Correlation Decay *up to* Uniqueness *in* Spin Systems

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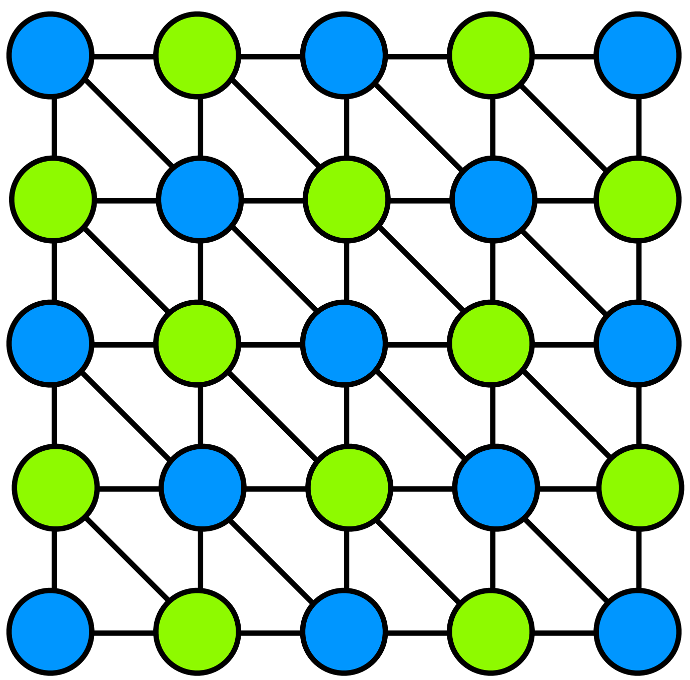
Two-State Spin System



graph $G=(V,E)$ 2 states {0,1}

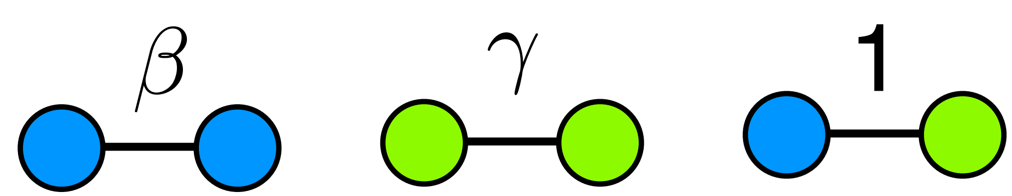
configuration $\sigma : V \to \{0,1\}$

$$A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$$

$$b = (b_0, b_1) = (\lambda, 1)$$

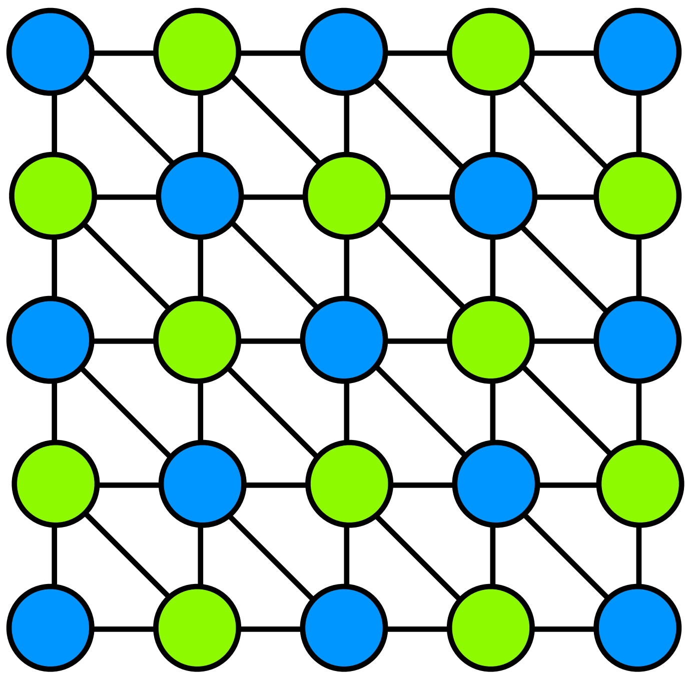
$$w(\sigma) = \prod_{(u,v)\in E} A_{\sigma(u),\sigma(v)} \prod_{v\in V} b_{\sigma(v)}$$

edge activity:



external field:

Two-State Spin System



graph $G=(V,E)$ 2 states {0, 1}

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$$w(\sigma) = \prod_{(u,v)\in E} A_{\sigma(u),\sigma(v)} \prod_{v\in V} b_{\sigma(v)}$$

Gibbs measure: $\Pr(\sigma) = \frac{w(\sigma)}{Z(G)}$

partition function: $Z(G) = \sum_{\sigma\in\{0,1\}^V} w(\sigma)$

$$A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix} \qquad b = (b_0, b_1) = (\lambda, 1)$$

$$w(\sigma) = \prod_{(u,v)\in E} A_{\sigma(u),\sigma(v)} \prod_{v\in V} b_{\sigma(v)}$$

partition function: $$Z(G) = \sum_{\sigma\in\{0,1\}^V} w(\sigma)$$

marginal probability: $$\Pr_{\sigma}[\sigma(v) = 0 \mid \sigma(v_1), \ldots, \sigma(v_k)]$$

$$\Pr(\tau) = \prod_{k=1}^{n} \Pr_{\sigma}[\sigma(v_k) = \tau(v_k) \mid \sigma(v_i) = \tau(v_i), 1 \leq i < k]$$

$$= \frac{w(\tau)}{Z}$$

1/poly(n) additive error for marginal in poly-time ⇒ FPTAS for $Z(G)$

ferromagnetic: $\beta\gamma > 1$

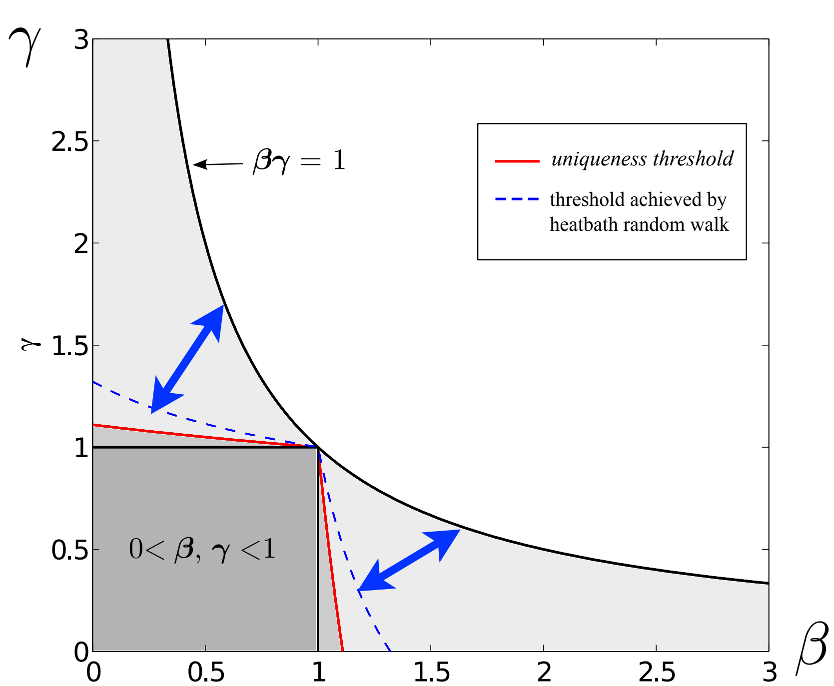
FPRAS: [Jerrum-Sinclair'93] [Goldberg-Jerrum-Paterson'03]

anti-ferromagnetic: $\beta\gamma < 1$

hardcore model: $\beta = 0, \gamma = 1$ [Weitz'06]

Ising model: $\beta = \gamma$ [Sinclair-Srivastava-Thurley'12]

(β, γ, λ) lies in the interior of uniqueness region of Δ-regular tree ⇒ ∃ FPTAS for graphs of max-degree Δ



[Goldberg-Jerrum-Paterson'03]

FPRAS for arbitrary graphs

[Li-Lu-Y. '12]: no external field

FPTAS for arbitrary graphs

anti-ferromagnetic: $\beta\gamma < 1$

bounded Δ or $\Delta=\infty$

(β, γ, λ) lies in the interiors of uniqueness regions of d-regular trees for all $d \leq \Delta$.

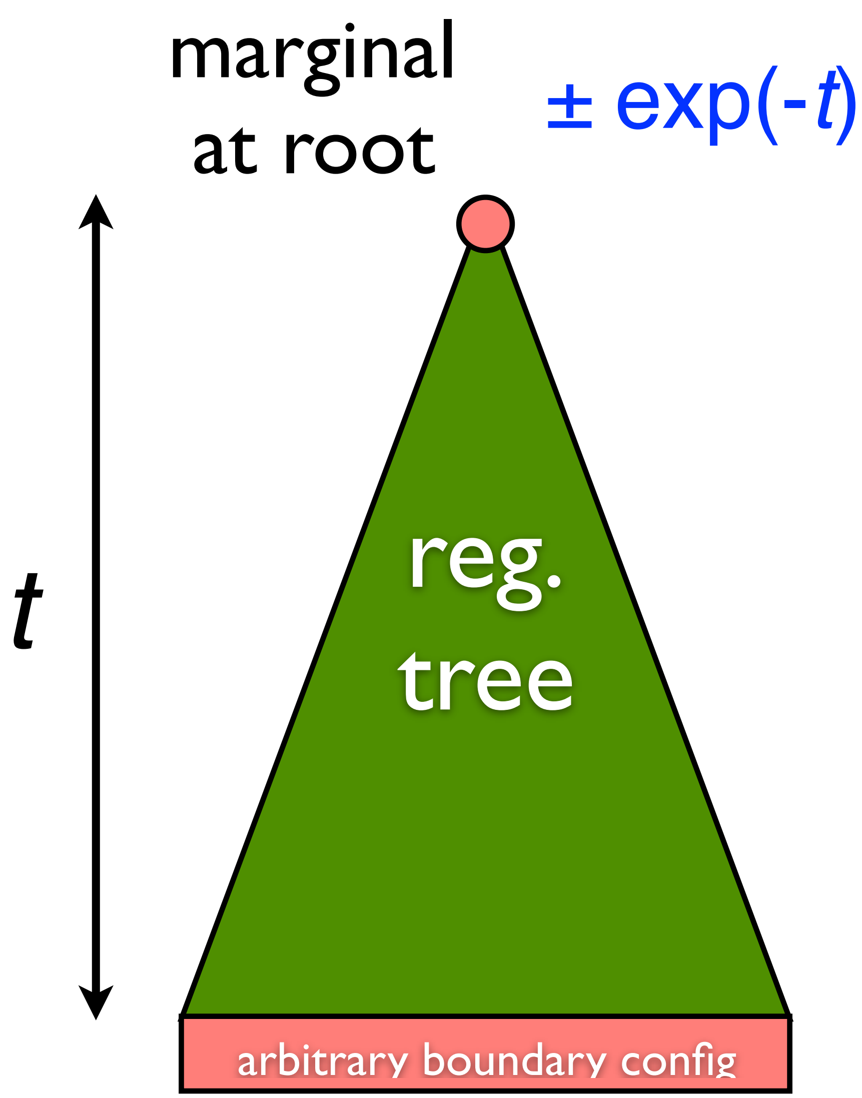
⇒ $\exists$ FPTAS for graphs of max-degree Δ

[Sly-Sun'12] [Galanis-Stefankovic-Vigoda'12]:

(β, γ, λ) lies in the interiors of non-uniqueness regions of d-regular trees for some $d \leq \Delta$.

assuming NP $\neq$RP ⇒ $\nexists$ FPRAS for graphs of max-degree Δ

Uniqueness Condition



(d+1)-regular tree

$$f_d(x) = \lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d$$

$$\hat{x}_d = f_d(\hat{x}_d)$$

$$|f'_d(\hat{x}_d)| < 1$$

anti-ferromagnetic: $\beta\gamma < 1$

bounded Δ or Δ=∞

$$f_d(x) = \lambda\left(\frac{\beta x + 1}{x + \gamma}\right)^d$$

$$\forall d < \Delta, |f'_d(\hat{x}_d)| < 1$$

⇒ ∃ FPTAS for graphs of max-degree Δ

[Sly-Sun'12] [Galanis-Stefankovic-Vigoda'12]:

$$\exists d < \Delta, |f'_d(\hat{x}_d)| > 1$$

assuming NP ≠RP ⇒ ∄ FPRAS for graphs of max-degree Δ

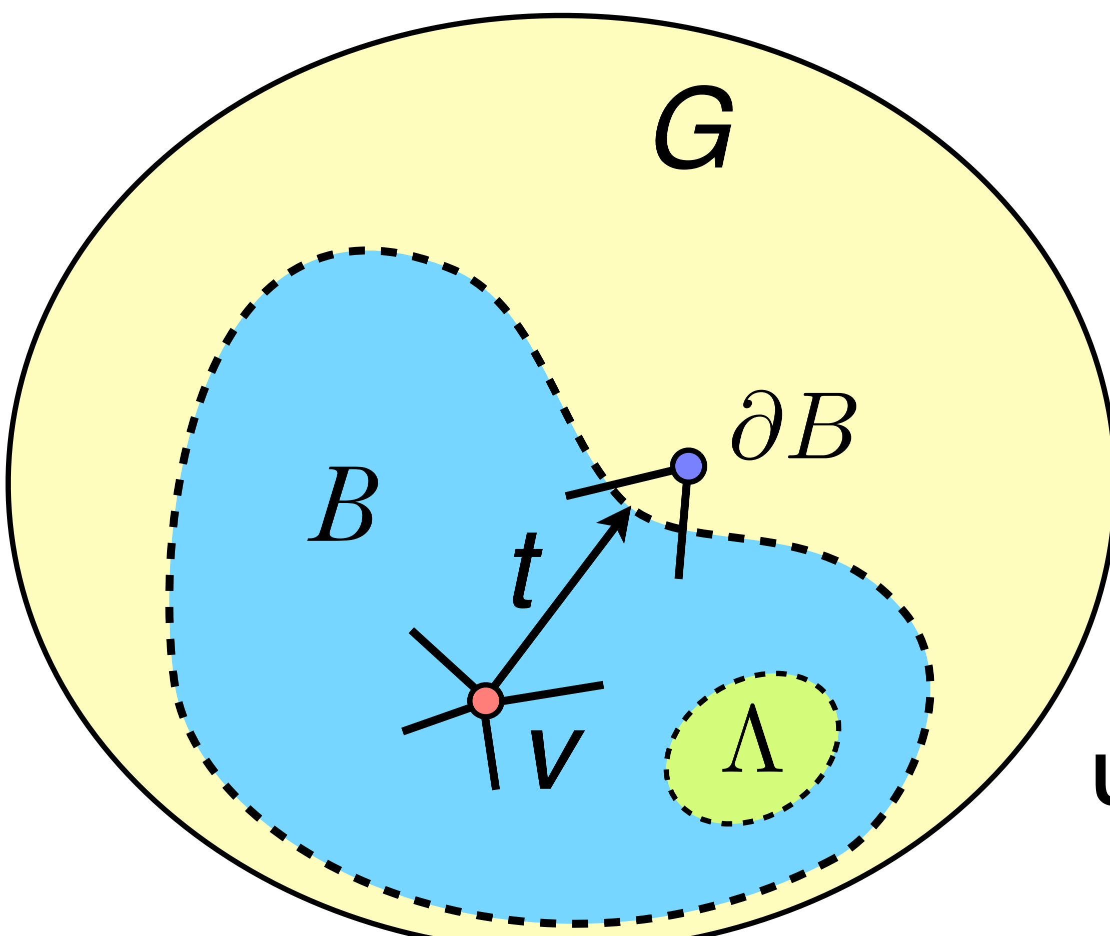
Correlation Decay

weak spatial mixing (WSM): $\forall \sigma_{\partial B}, \tau_{\partial B} \in \{0,1\}^{\partial B}$

$$\Pr_{\sigma}[\sigma(v) = 0 \mid \sigma_{\partial B}] \approx \Pr_{\sigma}[\sigma(v) = 0 \mid \tau_{\partial B}]$$

strong spatial mixing (SSM):

$$\Pr_{\sigma}[\sigma(v) = 0 \mid \sigma_{\partial B}, \sigma_{\Lambda}] \approx \Pr_{\sigma}[\sigma(v) = 0 \mid \tau_{\partial B}, \sigma_{\Lambda}]$$



error < exp (-*t*)

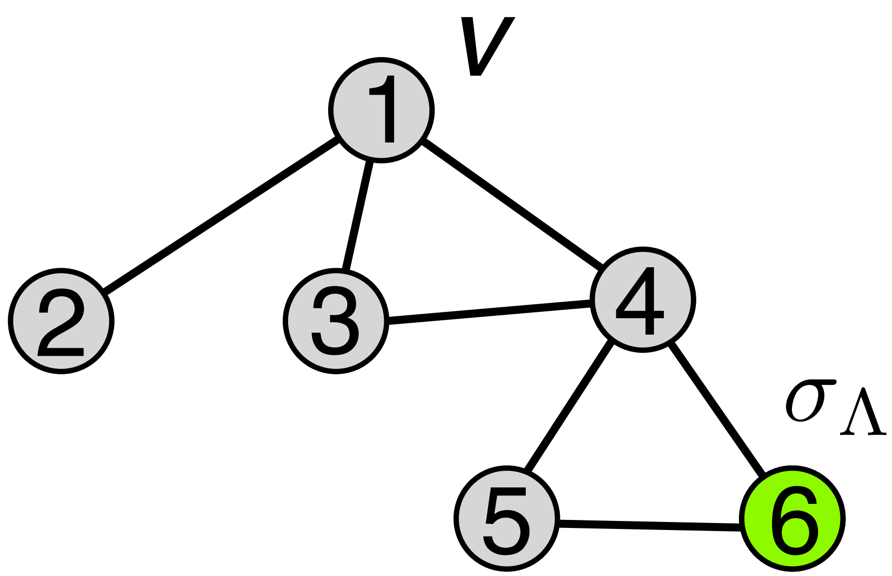
exponential correlation decay

uniqueness: WSM in reg. tree

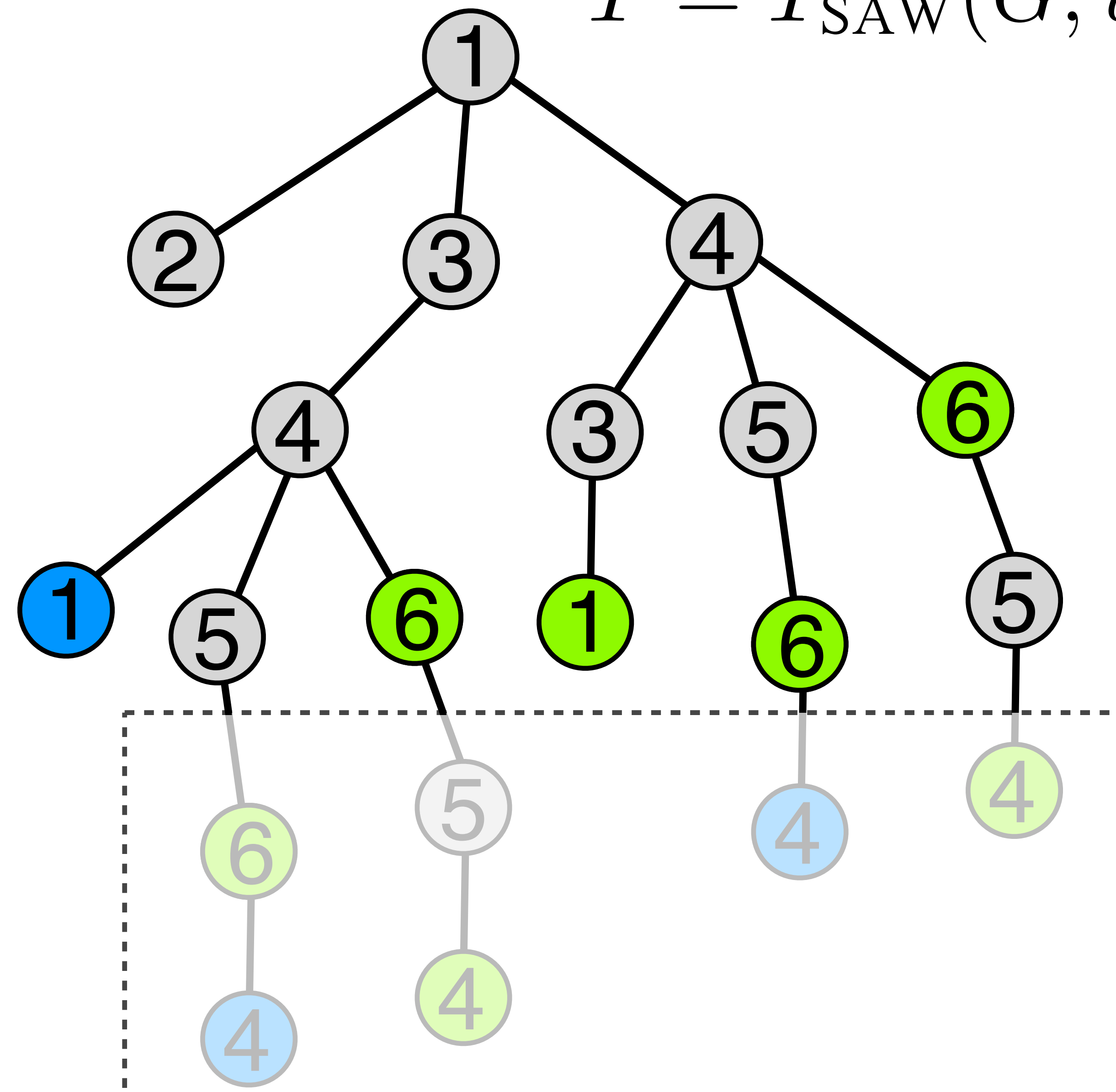
Self-Avoiding Walk Tree

due to Weitz (2006)

$G=(V,E)$



$T = T_{\mathrm{SAW}}(G, v)$



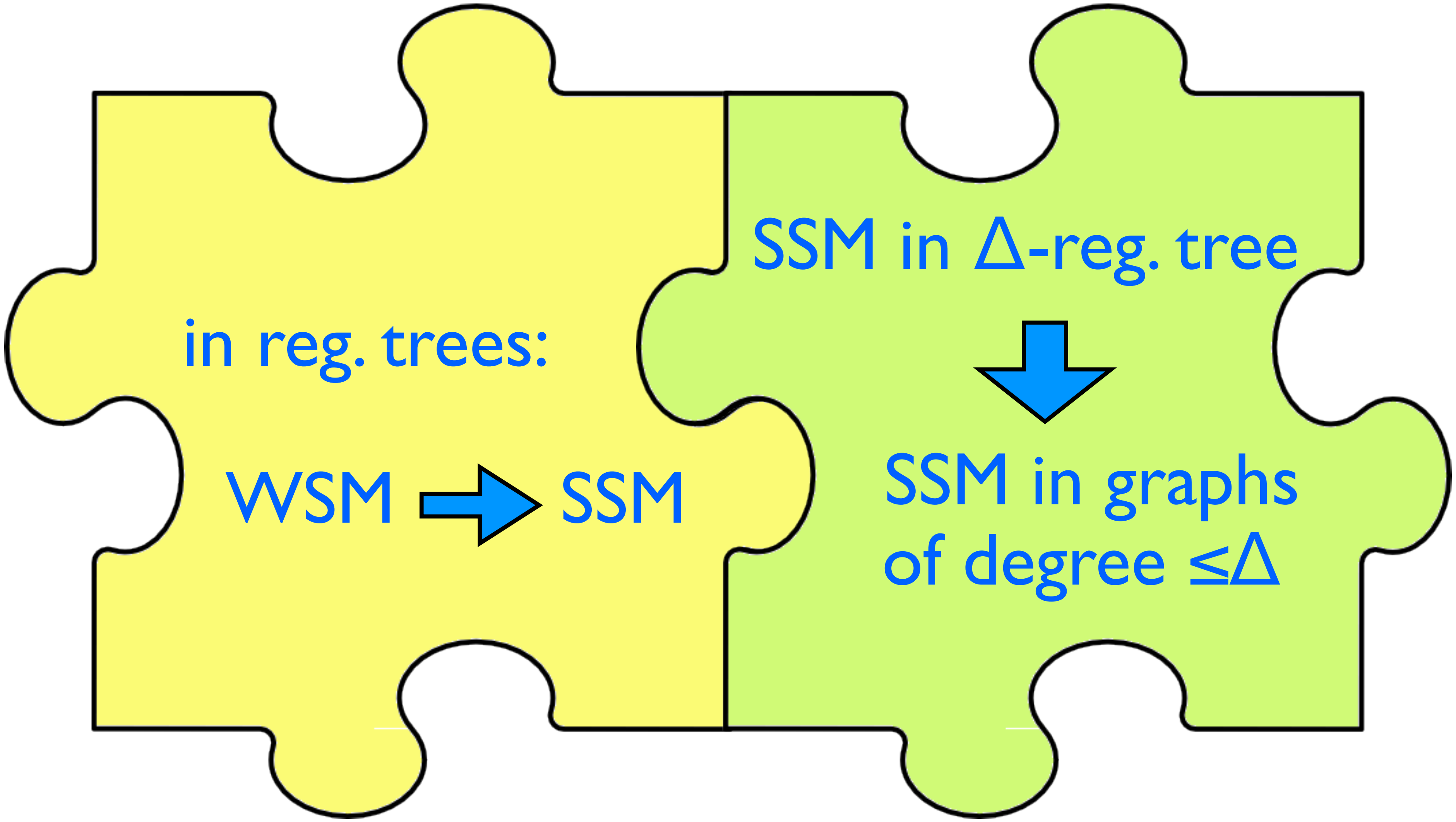
preserve the marginal dist. at v

on bounded degree graphs:

SSM → FPTAS

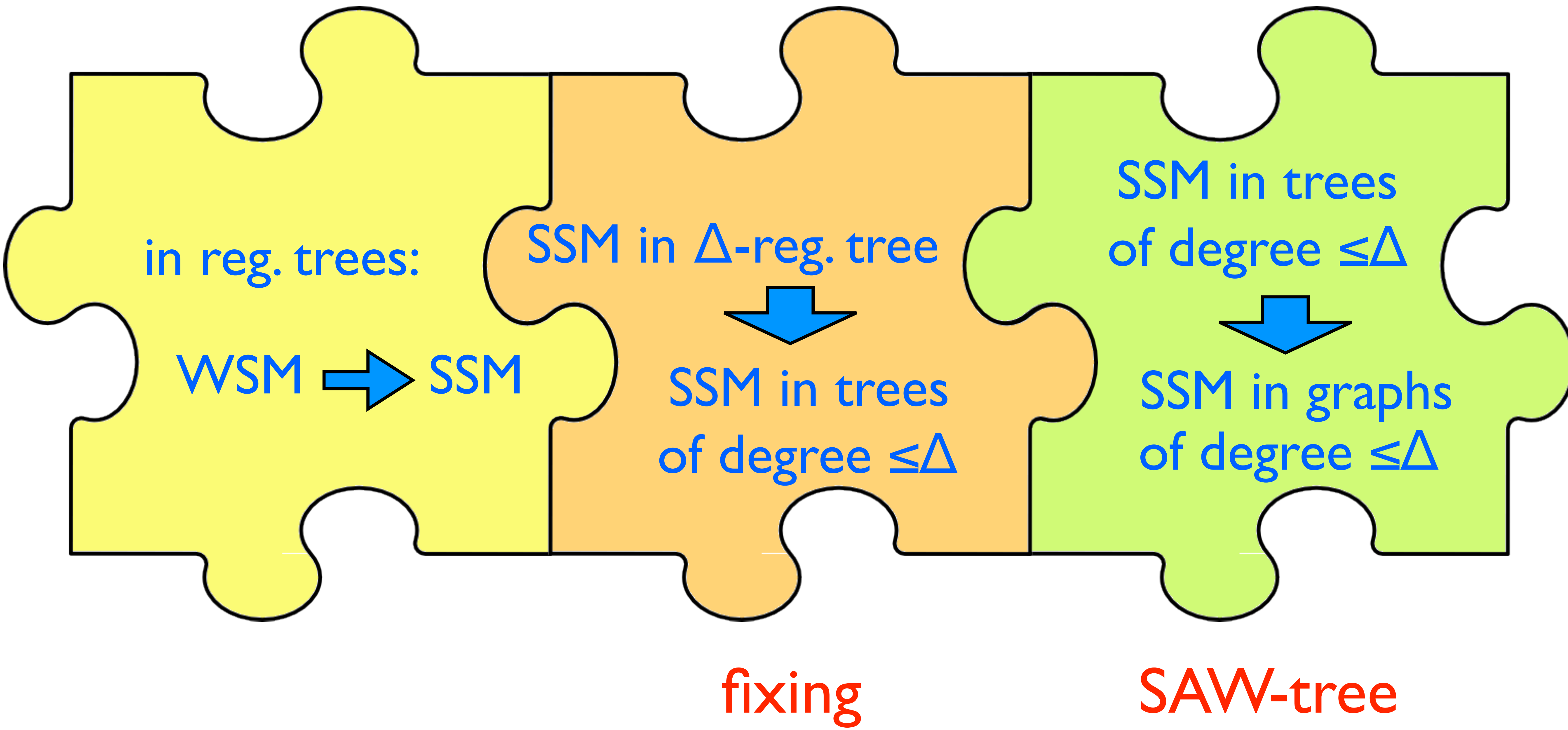
hardcore model, anti-ferro Ising model:

(for $\beta, \gamma < 1$)

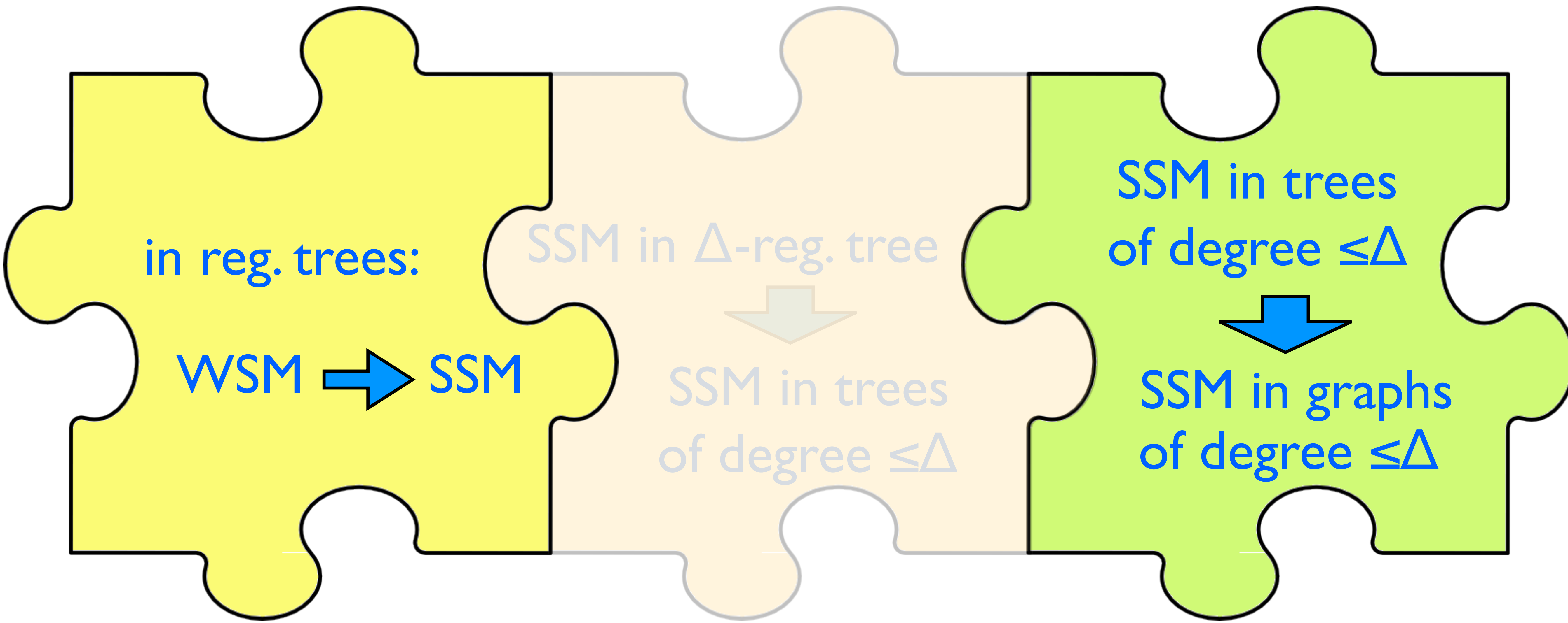


WSM in Δ-reg. tree → SSM in graphs of degree ≤Δ hardcore model, anti-ferro Ising model:

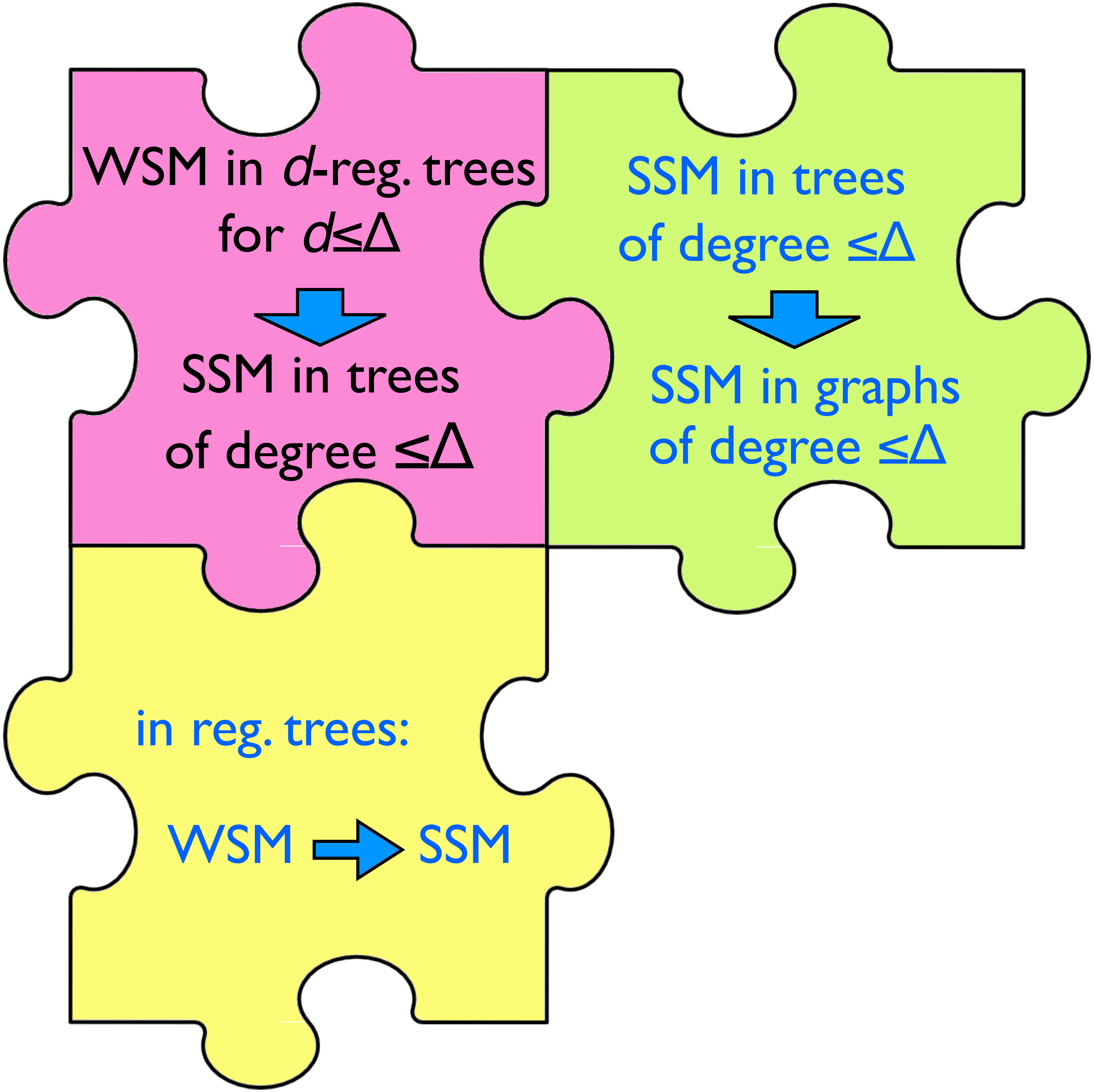
(for $\beta, \gamma < 1$)



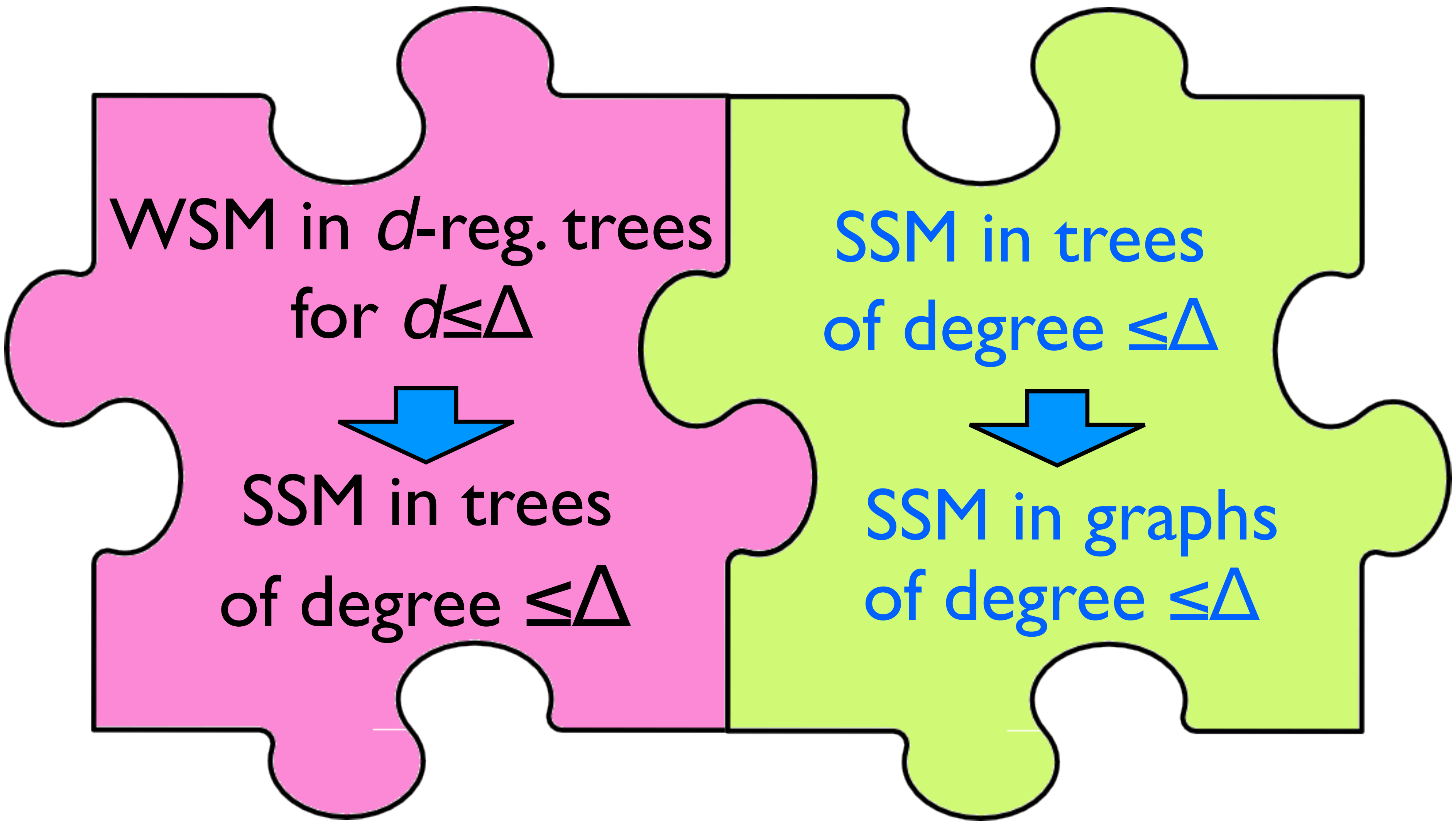
for general anti-ferro 2-state spin systems:



SAW-tree

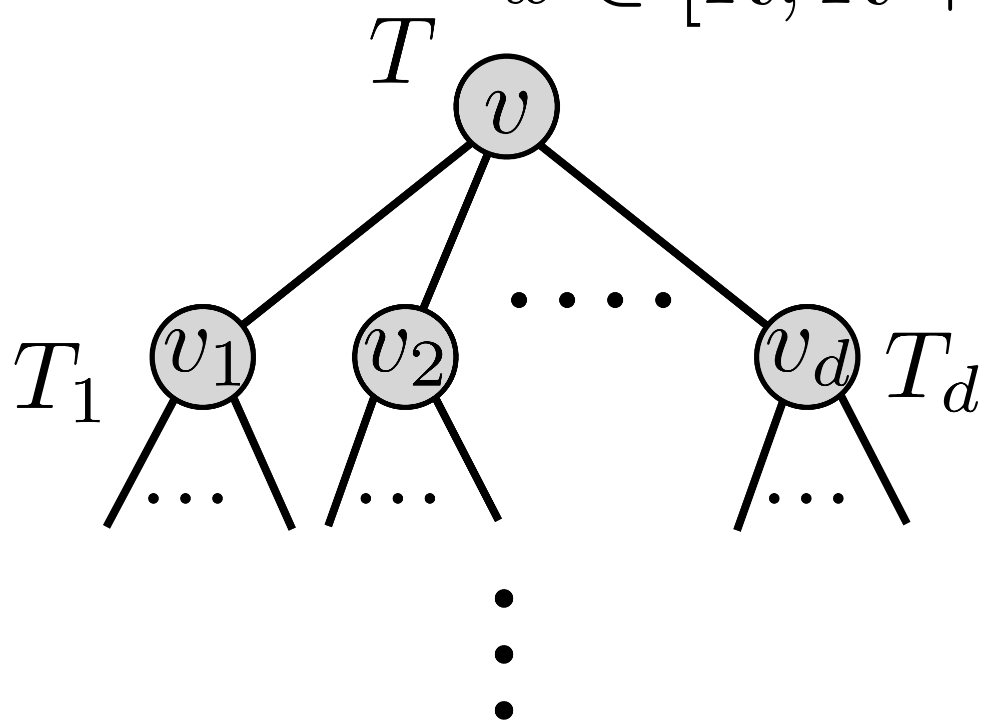

WSM in d-reg. trees
for d≤Δ
SSM in trees
of degree ≤Δ
SSM in trees
of degree ≤Δ
SSM in graphs
of degree ≤Δ
in reg. trees:
WSM
SSM

for general anti-ferro 2-state spin systems:



WSM in *d*-reg. trees for $d \leq \Delta$ → SSM in graphs of degree $\leq \Delta$

$x \in [R, R+\delta] \qquad \delta = \exp(-\Omega(n))$



$$x = \frac{\Pr[\sigma(v) = 0 \mid \sigma_\Lambda]}{\Pr[\sigma(v) = 1 \mid \sigma_\Lambda]}$$

x's in level $n \in [0, \infty)$

$$x = f(x_1, \ldots, x_d) = \lambda \prod_{i=1}^{d} \left(\frac{\beta x_i + 1}{x_i + \gamma} \right)$$

Potential Analysis

$$\begin{array}{c} f(x) \\ f \Big\uparrow \\ x \end{array}$$

$$F_n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n}(x)$$

$$F_n(x+\delta) - F_n(x) = F_n'(x_0) \cdot \delta$$

$$= \delta \cdot \prod_{t=0}^{n-1} f'(x_t) \qquad\qquad x_t = f(x_{t-1})$$

$$= \delta \cdot \frac{\Phi(x_0)}{\Phi(x_n)} \cdot \prod_{t=0}^{n-1} \frac{\Phi(f(x_t))}{\Phi(x_t)} f'(x_t)$$

Potential Analysis

$$
\begin{array}{ccc}
f(x) & \xrightarrow{\phi} & g(y) \\
f \uparrow & & \uparrow g \\
x & \xleftarrow{\phi^{-1}} & y
\end{array}
$$

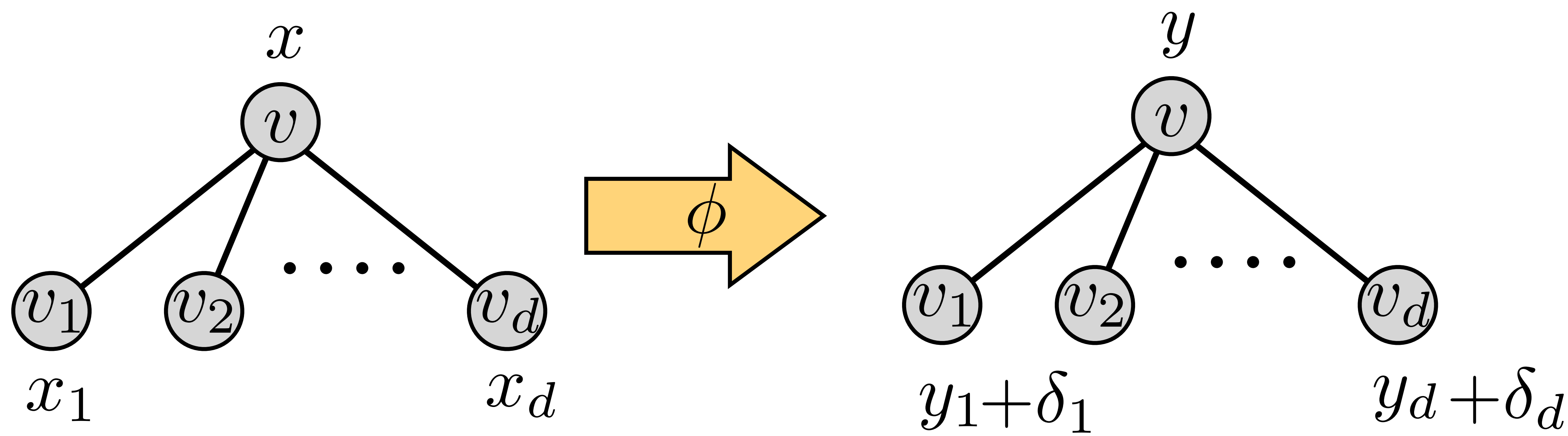
$$G_n(x) = \underbrace{g \circ g \circ \cdots \circ g}_{n}(x)$$

$$G_n(x+\delta) - G_n(x) = G_n'(x_0) \cdot \delta$$

$$
\begin{aligned}
G_n'(x_0) &= \prod_{t=0}^{n-1} g'(x_t) \\
&= \prod_{t=0}^{n-1} [\phi(f(\phi^{-1}(y_t))]' \\
&= \prod_{t=0}^{n-1} \frac{\Phi(f(x_t))}{\Phi(x_t)} f'(x_t)
\end{aligned}
$$

$$\phi'(x) = \Phi(x)$$

$$\phi'(x) = \Phi(x) = \frac{1}{\sqrt{x(\beta x + 1)(x + \gamma)}}$$



$$f(x_1, \ldots, x_d) = \lambda \prod_{i=1}^{d} \left(\frac{\beta x_i + 1}{x_i + \gamma} \right)$$

$$g(y_1, \ldots, y_d) = \phi(f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_d)))$$

$$g(y_1, \ldots, y_d) - g(y_1 + \delta_1, \ldots, y_d + \delta_d)$$

$$= -\nabla \phi(f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_d))) \cdot (\delta_1, \ldots, \delta_d)$$

$$\leq \alpha(d; x_1, \ldots, x_d) \cdot \max_{1 \leq i \leq d} \{\delta_i\}$$

amortized decay rate

$\alpha(d; x_1, ..., x_d)$ amortized decay rate

$$= \frac{(1-\beta\gamma)\left(\lambda\prod_{i=1}^{d}\frac{\beta x_i+1}{x_i+\gamma}\right)^{\frac{1}{2}}}{\left(\beta\lambda\prod_{i=1}^{d}\frac{\beta x_i+1}{x_i+\gamma}+1\right)^{\frac{1}{2}}\left(\lambda\prod_{i=1}^{d}\frac{\beta x_i+1}{x_i+\gamma}+\gamma\right)^{\frac{1}{2}}}\cdot\sum_{i=1}^{d}\frac{x_i^{\frac{1}{2}}}{(\beta x_i+1)^{\frac{1}{2}}(x_i+\gamma)^{\frac{1}{2}}}$$

Cauchy-Schwarz arithmetic and geometric means

$$\leq \alpha_d(x) \triangleq \alpha(d; \underbrace{x, \ldots, x}_{d})$$

$$= \sqrt{\frac{d(1-\beta\gamma)x}{(\beta x+1)(x+\gamma)}}\sqrt{\frac{d(1-\beta\gamma)\lambda\left(\frac{\beta x+1}{x+\gamma}\right)^{d}}{\left(\beta\lambda\left(\frac{\beta x+1}{x+\gamma}\right)^{d}+1\right)\left(\lambda\left(\frac{\beta x+1}{x+\gamma}\right)^{d}+\gamma\right)}}$$

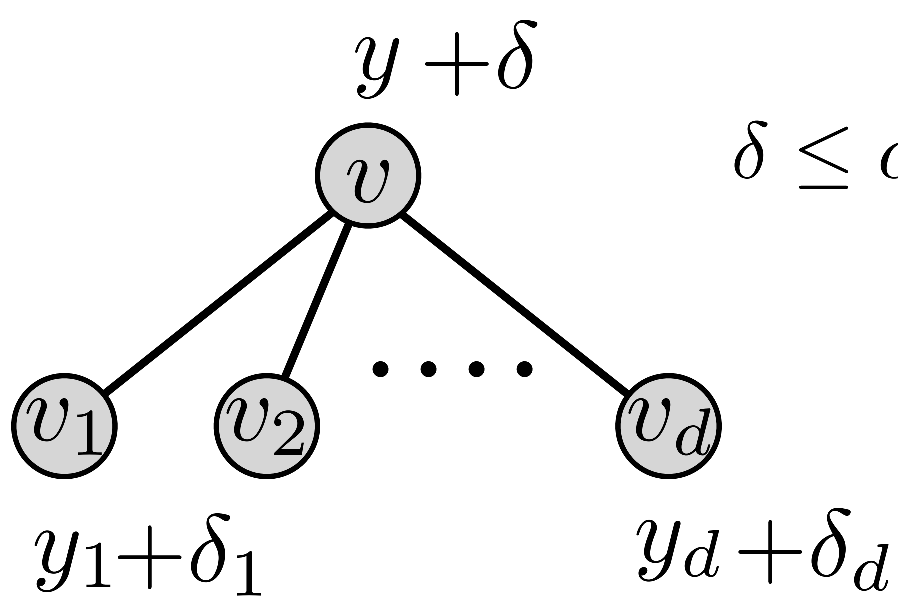
$$= \frac{\Phi(f(x))}{\Phi(x)}|f'(x)|$$

$$\alpha_d(x) = \sqrt{\frac{d(1-\beta\gamma)x}{(\beta x+1)(x+\gamma)}}\sqrt{\frac{d(1-\beta\gamma)\lambda\left(\frac{\beta x+1}{x+\gamma}\right)^d}{\left(\beta\lambda\left(\frac{\beta x+1}{x+\gamma}\right)^d+1\right)\left(\lambda\left(\frac{\beta x+1}{x+\gamma}\right)^d+\gamma\right)}}$$

$$= \sqrt{\frac{d(1-\beta\gamma)x}{(\beta x+1)(x+\gamma)}}\sqrt{\frac{d(1-\beta\gamma)f_d(x)}{(\beta f_d(x)+1)\left(f_d(x)+\gamma\right)}}$$

$$\leq \sqrt{\frac{d(1-\beta\gamma)\hat{x}}{(\beta\hat{x}+1)(\hat{x}+\gamma)}}$$

$$= \sqrt{|f_d'(\hat{x}_d)|}$$



$$\delta \leq \alpha \cdot \max_{1\leq i\leq d}\{\delta_i\}$$

$$\alpha < 1$$

$$f_d(x) = \lambda\left(\frac{\beta x+1}{x+\gamma}\right)^d$$

$$\hat{x}_d = f_d(\hat{x}_d)$$

$$|f_d'(\hat{x}_d)| < 1$$

anti-ferromagnetic: $\beta\gamma < 1$

$$f_d(x) = \lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d$$

$\forall d < \Delta, |f'_d(\hat{x}_d)| < 1$

- SSM in trees of max-degree Δ
- SSM in graphs of max-degree Δ
- $\exists$ FPTAS for graphs of max-degree Δ

bounded Δ

$|f'_\Delta(\hat{x}_\Delta)| < 1$ → SSM in Δ-reg. tree

in reg. trees: → WSSM → SSM

[Weitz'06] + [Sinclair-Srivastava-Thurley'12] + translation

requirement of potential function:

$$f(x) = \lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d \qquad \hat{x} = f(\hat{x})$$

uniqueness: $|f'(\hat{x})| < 1$

amortized decay: $|f'(x)| \cdot \dfrac{\Phi\left(f(x)\right)}{\Phi(x)} < 1$

requirement of potential function:

$$f(x) = \lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d \qquad \hat{x} = f(\hat{x})$$

phase-trans: $|f'(\hat{x})| = 1$

amortized decay: $|f'(x)| \cdot \dfrac{\Phi\left(f(x)\right)}{\Phi(x)}$

$$|f'(\hat{x})| \cdot \frac{\Phi\left(f(\hat{x})\right)}{\Phi(\hat{x})} = 1$$

$$\left[f'(x) \cdot \frac{\Phi\left(f(x)\right)}{\Phi(x)} \right]' \Bigg|_{x=\hat{x}} = 0$$

$$\Rightarrow \quad (\ln(\Phi(\hat{x})))' = -\frac{f''(\hat{x})}{2} = \frac{1}{2} \left(\frac{1}{\hat{x}} + \frac{1}{\hat{x} + \gamma} + \frac{\beta}{\beta \hat{x} + 1} \right)$$

requirement of potential function:

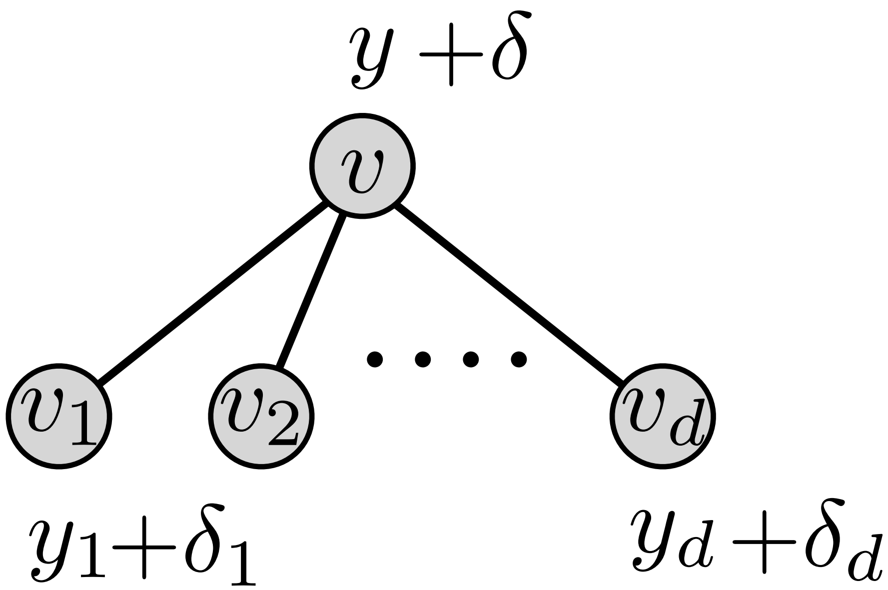
$$(\ln(\Phi(\hat{x})))' = \frac{1}{2}\left(\frac{1}{\hat{x}} + \frac{1}{\hat{x}+\gamma} + \frac{\beta}{\beta\hat{x}+1}\right)$$

strengthen the requirement:

$$(\ln(\Phi(x)))' = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+\gamma} + \frac{\beta}{\beta x+1}\right)$$

$$\Rightarrow \quad \Phi(x) = \frac{C}{\sqrt{x(\beta x+1)(x+\gamma)}}$$

Computationally Efficient Correlation Decay

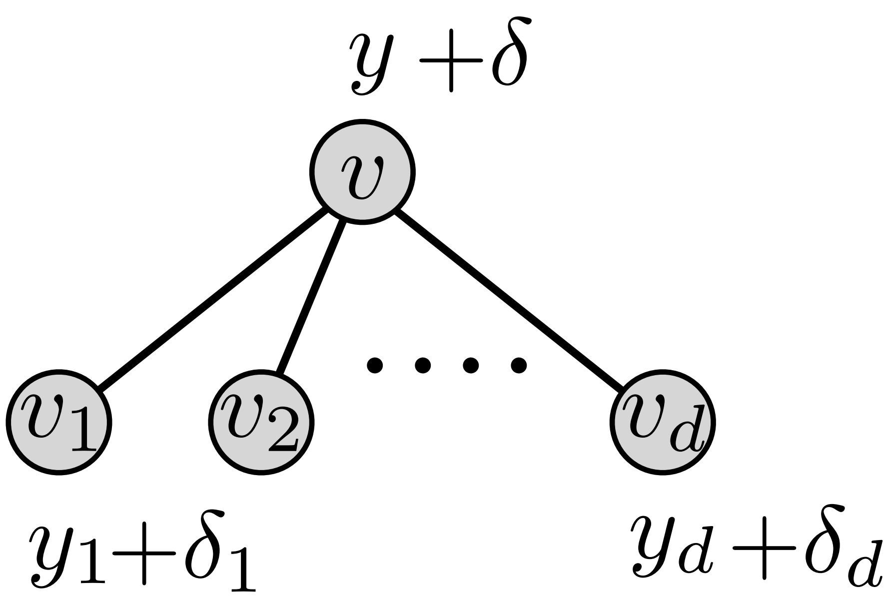


$$\delta \leq \alpha_d(x) \cdot \max_{1 \leq i \leq d} \{\delta_i\}$$

$$\alpha_d(x) = \sqrt{\frac{d(1-\beta\gamma)x}{(\beta x+1)(x+\gamma)}}\sqrt{\frac{d(1-\beta\gamma)f_d(x)}{(\beta f_d(x)+1)\left(f_d(x)+\gamma\right)}}$$

$$\leq \alpha^{\lceil \log_M (d+1) \rceil}$$ for some $\alpha < 1$, $M > 1$

Computationally Efficient Correlation Decay



$$\delta \le \alpha_d(x) \cdot \max_{1 \le i \le d} \{\delta_i\}$$

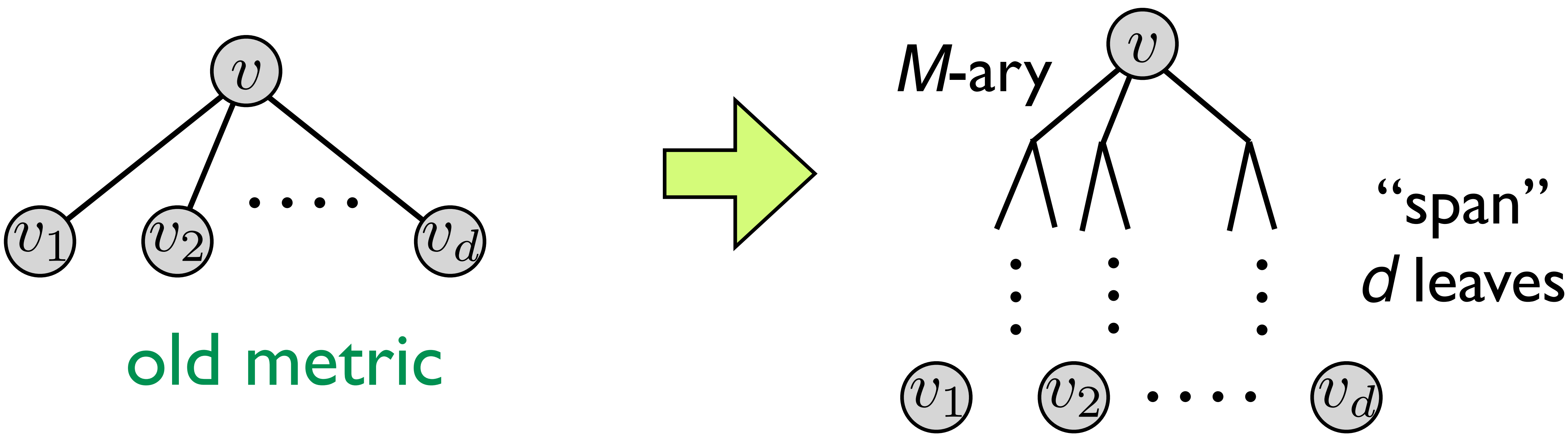
$\alpha_d(x) \le \alpha^{\lceil \log_M (d+1) \rceil}$ for some $\alpha < 1$ $M > 1$

for small $d < M$ one-step recursion decays α

for large $d \ge M$ one-step recursion decays $\alpha^{\lceil \log_M (d+1) \rceil}$

behaves like $\lceil \log_M (d+1) \rceil$ steps!

Computationally Efficient Correlation Decay



correlation decay in new metric α^{distance}

size grows exponentially: M^{distance}

distance = O(log *n*) 1/poly-precision in poly-time

anti-ferromagnetic: $\beta\gamma < 1$

WSM in d-reg. trees for $d \leq \Delta$

⇒ $\exists$ FPTAS for graphs of max-degree Δ

bounded Δ or $\Delta = \infty$

[Weitz'06] hardcore model

[Sinclair-Srivastava-Thurley'12] Ising model

uniqueness condition:

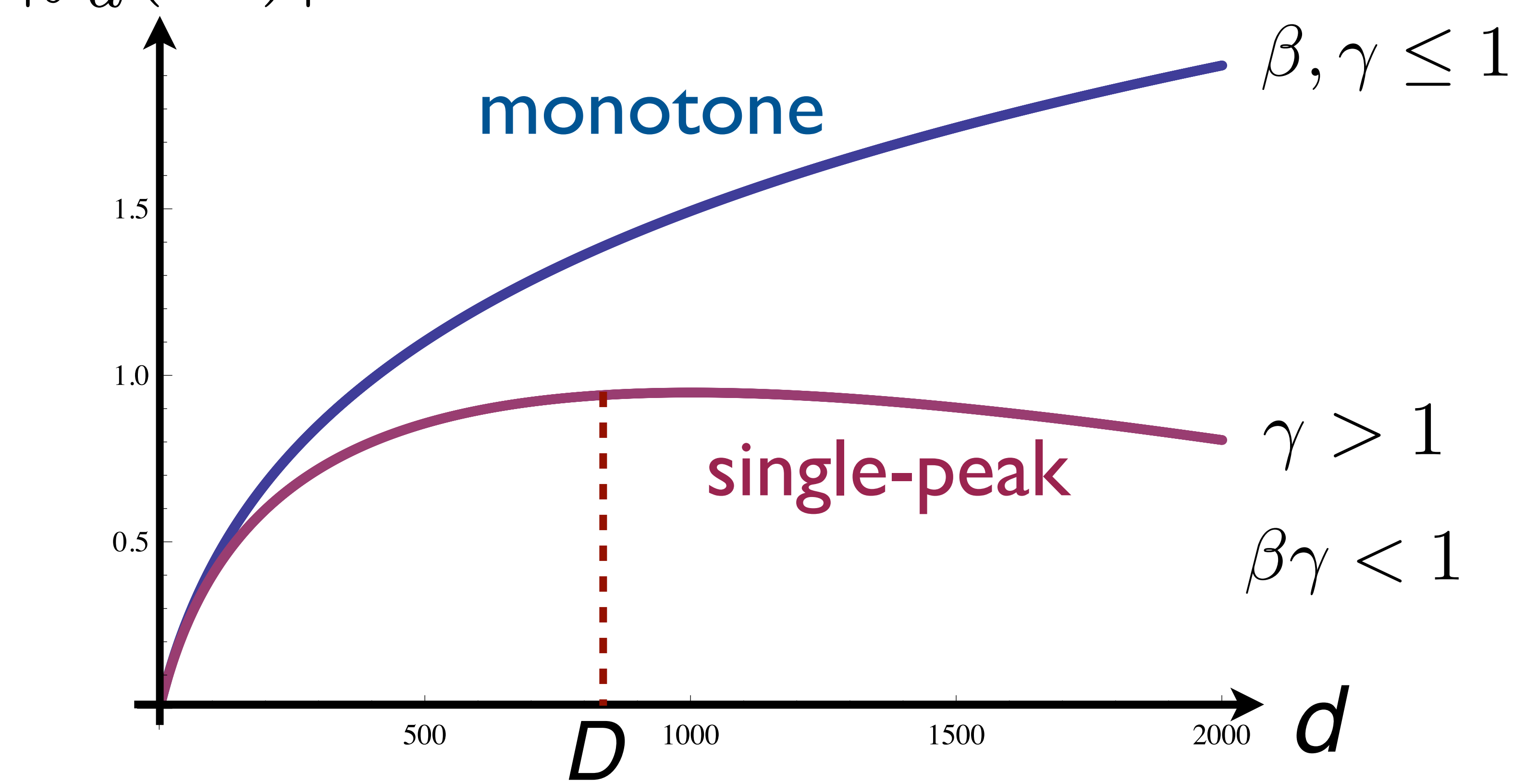
WSM in Δ-reg. tree

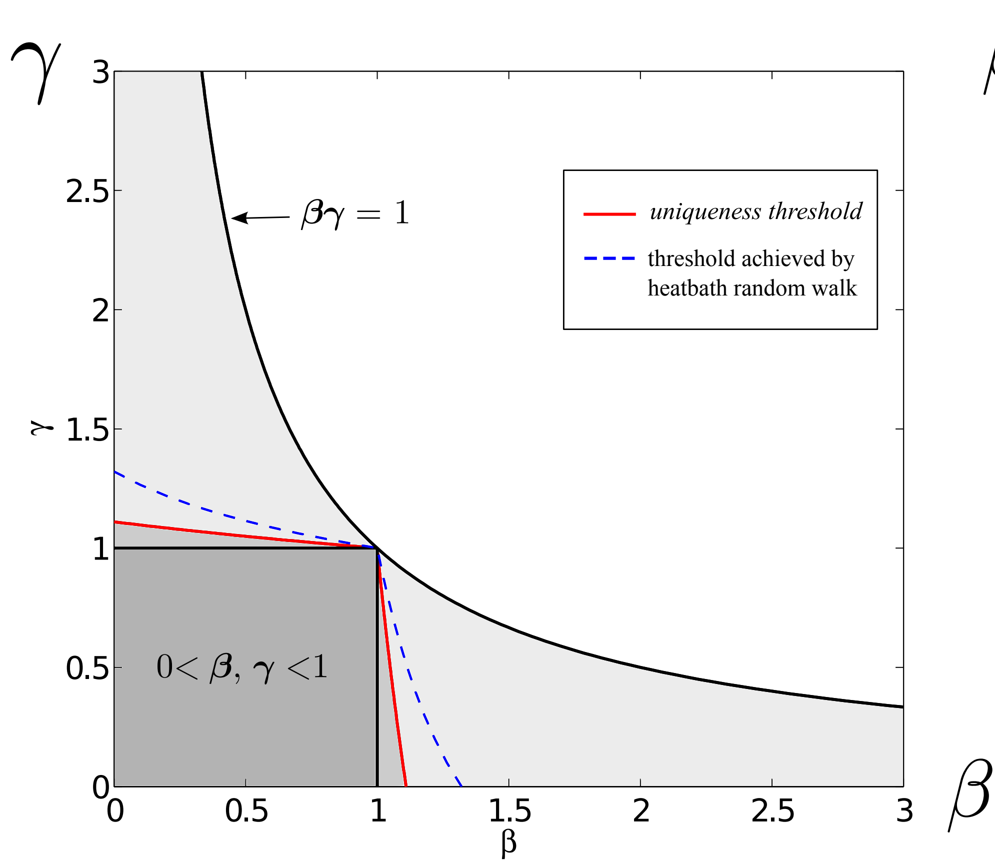
$$f_d(x) = \lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d$$

$$\hat{x}_d = f_d(\hat{x}_d)$$

$$|f'_d(\hat{x}_d)| < 1$$

due to [Guo'12]





$\beta, \gamma \leq 1$

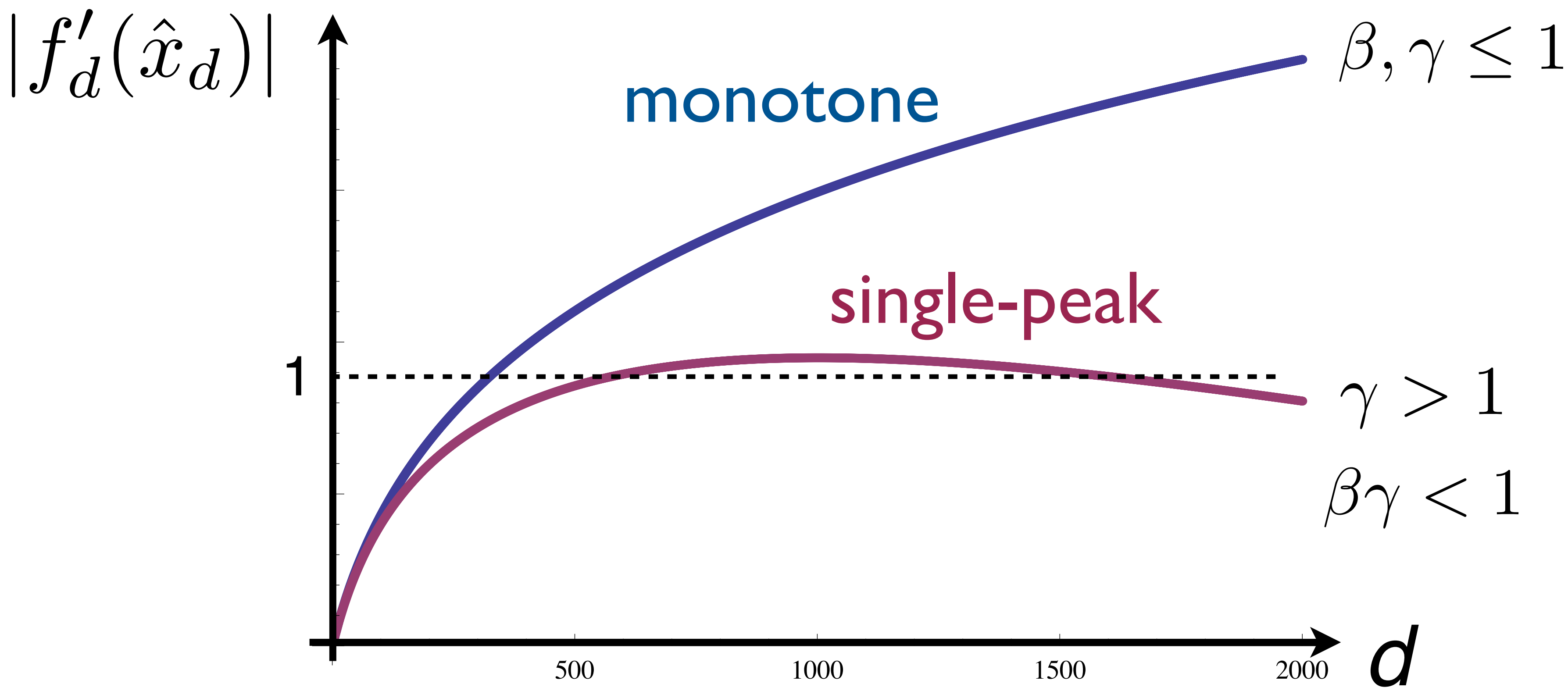
WSM in Δ-reg. tree ⇒

WSM in *d*-reg. tree for *d* ≤ Δ

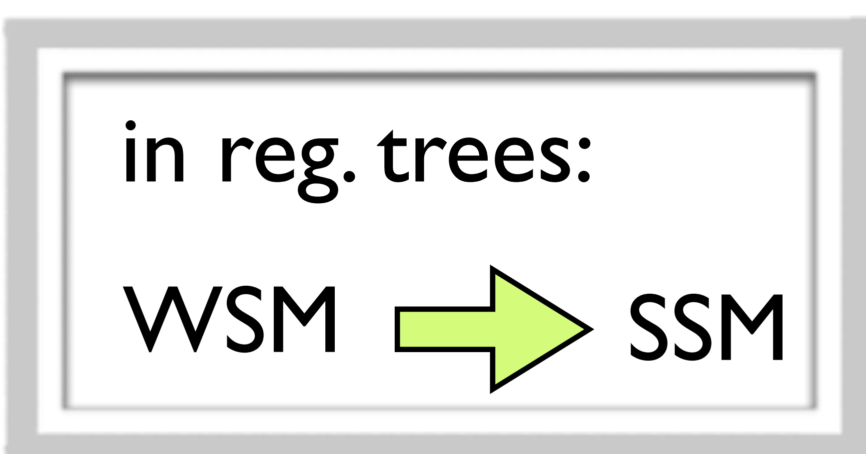
$\gamma > 1 \quad \beta\gamma < 1$

WSM in *D*-reg. tree ⇒

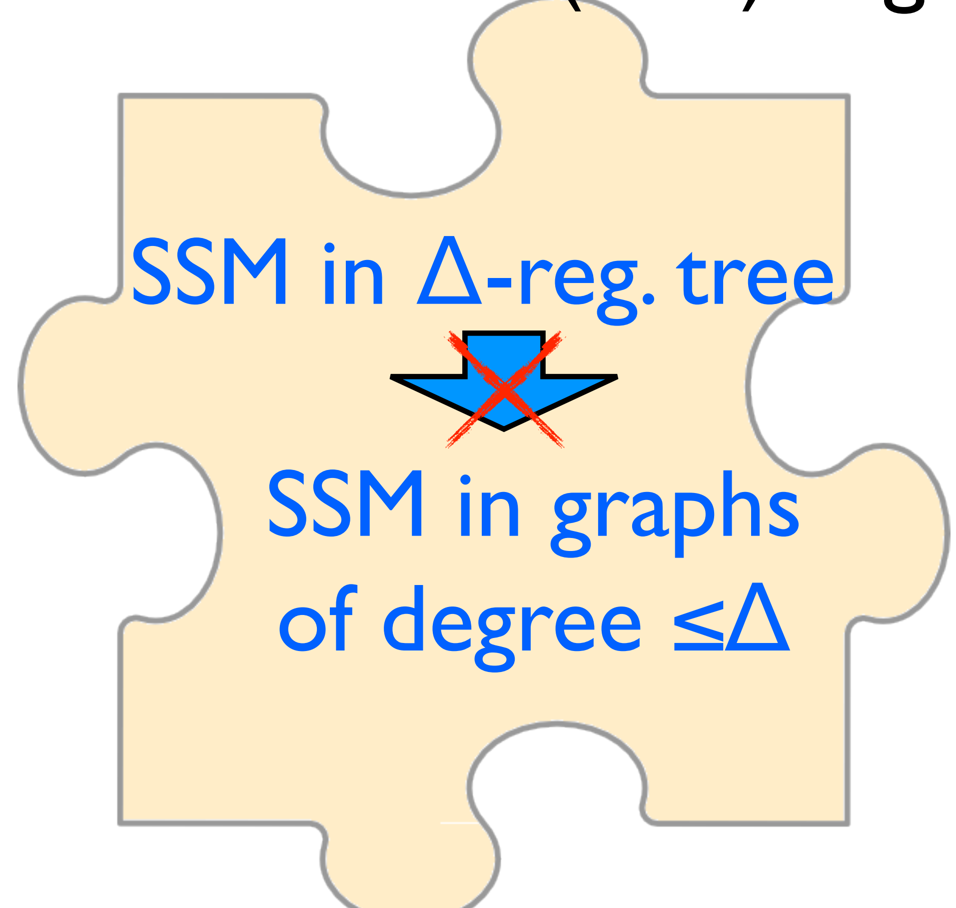
WSM in all *d*-reg. trees



(β, γ, λ) that WSM in Δ-reg. tree but not in (Δ-1)-reg. tree

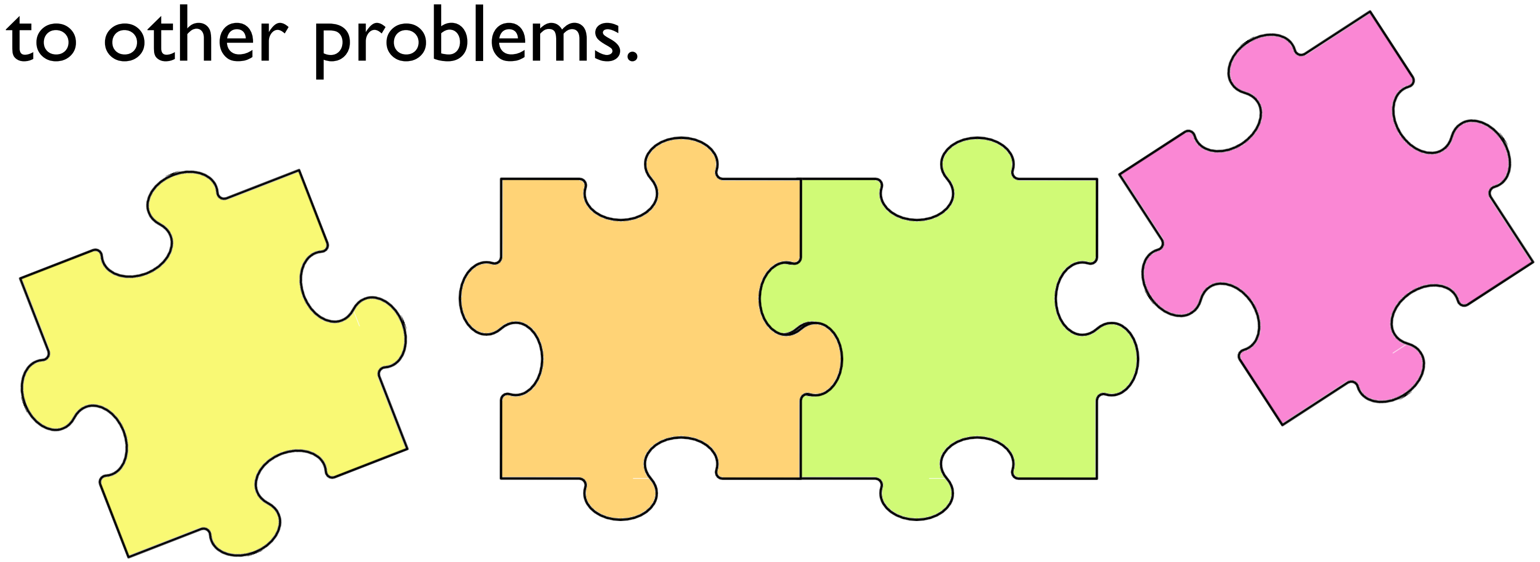


[Weitz'06] +
[Sinclair-Srivastava-Thurley'12] +
translation



Open Problems

- Characterization of SSM in ferromagnetic 2-state spin systems.
- SSM in multi-state spin systems:
    - difficulty: no SAW-tree;
    - implications: WSM vs. SSM in reg. trees, monotonicity of WSM/SSM w.r.t degree.
- Apply potential analysis and computationally efficient correlation decay to other problems.



Thank you!