Advanced Algorithm 26 Homework 2

November 2026

1 Adjacency Matrix

Let A be the adjacency matrix of a simple undirected graph G, with largest eigenvalue α_1.

1. **Lowerbounding α_1.** We proved $\alpha_1 \leq d_{\max}$ in class. Show that $\alpha_1 \geq d_{\text{avg}}$, where

$$d_{\text{avg}} := \frac{2|E|}{|V|}$$

   is the average degree of the graph.

2. **Monotonicity of the spectrum.** Let A' be the adjacency matrix of any subgraph of G produced by deleting vertices, and let α_1' be the largest eigenvalue of A'. Show that $\alpha_1' \leq \alpha_1$.

3. **Chromatic number.** The chromatic number $\chi(G)$ of a graph is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. Prove

$$\chi(G) \leq \lfloor \alpha_1 \rfloor + 1.$$

2 Graph Laplacian

1. **Spectrum of special graphs.** Find the eigenvalues of the Laplacian matrices of the following graphs:

   (a) The complete graph with n vertices.

   (b) The star graph with n vertices.

2. **Number of connected components.** Let G be an undirected graph, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of its Laplacian matrix L. Prove that $\lambda_k = 0$ if and only if G has at least k connected components.

3. **Lowerbounding λ_2.** Let G be a connected, undirected, unweighted graph on $n \geq 2$ vertices whose Laplacian is L, with second-smallest eigenvalue λ_2. We know that $\lambda_2 > 0$. Prove that

$$\lambda_2 \geq \Omega\left(\frac{1}{rn}\right) \geq \Omega\left(\frac{1}{n^2}\right)$$

by analyzing the Rayleigh quotient on all test vectors $x \perp \mathbf{1}$. Here, r is the *diameter* of the graph (i.e., the maximum shortest-path distance between pairs of vertices in the graph). Further, show that if G is additionally simple and d-regular, we have

$$\lambda_2 \geq \Omega\left(\frac{d}{n^2}\right).$$

3 Cheeger's Inequality

Let $G = (V, E)$ be a d-regular graph with $|V| = n \geq 2$ and $d \geq 1$. Let S, T, U be a partition of V, with $S, T \neq \varnothing$ and U possibly empty, such that $|S| \leq |T|$ and $|U| \leq \varepsilon \cdot \min\{|S|, |T|\}$. We define the projected expansion by

$$\phi_G(S, T \| U) := \frac{|E(S, T)|}{d|S|}.$$

Let λ_2 be the second-smallest eigenvalue of the normalized Laplacian $\mathcal{L} := (D - A)/d$. Show that there exists an efficient algorithm that, for every $\varepsilon > 0$, finds sets S, T, U such that

$$0 < |S| \leq |T|, \qquad |U| \leq \varepsilon \min\{|S|, |T|\},$$

and

$$\phi_G(S, T \| U) \leq O\left(\lambda_2 \cdot \left(1 + \frac{2}{\varepsilon}\right)\right).$$

(*Hint*: Try to modify the randomized thresholding in Cheeger's proof.)

4 Higher Eigenvalues and Sparse Cuts

Let $G = (V, E)$ be a d-regular graph with $|V| = n$ and $d \geq 1$, and let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of its normalized Laplacian $\mathcal{L} = I - A/d$. Suppose there exist k pairwise disjoint nonempty subsets $S_1, \ldots, S_k \subseteq V$, where $2 \leq k \leq n$, such that

$$\frac{|E(S_i, V \setminus S_i)|}{d|S_i|} \leq \varepsilon \qquad \text{for all } i = 1, \ldots, k.$$

Prove that $\lambda_k \leq 2\varepsilon$.

5 Random Walk on a Graph

In class, we showed that large conductance implies rapid mixing of random walks. In this problem, we show that this is also necessary to some extent.

Consider a connected, undirected, unweighted d-regular graph $G = (V, E)$ with $|V| = n$ and $d \geq 1$, and let A be its adjacency matrix. At each step, the *lazy random walk* stays at its current vertex with probability $1/2$ and moves to a uniformly random neighbor otherwise. Its transition matrix

and stationary distribution are

$$P = \frac{1}{2}\left(I + \frac{A}{d}\right), \qquad \pi(v) = \frac{1}{n} \quad \text{for all } v \in V.$$

1. For a nonempty set $S \subseteq V$, let π_S be π conditioned on S. Prove that the conductance of S for the lazy walk satisfies

$$\phi_P(S) := \frac{|E(S, V \setminus S)|}{2d|S|} = \mathbb{P}\left[X_1 \notin S \mid X_0 \sim \pi_S\right].$$

   Thus $\phi_P(S)$ is the probability that the walk started from π_S escapes from S in one step.

2. Show that P is positive semidefinite (PSD), and prove that

$$\mathbb{P}\left[X_t \in S \mid X_0 \sim \pi_S\right] \geq (1 - \phi_P(S))^t$$

   for all integers $t \geq 1$.

   (*Hint*: First show that $\langle x, M^t x\rangle \geq \langle x, Mx\rangle^t$ for any unit vector x, any PSD matrix M, and any integer $t \geq 1$.)

3. Let $p^{(t)}$ be the distribution of X_t. The ℓ_2 mixing time of this lazy walk is the smallest integer $t \geq 0$ such that

$$\left\| \frac{p^{(t)}}{\pi} - 1 \right\|_{2,\pi}^2 := \sum_{i=1}^{n} \pi_i \left(\frac{p_i^{(t)}}{\pi_i} - 1 \right)^2 \leq \frac{1}{4}, \qquad \forall p^{(0)}.$$

   Suppose there exists a set $S \subseteq V$ with $\pi(S) := \sum_{v \in S} \pi(v) = 1/\sqrt{n}$ and $0 < \phi_P(S) \leq 1/4$. Show that the ℓ_2 mixing time is at least

$$\Omega\left(\frac{\log n}{\phi_P(S)}\right).$$

6 Glauber Dynamics for Graph Colorings

Let $G = (V, E)$ be a simple undirected graph with maximum degree $\Delta \geq 1$. For an integer $q \geq \Delta + 1$, let Ω_q denote the set of all proper q-colorings of G, using colors from $[q] = \{1, \ldots, q\}$.

For a current coloring σ, one step of the Glauber dynamics on Ω_q is as follows:

- Choose a vertex $v \in V$ uniformly at random.
- Recolor v with a uniformly random color from $A_\sigma(v) = [q] \setminus \{\sigma(u) : u \in N(v)\}$, where $N(v)$ is the set of neighbors of v.

1. **A counterexample.** Show that the Markov chain need not be irreducible when $q = \Delta + 1$.

2. **Irreducibility.** Prove that the Markov chain is irreducible whenever $q = \Delta + 2$.

7 Random Walk with Teleportation

Let P be the transition matrix of an arbitrary Markov chain on $[n] = \{1, \ldots, n\}$. Fix $\eta \in (0, 1)$. At each step, the modified chain jumps to a uniformly random state with probability η and follows P otherwise. Its transition matrix is

$$\widetilde{P} = (1 - \eta)P + \frac{\eta}{n}\mathbf{1}\mathbf{1}^\top.$$

1. **Stationary distribution.** Show that the modified chain has a unique stationary distribution $\widetilde{\pi}$.

2. **Mixing time.** For $0 < \varepsilon \leq 1/2$, let $t_{\text{mix}}(\varepsilon)$ be the smallest integer $t \geq 0$ such that

$$\max_{x \in [n]} \left\| \widetilde{P}^t(x, \cdot) - \widetilde{\pi} \right\|_{\text{TV}} \leq \varepsilon,$$

where $\|\mu - \nu\|_{\text{TV}} := \frac{1}{2}\sum_{y=1}^{n} |\mu(y) - \nu(y)|$. Use a coupling argument to prove that

$$t_{\text{mix}}(\varepsilon) = O\left(\frac{\log(1/\varepsilon)}{\eta}\right).$$

8 MCMC and Coupling

In this problem, you will analyze the MCMC sampler over independent sets of a fixed size $k \geq 1$. Given a simple graph $G = (V, E)$ with $|V| = n$ and maximum degree Δ, let I_k be the set of all independent sets in G of size k. Consider a random walk $\{X_i\}_{i \geq 0}$ on I_k defined by the following process:

- Choose a vertex $v \in X_t$ uniformly at random and a vertex $w \in V$ uniformly at random.
- If $w \notin X_t$ and $(X_t \setminus \{v\}) \cup \{w\}$ is independent, set

$$X_{t+1} = (X_t \setminus \{v\}) \cup \{w\}.$$

- Otherwise, set $X_{t+1} = X_t$.

Recall the definition of total variation mixing time $t_{\text{mix}}(\varepsilon)$ from the previous problem, here with the uniform distribution on I_k as the target distribution.

Use a coupling argument to show that if $k \leq n/(3\Delta + 3)$, then $t_{\text{mix}}(\varepsilon)$ is bounded by a polynomial in n and $\log(1/\varepsilon)$.