Approximate Counting *via* Correlation Decay *on* Planar Graphs

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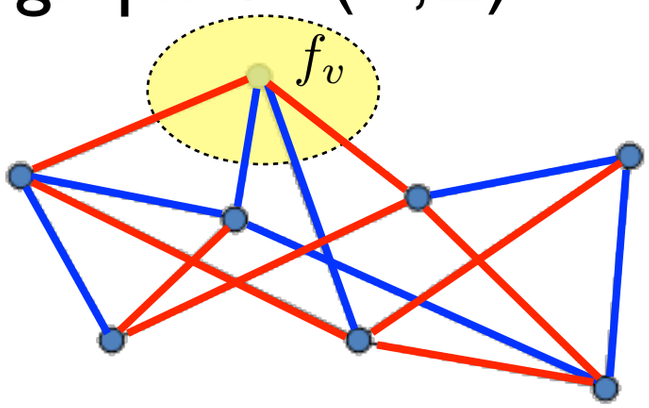
Holant Problems

(Valiant 2004)

instance: $\Omega = (G(V,E), \{f_v\}_{v \in V})$

graph G=(V,E)



edges: *variables* (domain [q])

vertices: *constraints* (arity=degree)

symmetric $f_v : [q]^{\deg(v)} \to \mathbb{C}$

configuration (*solution, coloring, ...*): $\sigma \in [q]^E$

holant (*counting*):

$$\mathrm{holant}(\Omega) = \sum_{\sigma \in [q]^E} \prod_{v \in V} f_v\left(\sigma\,|_{E(v)}\right)$$

#matchings: q=2 $\sigma \in \{0,1\}^E$ $f_v \equiv$ At-Most-One

Holant problem: $\mathrm{Holant}(\mathcal{G}, \mathcal{F})$

graph family function family

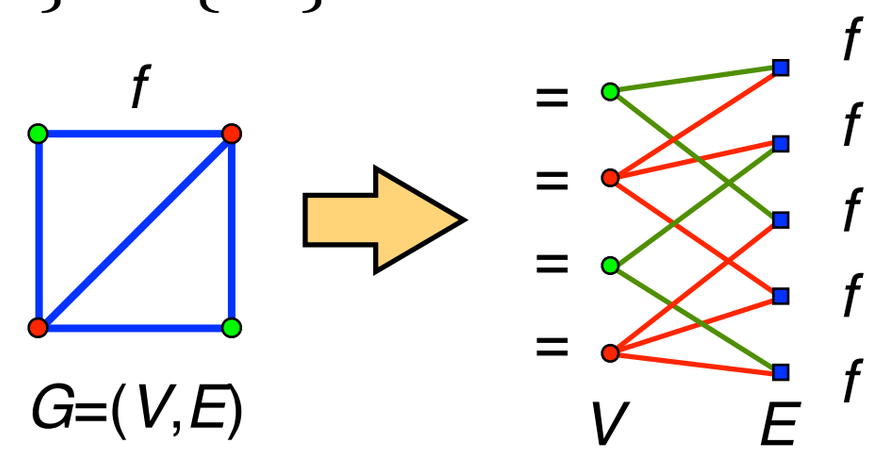
input: $\Omega = (G(V,E), \{f_v\}_{v \in V})$ with $\begin{cases} G \in \mathcal{G} \\ f_v \in \mathcal{F} \end{cases}$

output: $$\mathrm{holant}(\Omega) = \sum_{\sigma \in [q]^E} \prod_{v \in V} f_v\left(\sigma\,|_{E(v)}\right)$$

spin system / graph homomorphism (*G.H.*):

$$\mathcal{F} = \{f : [q]^d \to \mathbb{C}, d \leq 2\} \cup \{=\}$$

- #IS, #VC
- #q-colorings, #*H*-colorings
- *hardcore/Ising/Potts* models, MRF



spin model holant

Holant Problems

Holant problem: $\mathrm{Holant}(\mathcal{G}, \mathcal{F})$

graph family function family

characterize the *tractability* of $\mathrm{Holant}(\mathcal{G}, \mathcal{F})$ by $\mathcal{G}$ and $\mathcal{F}$

Bad news: for general/planar $\mathcal{G}$, almost all nontrivial $\mathcal{F}$: #P-hard

(Dyer-Greenhill'00, Bulatov-Grohe'05, Dyer-Goldberg'07, Bulatov'08, Goldberg-Grohe-Jerrum'10, Cai-Chen'10, Cai-Chen-Lu'10, Cai-Lu-Xia'10, Dyer-Richerby'10, Dyer-Richerby'11, Cai-Chen'12, ***Cai-Lu-Xia'13***)

Good news: tractable if $\mathcal{G}$ is *tree*, $\mathcal{F}$ is *Spin* or *Matching*

(arity$\leq$2 and =) (At-Most-One)

Our result:

$\mathcal{G}$ is *planar* (locally like a tree)

$\mathcal{F}$ is *regular* (like spin/matching)

correlation decay (local info is enough)

FPTAS

Gibbs Measure

$$\Omega = (G(V,E), \{f_v\}_{v\in V}) \qquad f_v : [q]^{\deg(v)} \to \mathbb{R}_{\geq 0}$$

$$\mathrm{holant}(\Omega) = \sum_{\sigma\in[q]^E} \prod_{v\in V} f_v\left(\sigma\mid_{E(v)}\right)$$

Gibbs measure: $\Pr(\sigma) = \dfrac{\prod_{v\in V} f_v(\sigma|_{E(v)})}{\mathrm{holant}}$

marginal probability: $\sigma_A \in [q]^A \quad A \subset E$

$$\Pr(\sigma(e) = c \mid \sigma_A)$$

compute $\Pr(\sigma(e) = c \mid \tau_A) \pm \frac{1}{n}$ in time $\mathrm{poly}\,(n)$

self-reduction

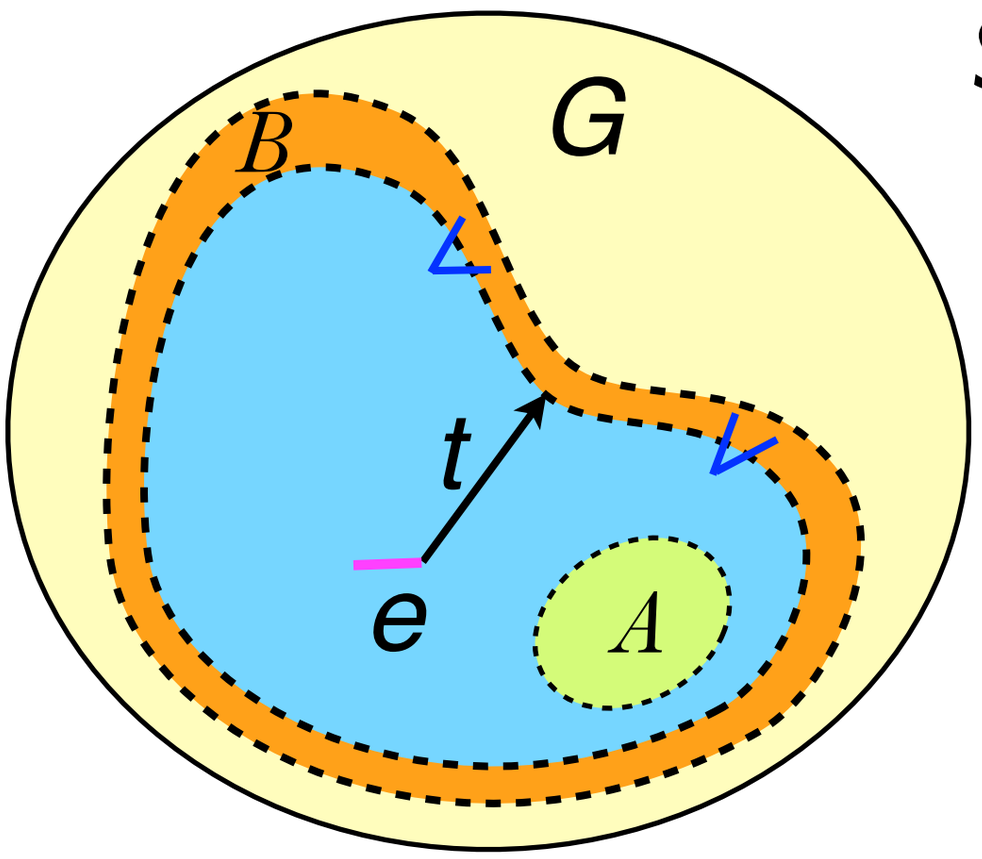
FPTAS for $\mathrm{holant}(\Omega)$

Correlation Decay

strong spatial mixing (SSM): $\forall \sigma_B \in [q]^B$

$$\left|\Pr(\sigma(e) = c \mid \sigma_A) - \Pr(\sigma(e) = c \mid \sigma_A, \sigma_B)\right|$$

$$\leq \mathrm{poly}(|V|)\exp(-\Omega(t))$$



SSM: sufficiency of *local information*

for approx. of $\Pr(\sigma(e) = c \mid \sigma_A)$

efficiency of *local computation* (FPTAS)

such implication was known for:

q=2, $\mathcal{F}$ is $\begin{cases} \text{Spin (Weitz'06)} \\ \text{Matching} \end{cases}$

(Bayati-Gamarnik-Katz-Nair-Tetali'08)

Regularity

Pinning: *symmetric* $f : [q]^d \to \mathbb{C}$ $\quad \tau \in [q]^k$

$$\mathrm{Pin}_\tau(f) = g \quad \text{where } g : [q]^{d-k} \to \mathbb{C}$$

$$\forall \sigma \in [q]^{d-k}, \quad g(\sigma) = f(\sigma_1, \ldots, \sigma_{d-k}, \tau_1, \ldots, \tau_k)$$

when q=2 write $f = [f_0, f_1, \ldots, f_d]$

where $f_i = f(\sigma)$ that $\|\sigma\|_1 = i$

C

a family $\mathcal{F}$ of symmetric functions is *regular* if

$\exists$ a finite C s.t. $\forall f \in \mathcal{F}$, f is C-regular

$$[\underbrace{f_0, f_1, f_2}_{d-k+1}, \ldots, f_i, \ldots, f_{d-1}, f_d]$$

counterexample: $[\underbrace{0, \ldots, 0}_{\frac{d}{2}}, 1, \underbrace{0, \ldots, 0}_{\frac{d}{2}}]$

examples: *bounded-arity*

equality [1,0,...,0,1]

at-most-one [1,1,0,...,0]

cyclic [*a*,*b*,*c*,*a*,*b*,*c*,...]

Holant can be computed in time $\mathrm{poly}(n) \cdot 2^{\mathrm{tw}}$ if $\begin{cases} \mathcal{F} \text{ is Spin (}\textit{junction-tree BP}\text{)} \\ \mathcal{F} \text{ has bounded-arity} \end{cases}$

(*tensor network*, Markov-Shi'09)

Theorem I

If $\mathcal{F}$ is regular, then $\mathrm{holant}(G, \{f_v\}_{v \in V} \subset \mathcal{F})$ can be computed in time $\mathrm{poly}(|V|) \cdot 2^{O(\mathrm{treewidth}(G))}$

Theorem II

If $\mathcal{G}$ is planar (apex-minor-free), $\mathcal{F}$ is regular, then

SSM $\Rightarrow$ FPTAS for $\mathrm{Holant}(\mathcal{G}, \mathcal{F})$

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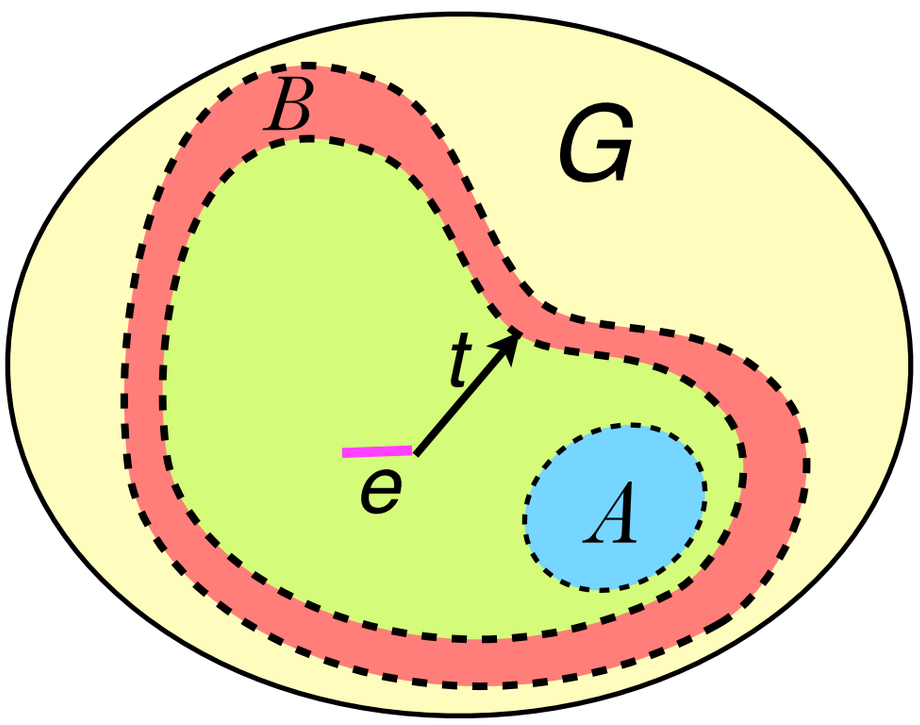
SSM: $\left|\Pr(\sigma(e) = c \mid \sigma_A) - \Pr(\sigma(e) = c \mid \sigma_A, \sigma_B)\right| \leq \mathrm{poly}(|V|) \exp(-t)$



compute $\Pr(\sigma(e) = c \mid \tau_A) \pm \frac{1}{n}$ in time $\mathrm{poly}\,(n)$ ⇒ FPTAS for Holant

Theorem (Demaine-Hajiaghayi'04)

For apex-minor-free graphs, treewidth of *t*-ball is O(*t*).

Theorem II

If $\mathcal{G}$ is planar (apex-minor-free), $\mathcal{F}$ is regular, then SSM ⇒ FPTAS for $\mathrm{Holant}(\mathcal{G}, \mathcal{F})$

Theorem I

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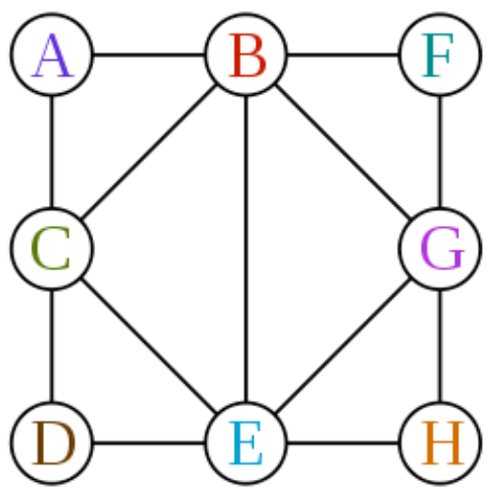
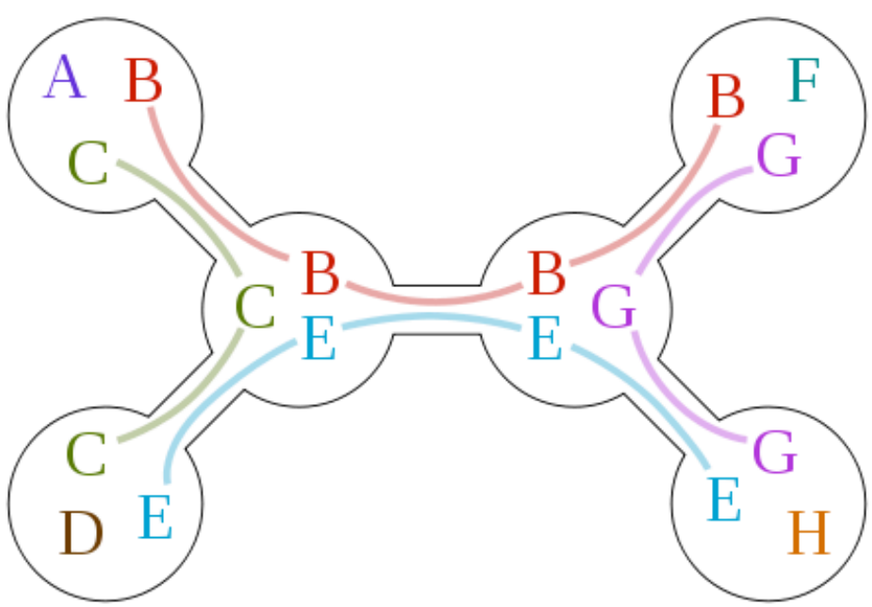
tree-decomposition:





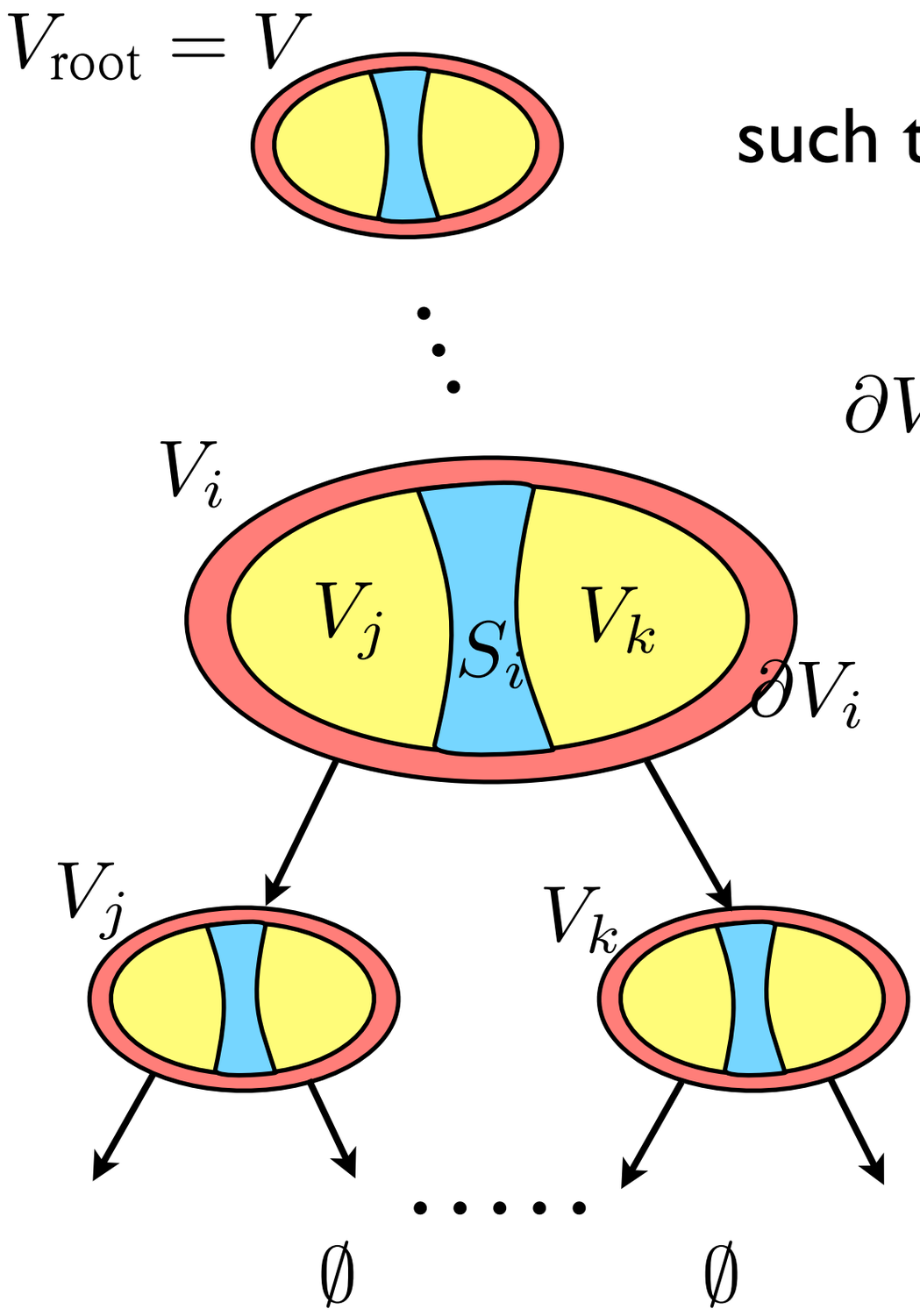
a tree of "bags" of vertices:

1. Every vertex is in some bag.
2. Every edge is in some bag.
3. If two bags have a same vertex, then all bags in the path between them have that vertex.

width: max bag size -1

treewidth: width of optimal tree decomposition

Separator-Decomposition T_G of $G(V,E)$:



each node $i \in T_G$ corresponds to (V_i, S_i)

such that $\begin{cases} V_{\text{root}} = V \text{ and } V_{\text{leaf}} = \emptyset \\ S_i \neq \emptyset \text{ is a vertex separator of } V_j, V_k \subset V_i \text{ in } G[V_i] \end{cases}$

∂V_i is vertex boundary of V_i in $G[V_i]$

width: $\max_{i \in T_G} \{|S_i|, |\partial V_i|\}$

separator-width *sw*(*G*) :

width of optimal T_G

Theorem:

$sw(G) = \Theta(tw(G))$ and

T_G can be constructed

in time $\text{poly}(n) \cdot 2^{O(tw(G))}$

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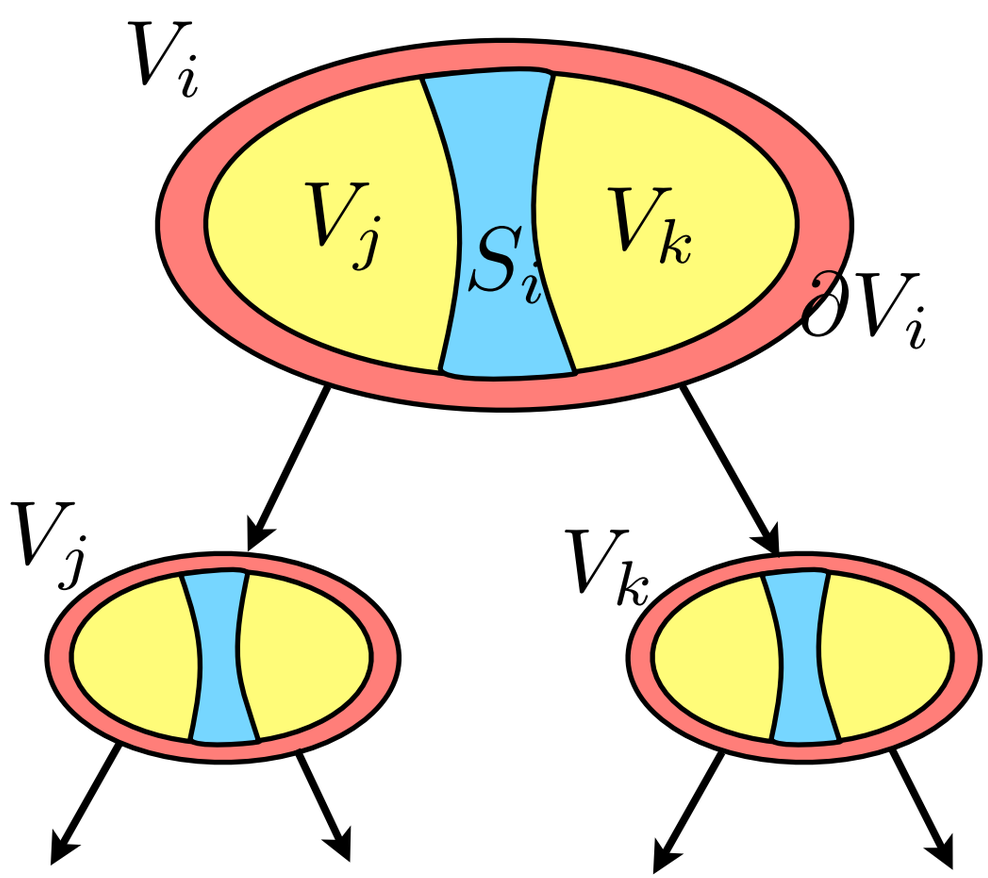
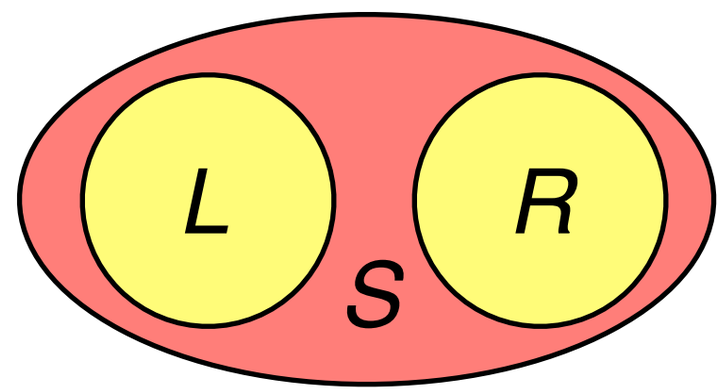


conditional independence:

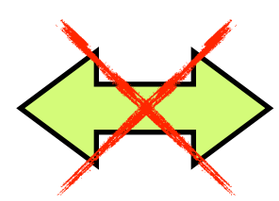


$\Pr(\sigma_L \mid \sigma_S)$ and $\Pr(\sigma_R \mid \sigma_S)$

are independent for fixed σ_S

S_i : vertex separator

∂V_i : vertex boundary



S : edge separator

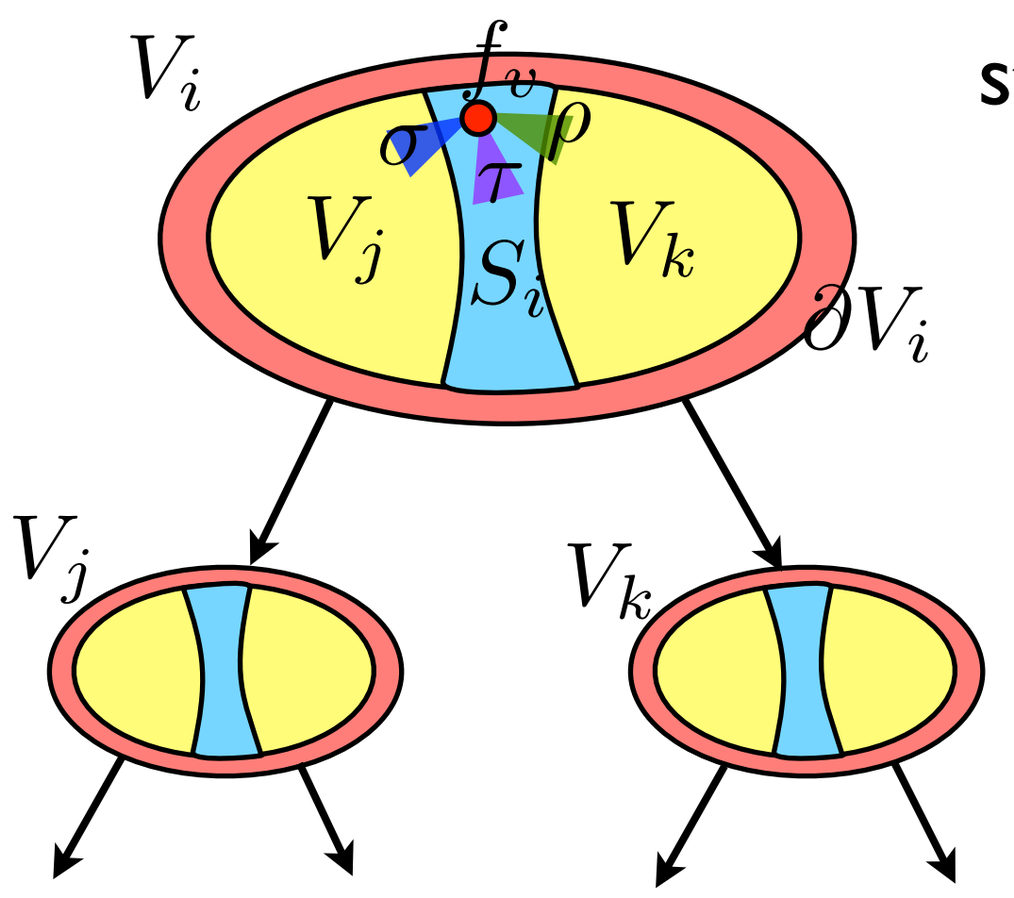
Peering: given $f : [q]^d \to \mathbb{C}$ $\quad \tau \in [q]^k$

$\mathrm{Peer}_\tau(f) : [q]^k \to \{0, 1\}$ defined as

$$\forall \sigma \in [q]^k, \quad \mathrm{Peer}_\tau(f)(\sigma) = \begin{cases} 1 & \mathrm{Pin}_\sigma(f) = \mathrm{Pin}_\tau(f) \\ 0 & \text{o.w.} \end{cases}$$

$$\mathrm{Peer}_\tau(f) = \{\sigma \in [q]^k \mid \mathrm{Pin}_\sigma(f) = \mathrm{Pin}_\tau(f)\}$$

peering classifies configurations around a vertex into equivalent classes



states of a vertex: peer classes

$f_v(\sigma\tau\rho)$ depends only on peer classes of σ, τ, ρ

Holant value can be figured out by keeping track of only peer classes

for regular *f*, # of peer classes is always finite

Algorithmic Implications

applying the SSM obtained by a "decay-only" technique *recursive coupling* (Goldberg-Martin-Paterson'05),

we have FPTAS for:

- #q-coloring of triangle-free planar graphs of max-degree Δ for $q > 1.76322\,\Delta - 0.47031$
- ferromagnetic Ising model with temperature β and field B on planar graphs of max-degree Δ, when
$$\Delta < \frac{1}{4}\left(\frac{e^{2\beta B}+e^{-2\beta B}}{e^{\beta B}+e^{-\beta B}}\right)^2$$
- ferromagnetic Potts model with temperature β on planar graphs of max-degree Δ for $\beta = O(\frac{1}{\Delta})$

(conjectured by Gamarnik-Katz'06)

Conclusions and Open Problems

for *Holant* problems defined by *regular* constraints:

- a $\mathrm{poly}(n) \cdot 2^{\mathrm{treewidth}}$ time algorithm for exact computation;
- SSM implies FPTAS on planar graphs.

open problems:

- in terms of reliance on treewidth, tightness of 2^{tw} for regular Holant and n^{tw} for all symmetric Holant (under some assumption);
- using SSM for FPTAS on general graphs.