \end{aligned}$$

$$x^\top x = (a_1 v_1 + \cdots + a_n v_n)^\top (a_1 v_1 + \cdots + a_n v_n) = a_1^2 + \cdots + a_n^2$$

Thus, we have $\frac{x^\top A\, x}{x^\top x} \leq \lambda_n$

# Courant-Fischer Theorem

For a real symmetric matrix $A$, the maximum eigenvalue:

$$\lambda_n(A) = \max_{x \neq 0} \frac{x^\top A\, x}{x^\top x}$$

The smallest eigenvalue:

$$\lambda_1(A) = \min_{x \neq 0} \frac{x^\top A\, x}{x^\top x}$$

More generally,

$$\lambda_k(A) = \min_{x \neq 0, x^\top v_i = 0, \forall i \in \{1, \dots, k-1\}} \frac{x^\top A\, x}{x^\top x}$$

$$\lambda_k(A) = \max_{x \neq 0, x^\top v_i = 0, \forall i \in \{k+1, \dots, n\}} \frac{x^\top A\, x}{x^\top x}$$

# Cheeger's inequality

In a $d$**-regular graph,** the **normalized Laplacian** $\mathcal{L} = \frac{L}{d}$

Cheeger's Inequality [Cheeger 70, Alon-Milman 85]

$$\frac{\lambda_2(\mathcal{L})}{2} \leq \varphi(G) \leq \sqrt{2\lambda_2(\mathcal{L})}$$

The first inequality is called the easy direction, and the second is called the hard direction
We start with some **intuition** in the case when $G$ is a $d$-regular graph.

**For the easy direction**: think of $\lambda_2$ as a "relaxation" of the graph conductance problem.

$$\varphi(G) = \min_{x \in \{0,1\}^n, |x| \leq \frac{n}{2}} \frac{\sum_{ij \in E}(x_i - x_j)^2}{d \sum_{i \in V} x_i^2} \quad \text{and} \quad \lambda_2(\mathcal{L}) = \min_{x \perp 1} \frac{\sum_{ij \in E}(x_i - x_j)^2}{d \sum_{i \in V} x_i^2}.$$

**Question**: What does the second eigenvector $\boldsymbol{x}$ look like when the graph $G$ is disconnected, i.e., $G = G_1 \uplus G_2$ ?

# Easy direction

Think of $\lambda_2$ as a "relaxation" of the graph conductance problem.

$$\varphi(G) \approx \min_{x \perp 1 \,:\, x \text{ is binary}} \frac{\sum_{ij \in E} (x_i - x_j)^2}{d \sum_{i \in V} x_i^2} \qquad \text{and} \qquad \lambda_2(\mathcal{L}) = \min_{x \perp 1} \frac{\sum_{ij \in E} (x_i - x_j)^2}{d \sum_{i \in V} x_i^2}.$$

Proof: Given a set $S$ with $\varphi(S) = \varphi(G)$, we try to find $x \perp 1$ with $R_{\mathcal{L}}(x) \leq 2\varphi(S)$

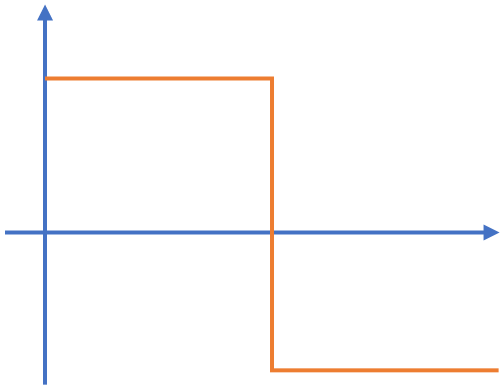
- Consider $x_i = \begin{cases} \frac{1}{|S|}, & \text{if } i \in S \\ -\frac{1}{n-|S|}, & \text{otherwise} \end{cases}$
- Then $\lambda_2(\mathcal{L}) \leq R_{\mathcal{L}}(x) = \frac{\sum_{ij \in E} (x_i - x_j)^2}{d \sum_{i \in V} x_i^2} = \frac{E(S, \bar{S})}{d} \left( \frac{1}{|S|} + \frac{1}{n-|S|} \right) \leq 2\varphi(S)$

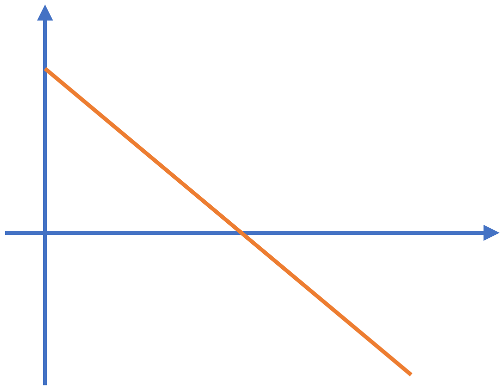
# Hard direction

In the hard direction, we are given the second eigenvector $x$, which has small Rayleigh quotient, and we need to find a binary vector $x'$

$$\varphi(G) \approx \min_{x \perp 1 \,:\, x \text{ is binary}} \frac{\sum_{ij \in E} (x_i - x_j)^2}{d \sum_{i \in V} x_i^2} \quad \text{and} \quad \lambda_2 = \min_{x \perp 1} \frac{\sum_{ij \in E} (x_i - x_j)^2}{d \sum_{i \in V} x_i^2}$$

To gain some intuition, consider sorting $x$





# Hard direction

WLOG, assume the number of positive entries in $x$ is at most the number of negative entries.

Zero out the negative entries of $x$ to obtain $y$.

Working with $y$, we ensure that the output set $S$ satisfies $|S| \leq n/2$.

**Lemma**. $R(y) \leq R(x) = \lambda_2$

**Proof**: Consider a row $i$ with $y_i > 0$. We have

$$(Ly)_i = \deg(i)\, y_i - \sum_{j \sim i} y_j \leq \deg(i)\, x_i - \sum_{j \sim i} x_j = (Lx)_i = \lambda_2 x_i$$

Then $y^\top L\, y = \sum_i y_i (Ly)_i \leq \sum_{i : x_i > 0} \lambda_2 x_i^2 = \lambda_2 \sum_i y_i^2$

# Hard direction

$\mathrm{supp}(y) \coloneqq \{i \mid y(i) \neq 0\}.$

**Claim**. Given any $y$, there exists a subset $S \subseteq \mathrm{supp}(y)$ such that

$$\varphi(S) \leq \sqrt{2R(y)} \leq \sqrt{2\lambda_2}$$

**Proof plan**. By scaling, assume that $\max_i y(i) = 1$.

For $0 < t \leq 1$, we consider a "threshold set" $S_t \coloneqq \{i \mid y(i)^2 \geq t\}$.

We want to prove that there exists a $t$ such that $\varphi(S_t) \leq \sqrt{2R(y)}$.

The **idea** is to choose $t$ uniformly randomly from $(0,1)$!

We will show that $\frac{\mathbb{E}_t[|E(S_t,\bar{S}_t)|]}{\mathbb{E}_t[d|S_t|]} \leq \sqrt{2R(y)}$.

This would imply that there exists $t$ such that $\frac{|E(S_t,\bar{S}_t)|}{d|S_t|} \leq \sqrt{2R(y)}$, as desired.

# Hard direction

$\mathrm{supp}(y) \coloneqq \{i \mid y(i) \neq 0\}.$

**Claim**. Given any $y$, there exists a subset $S \subseteq \mathrm{supp}(y)$ such that

$$\varphi(S) \leq \sqrt{2R(y)} \leq \sqrt{2\lambda_2}$$

**Proof**. Choose a random "threshold set" $S_t \coloneqq \{i \mid y(i)^2 \geq t\}$

We will show that $\frac{\mathbb{E}_t[|E(S_t,\bar{S}_t)|]}{\mathbb{E}_t[d|S_t|]} \leq \sqrt{2R(y)}$

Let's first calculate

$$\mathbb{E}_t[d|S_t|] = d\sum_i \Pr[i \in S_t] = d\sum_i \Pr[t \leq y(i)^2] = d\sum_i y(i)^2$$

# Hard direction

**Cauchy-Schwarz inequality**:
$$\langle u, v\rangle \le \sqrt{\langle u, u\rangle} \cdot \sqrt{\langle v, v\rangle}$$

**Proof** (cont.) Choose a random "threshold set" $S_t \coloneqq \{i \mid y(i)^2 \ge t\}$.

It remains to show that $\mathbb{E}_t[|E(S_t, \bar{S}_t)|] \le \sqrt{2R(y)} \cdot \left(d \sum_{i \in V} y_i^2\right)$, or equivalently, we want to show

$$\mathbb{E}_t[|E(S_t, \bar{S}_t)|] \le \sqrt{2 \sum_{ij \in E} (y_i - y_j)^2 \cdot d \sum_{i \in V} y_i^2}$$

$$\mathbb{E}_t[|E(S_t, \bar{S}_t)|] = \sum_{ij \in E} \Pr[i \in S_t, j \notin S_t] + \Pr[i \notin S_t, j \in S_t] = \sum_{ij \in E} |y_i^2 - y_j^2| = \sum_{ij \in E} |y_i - y_j| \cdot (y_i + y_j)$$

Apply Cauchy-Schwarz inequality:

$$\sum_{ij \in E} |y_i - y_j| \cdot (y_i + y_j) \le \sqrt{\sum_{ij \in E} |y_i - y_j|^2} \cdot \sqrt{\sum_{ij \in E} (y_i + y_j)^2} \le \sqrt{\sum_{ij \in E} |y_i - y_j|^2} \sqrt{2 \sum_{ij \in E} y_i^2 + y_j^2} = \sqrt{\sum_{ij \in E} |y_i - y_j|^2} \sqrt{2d \sum_{i \in V} y_i^2}$$

Combined, this concludes the proof of $\frac{\mathbb{E}_t[|E(S_t, \bar{S}_t)|]}{\mathbb{E}_t[d|S_t|]} \le \sqrt{2R(y)}$. Then we notice that this implies

$$\mathbb{E}_t\left[|E(S_t, \bar{S}_t)| - \sqrt{2R(y)} d |S_t|\right] \le 0$$

This means that there must be a choice of $t$, $|E(S_t, \bar{S}_t)| - \sqrt{2R(y)} d |S_t| \le 0 \Rightarrow \varphi(S_t) \le \sqrt{2R(y)}$

# Hard direction summary

Easy direction is to show that $\lambda_2 = \min_{\substack{x \in \mathbb{R}^n \\ x \perp 1}} \frac{\sum_{ij \in E} (x_i - x_j)^2}{d \sum_{i \in V} x_i^2}$ is a "relaxation" of $\varphi(G) \approx \min_{x: x \text{ "binary"}} \frac{\sum_{ij \in E} (x_i - x_j)^2}{d \sum_{i \in V} x_i^2}$.

For the hard direction, given an optimizer $x$ for $\lambda_2$, we want to produce a set $S$ with $\varphi(S) \leq \sqrt{2\lambda_2}$.

The idea is simply to try all "threshold" sets of $x$.

We truncate the vector $x$ to obtain $y$ to ensure that the output set is of size at most $n/2$.

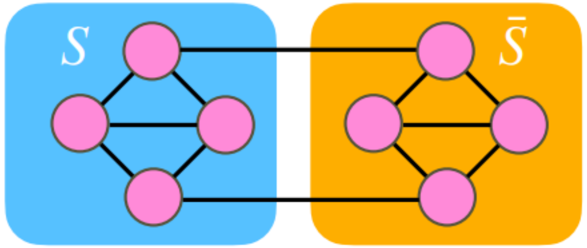
In the analysis, we choose a <u>random</u> threshold for $y$ and prove that $\frac{\mathbb{E}_t[|E(S_t, \bar{S}_t)|]}{\mathbb{E}_t[d|S_t|]} \leq \sqrt{2R(y)}$.

In general, this is called a "rounding" algorithm, where we turn a "fractional" solution to an integral solution.

This is the most common way to design approximation algorithms for NP-hard optimization problems.

Today we see an example of "randomized rounding", a useful technique in rounding algorithms.

# Summary: Spectral partitioning




$\varphi(S) := \frac{|E(S,\bar{S})|}{d|S|}$

To find a sparse cut with small **conductance** in a **general graph**, we

1. Compute the second smallest eigenvector $\boldsymbol{x} \in \mathbb{R}^n$ of the **normalized Laplacian** $\mathcal{L}(G) := D^{-\frac{1}{2}} L(G)\, D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$
2. Sort the vertices so that $x_1 \geq x_2 \geq \cdots \geq x_n$.
3. Let $S_i := \begin{cases} \{1, \dots, i\} & \text{if } i \leq \frac{n}{2} \\ \{i+1, \dots, n\} & \text{otherwise} \end{cases}$, and output $S_i = \underset{1 \leq i \leq n}{\operatorname{argmin}} \varphi(S_i)$.

**Theorem**: $\exists i, \varphi(S_i) \leq 2\sqrt{\varphi(G)}$

# Aside: Spectral Partitioning in Planar Graphs

**Planar graph separator theorem**: The removal of $O(\sqrt{n})$ vertices partitions the planar graph into disjoint subgraphs, each of which has at most $2n/3$ vertices

**Theorem** (Spielman-Teng'07). For bounded degree planar graphs, a recursive spectral partitioning finds a separator of size $O(\sqrt{n})$

# Recent Generalizations

Previously, spectral graph theory is mostly about the second eigenvalue.

In the past decade, there are a few interesting generalizations of Cheeger's inequality using other eigenvalues!

We will discuss some of them.

# Last Eigenvalue

$\mathcal{A} = D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$ is the **normalized adjacency matrix**

**Exercise**. $\lambda_1(A) = -\lambda_n(A)$ iff $G$ is bipartite; and $\lambda_n(\mathcal{A}) = -1$ iff $G$ is bipartite.

Let $\alpha_n$ be the smallest eigenvalue of $I + \mathcal{A}$. Then the above implies that $\alpha_n = 0$ iff $G$ is bipartite.

**Exercise**. $\alpha_n = \min_{x \in \mathbb{R}^n} \frac{\sum_{ij \in E} (x_i + x_j)^2}{d \sum_{i \in V} x_i^2}$ for $d$-regular graphs.

Define $\beta(G) = \min_{x \in \{-1,0,+1\}^n} \frac{\sum_{ij \in E} |x_i + x_j|}{d \sum_{i \in V} |x_i|}$. (By the way, we can also think of $\varphi(G)$ this way.)

This is called the bipartiteness ratio of $G$.

**Theorem**. [Trevisan 09] $\frac{1}{2}\alpha_n \leq \beta(G) \leq \sqrt{2\alpha_n}$.

Proof idea. Pick a random $t \in [0,1]$.

A vertex $i$ gets $-1$ if $x_i^2 \geq t$ and $x_i \leq 0$, gets $+1$ if $x_i^2 \geq t$ and $x_i \geq 0$, and gets $0$ otherwise.

This result is used to design a spectral algorithm for approximating maximum cut of a graph.

# $k$-th Eigenvalue

**Exercise**. $\lambda_k(\mathcal{L}) = 0$ if and only if $G$ has at least $k$ components.

There are two interesting ways to generalize this statement.

- If $\lambda_k$ is small, then there is a sparse cut $S$ with $|S| \lesssim \frac{n}{k}$.
- If $\lambda_k$ is small, then there are $k$ disjoint sparse cuts.

The second result is more general, but the first result is quantitatively stronger.

[Arora,Barak,Steurer,10] proved that when $k$ is large enough, there is a set $S$ with $\varphi(S) \lesssim \sqrt{\lambda_k}$ and $|S| \approx \frac{n}{k}$.

The proof uses ideas about random walks.

The algorithm is used for solving "**unique games**".

# Small set expansion, local graph partitioning

Note that the Cheeger rounding works with any vector with small Rayleigh quotient

One could try to run a random walk to find such vectors, this will be efficient in both time and space

Further, if we only care about finding a small sparse cut (e.g., a small community), the algorithm has a running time that only depends on the output size

The question of finding small set expansion is closely related to the Unique Games problem

# Higher Order Cheeger's Inequality

**Theorem**. [Lee, Oveis-Gharan, Trevisan 12] [Louis, Raghavendra, Tetali, Vempala 12]

$$\frac{\lambda_k}{2} \le \varphi_k(G) \le O(k^2 \cdot \mathrm{polylog}(k))\sqrt{\lambda_k}, \qquad \text{where } \varphi_k(G) \coloneqq \min_{\text{disjoint } S_1,\ldots,S_k} \max_{1 \le i \le k} \phi(S_i) .$$

Furthermore, $\varphi_k(G) \le O\left(\sqrt{\ln k}\right) \cdot \sqrt{\lambda_{1.01k}}$.

The proof is by a spectral embedding, where each vertex is mapped to a point in $\mathbb{R}^k$ using the $k$ eigenvectors.

The vectors are orthonormal, so the points are "well spread out".

The algorithm by [LRTV] is very simple:

(1) Generate $k$ random directions. (2) Put each point to its closest direction. (3) Run Cheeger on each direction.

The algorithm by [LOT] is similar to a clustering heuristic that was proposed in machine learning.

# Improved Cheeger's Inequality

**Theorem**. [Kwok, Lau, Lee, Oveis-Gharan, Trevisan 13] For any $k \geq 2$,

$$\frac{\lambda_2}{2} \leq \varphi(G) \leq O\left(\frac{k\lambda_2}{\sqrt{\lambda_k}}\right).$$

Cheeger's inequality is when $k = 2$.

Performance achieved by the same spectral partitioning algorithm.

Constant factor approximation when $\lambda_k$ is large for a small $k$,

which happens in image segmentation when there are only few outstanding objects in an image.

Tight up to a constant factor for any $k$.

The proof is by showing that if $\lambda_k$ is large, then the second eigenvector looks like a $k$-step function.

See Chapter 5.4 of Lap Chi Lau's book for more discussions.

# What next

<u>Random walks on undirected graphs</u>

- Fundamental theorem of Markov chains
- Spectral analysis
- Mixing time
- Random sampling